Theory of the backscattering of sound by phase-matched nonlinear interaction

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The nonlinear interaction of noncollinear acoustical waves is considered. Conditions of the resonant backscattering of one wave from the lattice produced by the other two are formulated in analogy with the four-wave mixing known in optics. The efficiency of the phase-matched interaction of acoustical waves is calculated in the resonant approximation for a gas media. Such approximation is constructed on the basis of the expansion of the sound equations preserving up to cubic terms. The amplitude of the backscattered wave is expressed as the product of the efficiency, the amplitudes of three waves, the wave number of the backscattered wave, and the size of the region of interaction. Such backscattering is proposed as an acoustical remote probe. The distance to the interaction region and the amplitude of initial waves are limited by nonlinear degradation of waves due to the second-order nonlinearity. For acoustical waves with wave number 10 m^{-1}, sources of size 1 m, and about 100 m to the interaction region, the amplitude of the backscattered wave can be about 10^{-10} of the atmospheric pressure. At the detection with a signal-to-noise ratio about of 10, the resolution of such method on the wind velocity may be about 1 m/s. © *1999 Acoustical Society of America.* [S0001-4966(99)04103-X]

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INTRODUCTION

Acoustic remote probes are efficient tools in atmospheric physics. Usually one registers a signal scattered from turbulent inhomogeneities of the atmosphere.[1,2] To get efficient phase-matched scattering, one uses waves of different origin, such as radiowaves and acoustical waves with appropriate relation of wave numbers.[3,4]

We propose an alternative technique which uses the interaction of only acoustical waves; here we present the deduction of formulas which appeared in Ref. 5 without proof. In this reference, the present results were proposed to be used as an acoustic remote probe. We briefly discuss such a proposal as well.

In a homogeneous isotropic medium, we need two acoustical waves to write a lattice, and one probe wave to be reflected from this lattice. Such processes are widely used in optics, where it is known as four-wave mixing.[6,7] It takes place for all types of nonlinearities, even in a gain medium.[8] Efficient interaction occurs at the phase-matching condition. This condition depends neither on the nature of waves nor on the type of nonlinearity. Here we deal with acoustical waves in a gas; the medium is isotropic and nondispersive, which simplifies the equations.

We should mention that recently J. Berntsen *et al.* published a series of papers (see Ref. 9 and references therein) in which they analyzed the interaction of Gaussian acoustical beams in the second-order approximation. However, effects of four-wave mixing mentioned above appear only in the third order. To keep equations simple we work here with plane waves. The consideration of these effects with well-defined beams could be a continuation of the present work.

The possible geometry of a phase-matched backscattering experiment is shown in Fig. 1. Let sources 1 and 2 emit two strong sound beams which produce a lattice within the region of interaction. Here we are interested in the case in which a third wave (the probe beam) emited by source 3 is backscattered from this lattice. In this case the reflected wave can be registered by a detector located at source 3.

We assume that the frequency of the third source is given. So, we need to calculate, for the given geometry, the frequencies of sources 1 and 2, and the placement of the window of frequency selection of the detector (Sec. I). Then, for the case of resonance, we need to calculate the efficiency of the nonlinear interaction (Sec. II). Finally, we need to estimate the maximum amplitude of sound, which could be registered with such a scheme (Sec. III), and, as an example, estimate the sensitivity of such a method to the wind velocity.

I. PHASE-MATCHING CONDITION

Consider three plane waves emitted by sources 1, 2, and 3 as shown in Fig. 1. Let the corresponding wave vectors be $\mathbf{p}$, $\mathbf{q}$, $\mathbf{k}$.

To have resonant interaction, we need to satisfy the conditions of phase synchronism:

$$\mathbf{p}-\mathbf{q}+\mathbf{k}=\mathbf{r}, \quad \omega_{\mathbf{p}}-\omega_{\mathbf{q}}+\omega_{\mathbf{k}}=\omega_{\mathbf{r}}, \tag{1}$$

where $\mathbf{r}$ is the wave vector of some scattered wave, and the ω's are the corresponding frequencies. Such conditions are well known in optics.[6,7] They are refered as ''phase-matching conditions,'' or ''phase synchronism conditions.''

