Temperature distribution in a uniform medium heated by linear absorption of a Gaussian light beam

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The linear-diffusion equation is considered for a positive half-space with heat sources represented by Gaussian functions in the transverse plane and by exponential decay along the longitudinal axis. The exact solution is presented as a single quadrature of the complementary error function (erfc). The approximate solution is suggested in the form of the product of two Gaussian functions and the hyperbolic secant function. Comparison with the exact solution shows that the error of this approximation is near 10%. The approximation may be used in different medical applications, e.g., laser angioplasty.

1. Introduction

A good understanding of some of the phenomena that are produced by laser radiation that hits an organic tissue are essential for the success of many applications. Some of them have been considered in past publications,[1–3] and testing of such interactions between radiation and tissue has been developed from various points of view.[4–11] Three-dimensional approximations have been presented,[6,9,10] but they are complicated and difficult to analyze.

The aim of this paper is to present a simple description of the process of the three-dimensional thermodiffusion of the energy of absorbed light in an approximation of a Gaussian transverse profile of an incident light beam. For the light we suppose an approximation of geometric optics, i.e., there is no diffraction in our approximation. The initial stage of the heating of the surface of a body by a laser beam can be described in the linear approximation, i.e., in the supposition that thermal properties do not depend on the temperature. In this approximation the coefficient of thermoconductivity, the thermocapacity, the density, and the absorption coefficient are assumed to be constant.

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2. Equations

In linear approximations the source of heat inside a medium irradiated by a laser beam may be described by the function

$$S(x, y, z) = S_m \exp[-(x^2 + y^2)/r_0{}^2 - \gamma z], \qquad (1)$$

where r_0 is the radius of the Gaussian beam, γ is the absorption coefficient, and x, y, and z are the Cartesian coordinates. The factor $S_m = \gamma' I_0/\rho c$ depends on the axial intensity I_0 of the initial beam; ρ is the density and c is the heat capacity of the medium. The γ' parameter usually depends on the absorption, scattering, and phase function.[4,5,7] For the first approximation we may set $\gamma = \gamma'$.

The surface of the medium is defined by the condition $z = 0$, and the medium is located at $z > 0$. The heat-conduction equation has the form

$$\mathrm{d}T/\mathrm{d}t = \mu \Delta T + S, \qquad (2)$$

where μ is the thermal diffusion coefficient, and $\Delta = \partial^2/\partial z^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian term.

The initial distribution of the temperature is assumed to be constant. Without losing generality we can put this constant equal to zero. Hence we may consider the relation $T(x, y, z, 0) = 0$ as the initial condition for Eq. (2).

At the surface there is no flux of the heat. Therefore the boundary condition may be written in the form $\partial T/\partial z = 0$ at $z = 0$. Hence Eq. (2) must be solved for the half space of $z > 0$. The symmetry of Eq. (2) enables us to consider the solution in the full space by the symmetrical continuation of the sources in Eq. (1) to negative values of z:

$$S(x, y, z) = S_m \exp[-(x^2 + y^2)/r_0{}^2 - \gamma|z|]. \quad (3)$$

2. Source Function and Fourier-Transform

The general solution of Eq. (2) in the three-dimensional Euclidean space may be written by the source function:

$$P(x,y,z,t) = (4\pi\mu t)^{-3/2}\exp[-(x^2 + y^2 + z^2)/(4\mu t)]. \quad (4)$$

This function is the solution of Eq. (2) for $S(x, y, z, t) = \delta(x)\delta(y)\delta(z)\delta(t)$. When function (4) is used in the solution of Eq. (2), the general case may be written in the form

$$T(x, y, z, t) = \int_0^t \mathrm{d}\tau \iiint_{-\infty}^{\infty} \mathrm{d}u\mathrm{d}v\mathrm{d}w S(u, v, w) \times P(x - u, y - v, z - w, \tau). \quad (5)$$