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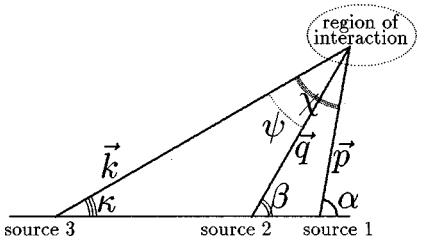

FIG. 1. Geometry of a nonlinear acoustical remote probe based on the backscattering due to four-wave interaction.

For acoustical waves in isotropic and nondispersive media, $\omega_{\mathbf{k}}=v_s/|\mathbf{k}|$, where the velocity of sound v_s is the same for all wave vectors.

Note that the phase synchronism conditions are satisfied for the collinear second-harmonic generation. If we set $\mathbf{q}=0$ and $\mathbf{p}=\mathbf{k}=\mathbf{r}/2$, Eqs. (1) hold. We see that only one incident wave is necessary for the phase-matched interaction, but no backscattering is possible in this case. In a dispersionless and isotropic medium we need to have at least three different waves to generate a wave with its wave vector anti-collinear to one of the incident waves.

For the purpose of an acoustical remote probe, we assume that all initial wave vectors have positive vertical components: all waves are produced by ground level sources and no initally counter-propagating waves can be realized.

Consider the condition that the wave vector $\mathbf{r}$ of the scattered sound is anti-collinear to one of inital wave vectors $\mathbf{k}$ (backscattering). Then Eqs. (1) imply that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{k}$, $\mathbf{r}$ are in the same plane. Therefore, the problem becomes two-dimensional. Using the angles α, β, κ defined in Fig. 1, we can represent these vectors in Cartesian coordinates:

$$\begin{aligned}\mathbf{p}&=\{p\cos(\alpha),p\sin(\alpha)\},\\ \mathbf{q}&=\{q\cos(\beta),q\sin(\beta)\},\\ \mathbf{k}&=\{k\cos(\kappa),k\sin(\kappa)\},\\ \mathbf{r}&=\{-r\cos(\kappa),-r\sin(\kappa)\}.\end{aligned}\tag{2}$$

(The last equality implies that $\mathbf{k}$ and $\mathbf{r}$ are anti-collinear.) We assume that the wave number of the probe beam, k, and the angles α, β, κ are given. These angles are defined by the location of the sources of sound and the region from which we want to get the backscattered signal.

The question is: for given α, β, κ and k, what frequencies should sources 1 and 2 emit so that the resonant scattered wave goes back to source 3?

Since the velocity of sound is constant, we have $r=p-q+k$. Substituting Eqs. (2) into the first of Eqs. (1), we get

$$p\cos(\alpha)-q\cos(\beta)+k\cos(\kappa)=-(p-q+k)\cos(\kappa),\tag{3}$$

$$p\sin(\alpha)-q\sin(\beta)+k\sin(\kappa)=-(p-q+k)\sin(\kappa).\tag{4}$$

From this system, we may readily express p and q in terms of the wave number k and the angles α, β, κ:

$$p=\frac{2k\sin(\beta-\kappa)}{\sin(\alpha-\beta)+\sin(\alpha-\kappa)-\sin(\beta-\kappa)},\tag{5}$$

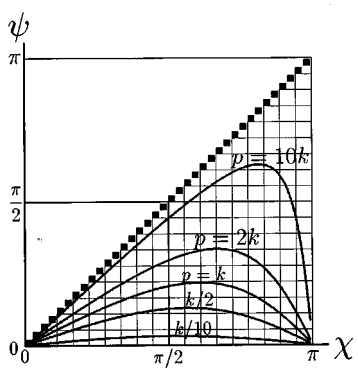

FIG. 2. Wave number p of one of two phase-matched waves which provide the effective backscattering of the wave with given wave number k as a function of angles χ and ψ between $\mathbf{k}$, $\mathbf{p}$ and between $\mathbf{k}$, $\mathbf{q}$, respectively [formula (7)].

$$q=\frac{2k\sin(\alpha-\kappa)}{\sin(\alpha-\beta)+\sin(\alpha-\kappa)-\sin(\beta-\kappa)}.\tag{6}$$

Note the rotational invariance: the frequencies $\omega_1=v_s p$ and $\omega_2=v_s q$ which give rise to the effective backscattering depend on the differences between the angles only. So, to represent wave numbers p,q graphically, we define $\chi=\alpha-\kappa$, $\psi=\beta-\kappa$ (see Fig. 1) and plot

$$\begin{aligned}p=p(\chi,\psi)&=\frac{2k\sin(\psi)}{\sin(\chi-\psi)+\sin(\chi)-\sin(\psi)}\\ &=\frac{k\sin(\psi/2)}{\sin(\chi/2-\psi/2)\cos(\chi/2)},\end{aligned}\tag{7}$$

$$\begin{aligned}q=q(\chi,\psi)&=\frac{2k\sin(\chi)}{\sin(\chi-\psi)+\sin(\chi)-\sin(\psi)}\\ &=\frac{k\sin(\chi/2)}{\sin(\chi/2-\psi/2)\cos(\psi/2)}.\end{aligned}\tag{8}$$

Equilines of $p(\chi,\psi)$, $q(\chi,\psi)$, and $r(\chi,\psi)=p(\chi,\psi)-q(\chi,\psi)+k$ are plotted in Figs. 2, 3, and 4 respectively. The line of small black squares represents infinite values at $\chi=\psi$. Thus, sources 1 and 2 cannot have the same location. Note the symmetry: $r(\pi-\psi,\pi-\chi)=k^2/r(\chi,\psi)$.

Equations (7) and (8) tell us the frequencies that sources 1 and 2 should emit in order to have synchronism and produce effective backscattering. Function $r(\chi,\psi)$ expresses the

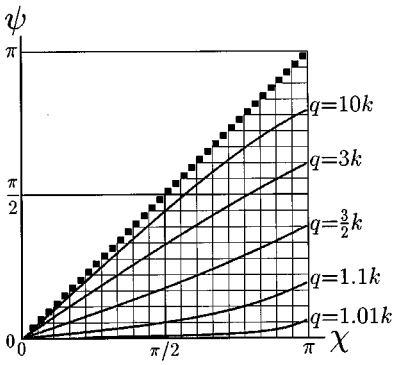

FIG. 3. Wave number q of the second phase-matched wave as a function of the angles χ and ψ [formula (8)].

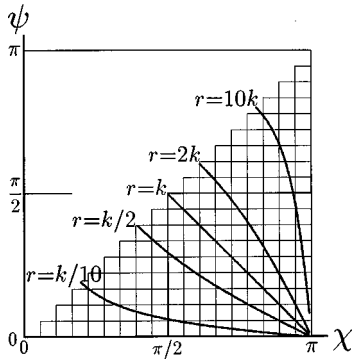

FIG. 4. Wave number $r=p-q+k$ of the reflected wave [formulas (7) and (8)].

wave number of the backscattered wave. To estimate its amplitude, we need first to calculate the efficiency of such backscattering. This is the subject of the following section.

II. SOUND EQUATIONS

Now we should calculate the efficiency of the resonant four-wave interaction of acoustical waves starting from the fundamental equations of sound propagation. In the numbered equations of this section, we use neither the assumption of backscattering ($\mathbf{k}\downarrow\uparrow\mathbf{r}$), nor that vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are coplanar, but we will return to these assumptions to plot figures.

Sound propagation in a fluid such as the atmosphere is described by Euler's equation and the continuity equation.[10,11]

$$\rho\frac{\partial\mathbf{v}}{\partial t}+\rho(\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{v}+\boldsymbol{\nabla}P=0,\quad \frac{\partial\rho}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}\rho+\rho\boldsymbol{\nabla}\cdot\mathbf{v}=0. \tag{9}$$

Here ρ is the density of the fluid, $\mathbf{v}$ is the wave velocity, P is the pressure, and t is time. Operator $\boldsymbol{\nabla}$ differentiates with respect to the vector $\mathbf{x}$ of spatial coordinates. We assume that the process is adiabatic, and

$$P=P_0\left(\frac{\rho}{\rho_0}\right)^{\gamma}, \tag{10}$$

where P_0 and ρ_0 are the pressure and density in the absence of waves. To be more concrete, we assume that $\gamma=\text{const}$.[12] For a monoatomic gas, $\gamma=5/3$; for a diatomic gas with a rigid molecule (the case of the atmosphere), $\gamma=7/5$. We assume that the vibrational degrees of freedom are not excited for air at room temperature. For the limit of multiatomic gas with soft molecules, $\gamma\approx1$. Note that the generalization to the case of any smooth function $P(\rho)$ is straightforward.