For a complicated function S, the integral Eq. (5) can be calculated numerically. Because this calculation is cumbersome it is better to represent the solution by Fourier integrals. Let

$$\tilde{T}(k,p,q,t) = \int_0^t \mathrm{d}\tau \exp[-(k^2 + p^2 + q^2)(t - \tau)\mu] \times \tilde{S}(k,p,q,t), \quad (6)$$

where

$$\tilde{S}(k,p,q,t) = (2\pi)^{-3/2} \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z \times \exp(ikx + ipy + iqz)S(x,y,z,t). \quad (7)$$

The solution of Eq. (2) may be presented in the form

$$T(x,y,z,t) = (2\pi)^{-3/2} \iiint_{-\infty}^{\infty} \mathrm{d}k\mathrm{d}p\mathrm{d}q \times \exp(-ikx - ipy - iqz)\tilde{T}(k,p,q,t). \quad (8)$$

Eqs. (6)–(8) are easier to calculate numerically than is Eq. (5): It is easier to calculate two 3-integral functions than to calculate one 4-integral function. In addition, fast Fourier-transform algorithms can be used to calculate Eqs. (7) and (8). Therefore, Eqs. (5) and (8) may both be considered as solutions of the problem. However, they require the use of a powerful computer to be realized. The purpose of Sections 3 and 4 is to describe effective algorithms for calculating the solution.

3. One-Quadrature Representation of the Exact Solution

In this section the solution of the heat-conduction equation, Eq. (2), in the form of the integral in Eq. (5) is calculated with the special form of the heat sources described by Eq. (3). Substituting Eq. (3) into Eq. (5) enables us to write the solution in the form

$$T(x, y, z, t) = \int_0^t \mathrm{d}\tau I(x, \mu\tau)I(y, \mu\tau)J(z, \mu\tau), \quad (9)$$

$$I(x, v) = \int_{-\infty}^{+\infty} \mathrm{d}u \exp[-u^2/r^2 - (x - u)^2/(4v)], \quad (10a)$$

$$J(x, v) = \int_{-\infty}^{+\infty} \mathrm{d}u \exp[-\gamma|u| - (x - u)^2/(4v)], \quad (10b)$$

Both integrals in Eqs. (10) are known[12]:

$$I(x, v) = \exp[-x^2/(r^2 + 4v)]/(1 + 4v/r^2)^{1/2}, \quad (11)$$

$$J(x, v) = \tfrac{1}{2}\exp(-\gamma^2 v + \gamma x)\mathrm{erfc}[\gamma\sqrt{v} + x/(2\sqrt{v})] + \tfrac{1}{2}\exp(-\gamma^2 v - \gamma x)\mathrm{erfc}[\gamma\sqrt{v} - x/(2\sqrt{v})]. \quad (12)$$

Where the complementary error function erfc is defined by Eq. (7.1.2) from Ref. 12:

$$\mathrm{erfc}(z) = \frac{2}{\sqrt{z}} \int_z^{+\infty} \exp(-t^2)\mathrm{d}t. \quad (13)$$

Note that to obtain Eq. (12) we used formula (7.4.2) from Ref. 12 with $c = 0$.

4. Approximation by the Elementary Function

The solution of Eqs. (9), (11), and (12) of the diffusion Eq. (2) is suitable for numerical calculations, but for practical applications it is interesting to calculate some simple characteristics of the solution: The total thermic energy is given by

$$T_0(t) = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t). \quad (14)$$

The moments of the temperature distribution are described by

$$T_{xx}(t) = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t)x^2, \quad (15)$$

$$T_{yy}(t) = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t) y^2, \tag{16}$$

$$T_{zz}(t) = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t) z^2. \tag{17}$$

It is also possible to calculate moments of higher order, but the presence of the one-quadrature exact solution makes it unnecessary: The approximation with second-order moments can be done with a microcalculator, while the calculation of higher-order moments requires the use of a computer. But with any computer the one-quadrature representation of Eqs. (9)–(11) of the exact solution may be used.

We now present formulas that describe the behavior of the moments from Eqs. (14)–(17) and construct the approximation of the solution with the elementary function.