It is convenient to define the normalized variables $\tau=t/v_s$, $\mathbf{u}=\mathbf{v}/v_s$, where

$$v_s=\sqrt{\gamma\frac{P_0}{\rho_0}} \tag{11}$$

is the velocity of sound. Let $\rho=\rho_0(1+\eta)$, where η is treated as a new variable. We assume that $|\eta|\ll1$. Then

$$\frac{1}{1+\eta}=1-\eta+\eta^2+\cdots, \tag{12}$$

$$\boldsymbol{\nabla}(1+\eta)^{\gamma}=\gamma\boldsymbol{\nabla}\eta+\gamma(\gamma-1)\eta\boldsymbol{\nabla}\eta+\tfrac{1}{2}\gamma(\gamma-1)\times(\gamma-2)\eta^2\boldsymbol{\nabla}\eta+\cdots. \tag{13}$$

Now we can rewrite (9) in the new variables:

$$\frac{\partial\eta}{\partial\tau}+\boldsymbol{\nabla}\cdot\mathbf{u}=-\eta\boldsymbol{\nabla}\cdot\mathbf{u}-\mathbf{u}\cdot\boldsymbol{\nabla}\eta, \tag{14}$$

$$\frac{\partial\mathbf{u}}{\partial\tau}+\boldsymbol{\nabla}\eta=-(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}-(\gamma-2)\eta\boldsymbol{\nabla}\eta-\frac{(\gamma-2)(\gamma-3)}{2}\eta^2\boldsymbol{\nabla}\eta-\cdots. \tag{15}$$

Searching for an approximate solution to the above equations we use the perturbation series:

$$\eta=\eta^{(1)}+\eta^{(2)}+\eta^{(3)}+\cdots,\qquad \mathbf{u}=\mathbf{u}^{(1)}+\mathbf{u}^{(2)}+\mathbf{u}^{(3)}+\cdots. \tag{16}$$

Let the first-order approximation be the superposition of the three plane waves emitted by the three sources:

$$\eta^{(1)}=A+B+C+A^*+B^*+C^*,\qquad \mathbf{u}^{(1)}=\hat{p}A+\hat{q}B+\hat{k}C+\hat{p}A^*+\hat{q}B^*+\hat{k}C^*, \tag{17}$$

where

$$A=a\exp(i\mathbf{p}\cdot\mathbf{x}-ip\tau),\quad B=b\exp(i\mathbf{q}\cdot\mathbf{x}-iq\tau),\quad C=c\exp(i\mathbf{k}\cdot\mathbf{x}-ik\tau), \tag{18}$$

and $\hat{p}=\mathbf{p}/p$, $\hat{q}=\mathbf{q}/q$, $\hat{k}=\mathbf{k}/k$. In what follows, we also use $\hat{r}=\mathbf{r}/r$ and treat a, b, c as constants. It is easy to see that $\eta^{(1)}$ and $\mathbf{u}^{(1)}$ satisfy the linearized equations

$$\frac{\partial\eta^{(1)}}{\partial\tau}+\boldsymbol{\nabla}\cdot\mathbf{u}^{(1)}=0,\quad \frac{\partial\mathbf{u}^{(1)}}{\partial\tau}+\boldsymbol{\nabla}\eta^{(1)}=0. \tag{19}$$

Four-wave mixing is due to the third-order terms, but first we need to construct the second-order approximation. To get the equations for $\eta^{(2)}$ and $\mathbf{u}^{(2)}$, we substitute the first-order approximations in the right-hand part of Eqs. (14) and (15), and keep only quadratic terms:

$$\frac{\partial\eta^{(2)}}{\partial\tau}+\boldsymbol{\nabla}\cdot\mathbf{u}^{(2)}=-\boldsymbol{\nabla}\cdot(\mathbf{u}^{(1)}\eta^{(1)}), \tag{20}$$

$$\frac{\partial\mathbf{u}^{(2)}}{\partial\tau}+\boldsymbol{\nabla}\eta^{(2)}=-(\mathbf{u}^{(1)}\cdot\boldsymbol{\nabla})\mathbf{u}^{(1)}-(\gamma-2)\eta^{(1)}\boldsymbol{\nabla}\eta^{(1)}. \tag{21}$$