Consider the marginal distributions

$$T_1(x, t) = \iint_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}z T(x, y, z, t), \tag{18}$$

$$T_2(y, t) = \iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}z T(x, y, z, t), \tag{19}$$

$$T_3(z, t) = \iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y T(x, y, z, t). \tag{20}$$

All three distributions satisfy the one-dimensional differential equation of the same form

$$\partial T_m(u, t)/\partial t = T_m''(u, t) + S_m(u, t), \tag{21}$$

where S_m is the marginal distribution of sources analogous to T_1, T_2, and T_3:

$$S_1(x, t) = \iint_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}z S(x, y, z),$$

$$S_2(y, t) = \iint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}z S(x, y, z),$$

$$S_3(z, t) = \iint_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}x S(x, y, z).$$

The prime represents differentiation by the first argument u (i.e., x, y, or z, respectively).

The total thermal energy is determined by the total integral of the source given by

$$T_0(t) = \iiint_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}x\mathrm{d}z T(x, y, z, t) = S_0 t,$$

where

$$S_0 = \iiint_{-\infty}^{\infty} \mathrm{d}y\mathrm{d}x\mathrm{d}z S(x, y, z). \tag{22}$$

Equation (22) uses the independence of the source on time. Let us calculate the time derivative T_{uu}:

$$\begin{aligned}
\mathrm{d}/\mathrm{d}t\, T_{\mathrm{uu}}(t) &= \int_{-\infty}^{\infty} \mathrm{d}/\mathrm{d}t\, T_m(u, t) u^2 \mathrm{d}u, \\
&= \int_{-\infty}^{\infty} [\mu T_m''(u, t) + S_m(u)] u^2 \mathrm{d}u, \\
&= -\int_{-\infty}^{\infty} \mu T_m'(u, t) 2u \mathrm{d}u + S_{\mathrm{uu}} \\
&= 2\int_{-\infty}^{\infty} \mu T_m(u, t) + S_{\mathrm{uu}} \\
&= 2S_0 \mu t + S_{\mathrm{uu}}.
\end{aligned} \tag{23}$$

Neither S_0 nor S_{uu} depends on time. Therefore we can integrate Eq. (23) by time:

$$T_{\mathrm{uu}}(t) = S_0 \mu t^2 + S_{\mathrm{uu}} t. \tag{24}$$

Eq. (3) give

$$S_0 = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z S(x,y,z) = 2\pi r^2/\gamma, \tag{25}$$

$$S_{\mathrm{xx}} = S_{\mathrm{yy}} = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z S(x,y,z) x^2 = S_0 r^2/2, \tag{26}$$

$$S_{zz} = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z S(x,y,z) z^2 = S_0 2/\gamma^2. \tag{27}$$

Thus we can calculate the mean-square radius of the heat spot:

$$R^2 = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t)(x^2 + y^2) \Big/ \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t)$$
$$= (T_{xx}{}^2 + T_{yy}{}^2)/S_0 = 2T_{xx}{}^2/S_0 = 2\mu t + 2S_{xx}/S_0, \quad (29)$$

and the main-square depth of heat is

$$Z^2 = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t)z^2 \Big/ \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t)$$
$$= \mu t + S_{zz}/S_0. \quad (30)$$

Equations (25)–(30) enable us to approximate the distribution of the temperature by any simple function with appropriate values of R and Z, the correct altitude and size, respectively. A good approximation may be provided by the function

$$T_d(x, y, z, t) = a \exp[-(x^2 + y^2)/r^2]/\cosh(z/b). \quad (31)$$

The values of the three parameters a, b, and r are determined by the relations

$$\iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T_d(x, y, z, t) = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t), \quad (32)$$

$$\iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T_d(x, y, z, t)x^2 = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t)x^2, \quad (33)$$

$$\iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T_d(x, y, z, t)z^2 = \iiint_{-\infty}^{\infty} \mathrm{d}x\mathrm{d}y\mathrm{d}z T(x, y, z, t)z^2. \quad (34)$$

Calculations of the left-hand sides of Eqs. (32)–(34) with the function in Eq. (31) and formulas (25)–(27) give

$$a \times \pi b \times \pi r^2 = S_0 t, \quad (35)$$

$$a \times \pi b \times \pi r^2 \times r^2/2 = S_0 \mu t^2 + S_{xx} t, \quad (36)$$

$$a \times \pi b \times \pi r^2 \times \pi^2 b^2/4 = S_0 \mu t^2 + S_{zz} t. \quad (37)$$

[Formulas (3.531) and (3.533) from Ref. 13 are used to calculate the integral by z.] Equations (35)–(37) give

$$r^2/2 = S_{xx}/S_0 + \mu t; \qquad \pi^2 b^2/4 = S_{zz}/S_0 + \mu t;$$
$$a = S_0 t/(\pi b r^2). \quad (38)$$

Equations (25)–(27), and (38) determine the approxi-

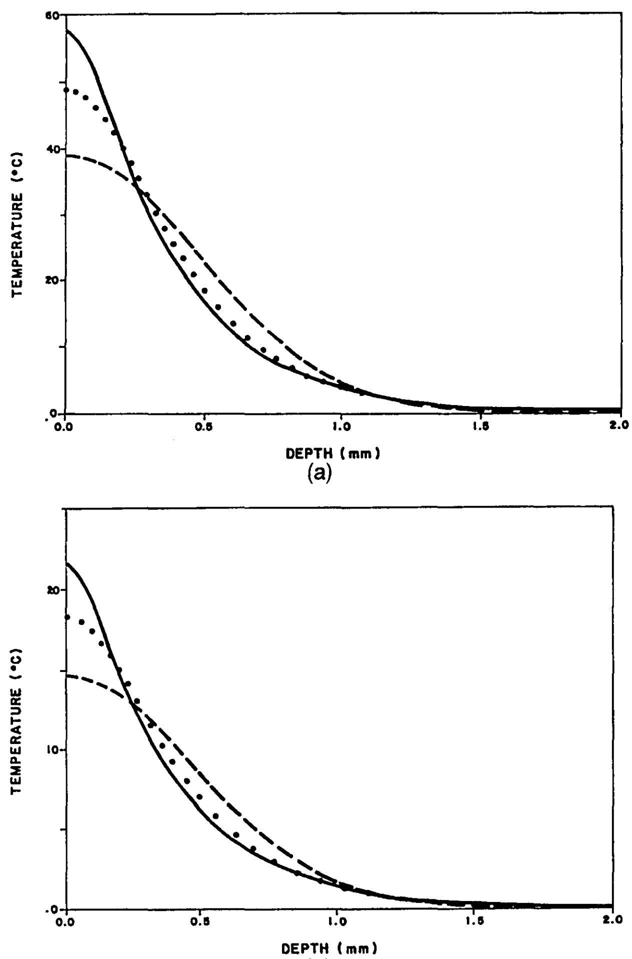

Fig. 1. Comparison of the temperature distributions predicted by the Gauss–hyperbolic secant [Eq. (31)] approximation (dotted curve) with the exact solution [Eqs. (9), and (10)] of Eq. (2) for thermodiffusion (solid curve) for the time t = 0.1 s: (a) the dependence of the temperature on depth at the axis of the beam, and (b) the same distribution at a distance of 1 mm off the axis. The dashed curve corresponds to the approximation by the Gauss–Gauss function [Eq. (39)]. Values used to plot functions: a = 49 °C, r = 1.01 mm, and b = 0.31 mm for Eq. (31), and u = 39 °C, r = 1.01 mm, v = 0.68 mm for Eq. (39).

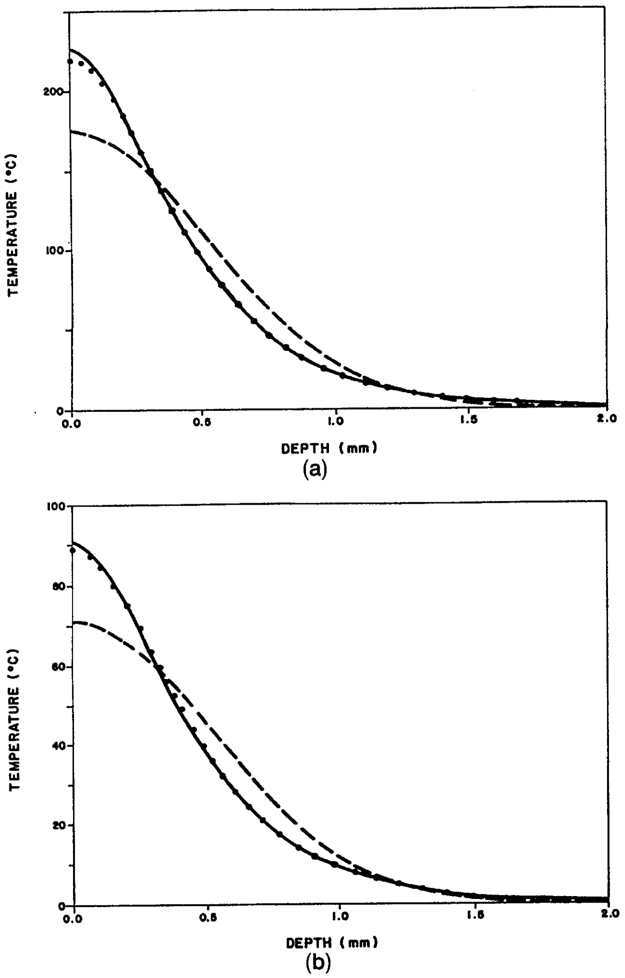

Fig. 2. The same distributions of temperature shown in Fig. 1 for time $t = 0.5$ s. Values used to plot functions: $a = 219$ °C, $r = 1.05$ mm, and $b = 0.33$ mm for Eq. (31), and $u = 174$ °C, $r = 1.05$ mm, and $v = 0.74$ mm for Eq. (39).

mation of the solution of Eq. (2) for the diffraction with the source from Eq. (3) by the elementary function in Eq. (31). It should be noted that for the small values of time, r is approximately the size of the radius r_0 of the light beam.

Note that other functions may be used instead of Eq. (31), for example, the function

$$T_r(x, y, z, t) = u \exp[-(x^2 + y^2)/r^2 - z^2/v^2]. \quad (39)$$

The values of parameters u, r, and v may be found in analogy with Eqs. (32)–(38). It is only necessary to substitute $T_d \rightarrow T_r$ in Eqs. (32)–(34) to obtain the set of equations for u, v, and r. Comparison of the approximations (31) and (39) with the exact solution described in the next section shows that the approximation obtained with Eq. (31) is better than the one from Eq. (39).

5. Numerical Example

To see the accuracy of the approximation of the solution from Eq. (31), it is interesting to compare it with the exact solution. The aim of this section is to present diagrams that illustrate the difference between the exact solution and the approximate solution obtained with Eq. (31).

Assume values for the parameters of the light and the medium given by[4,6,9,10]

$$\mu = 0.106\ \text{mm}^2/\text{s},$$
$$\gamma = 3\ \text{mm}^{-1},$$
$$\rho c = 3.96 \times 10^{-3} (\text{J}/\text{mm}^3\ \text{deg C}),$$
$$I_0 = 1\ \text{W}/\text{mm}^2. \quad (40)$$

The distribution of the temperature at several values for time is presented in Figs. 1 and 2 by the solid curve as the function of the depth z at the optical axis ($x = y = 0$) and at some distances of the axis. Approximations obtained with Eqs. (31) and (39) also are presented. It should be noted that the hyperbolic secant—Gauss function from Eq. (31) provides a better approximation than the Gauss–Gauss function from Eq. (39). The exponential asymptotic behavior of 1/cosh at large argument values is in agreement with the behavior of the solution of Eq. (2). The relative error of the approximation from Eq. (31) in typical cases is approximately 10%.

6. Results and Conclusions

The heating of a uniform medium by linear absorption of a light beam is described by the thermodiffusion equation. For the case of a Gaussian beam the exact solution is presented by a single quadrature [Eq. (9)]. The method of moments describes the evolution of the mean-square radius of the heat spot [Eq. (29)] and the mean-square depth [Eq. (30)]. With these parameters an approximation of the heat distribution by the elementary function is suggested [Eq. (31)]. Comparison with the exact solution (Figs. 1 and 2) shows the validity of this approximation. It can be used in different applications such as in calculations of the temperature distribution inside a blocked artery for laser angioplasty.

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