We take the divergence of (21) and subtract (20) differentiated with respect to τ. It gives

$$\nabla^2\eta^{(2)}-\frac{\partial^2\eta^{(2)}}{\partial\tau^2}=-\boldsymbol{\nabla}\cdot((\mathbf{u}^{(1)}\cdot\boldsymbol{\nabla})\mathbf{u}^{(1)}+(\gamma-2)\eta^{(1)}\boldsymbol{\nabla}\eta^{(1)})+\frac{\partial}{\partial\tau}\boldsymbol{\nabla}\cdot(\eta^{(1)}\mathbf{u}^{(1)}). \tag{22}$$

After solving this equation, we can get $\mathbf{u}^{(2)}$, upon integrating (21) with respect to τ. Similarly we find the equation to third order:

$$\nabla^2\eta^{(3)}-\frac{\partial^2\eta^{(3)}}{\partial\tau^2}=-\nabla\cdot((\mathbf{u}^{(2)}\cdot\nabla)\mathbf{u}^{(1)}+(\mathbf{u}^{(1)}\cdot\nabla)\mathbf{u}^{(2)})$$
$$+(\gamma-2)\nabla^2\left(\eta^{(2)}\eta^{(1)}+\frac{\gamma-3}{6}\eta^{(1)^3}\right)$$
$$+\nabla\cdot\left(\frac{\partial}{\partial\tau}(\eta^{(2)}\mathbf{u}^{(1)}+\eta^{(1)}\mathbf{u}^{(2)})\right). \quad (23)$$

Equations (22) and (23) are inhomogeneous Helmholtz equations where the right-hand side represent sources. Direct substitution of (17) in (22) results in many terms. Fortunately, we have no need to consider all these terms, but only those that yield resonating contributions to the third-order approximation.

As we shall see, the amplitude of the acoustical waves should be much less than the atmospheric pressure. The nonlinear income per wavelength is small; so, only the resonant terms are important.

The resonant terms arise from the product AB^*C, proportional to $\exp(i\mathbf{r}\mathbf{x}-ir\tau)$, and from its complex conjugate. Thus, on the right-hand side of (22) we may keep only terms containing the products AB^*, AC, B^*C and their complex conjugate. Such resonant approximation is used commonly in optics to calculate the efficiency of harmonics generation.[6,7]

In this resonant approximation we get from (22)

$$\nabla^2\eta^{(2)}-\frac{\partial^2\eta^{(2)}}{\partial\tau^2}$$
$$=((\hat{k}\cdot\hat{p}+\gamma-2)|\mathbf{k}+\mathbf{p}|^2+(1+\hat{p}\cdot\hat{k})(p+k)^2)AC$$
$$+((\hat{p}\cdot\hat{q}+\gamma-2)|\mathbf{p}-\mathbf{q}|^2+(1+\hat{p}\cdot\hat{q})(p-q)^2)AB^*$$
$$+((\hat{k}\cdot\hat{q}+\gamma-2)|\mathbf{k}-\mathbf{p}|^2+(1+\hat{k}\cdot\hat{q})(k-q)^2)B^*C$$
$$+\text{c.c.}, \quad (24)$$

where ''c.c.'' denotes the complex conjugate terms.

To calculate the contribution from each source term in (24) to $\eta^{(2)}$, consider equation

$$\nabla^2\eta^{(2)}-\frac{\partial^2\eta^{(2)}}{\partial\tau^2}=Q\exp(i\mathbf{s}\cdot\mathbf{x}-i\Omega\tau), \quad (25)$$

where $Q=\text{const}$. One can readily check that

$$\eta^{(2)}=\frac{Q}{\Omega^2-|\mathbf{s}|^2}\exp(i\mathbf{s}\mathbf{x}-i\Omega\tau) \quad (26)$$

is a solution of (25). To solve (24), we apply (26) as solution of (25) with $\mathbf{s}=\mathbf{k}+\mathbf{p}$, $\Omega=k+p$; $\mathbf{s}=\mathbf{p}-\mathbf{q}$, $\Omega=p-q$; $\mathbf{s}=\mathbf{k}-\mathbf{p}$, $\Omega=k-p$; and their complex congugations. We write no additional solution of the homogeneous equation, since it would not contribute to the resonant third-order terms.

Calculating the corresponding contributions to $\eta^{(2)}$ as indicated above, we get

$$\eta^{(2)}=\xi_b AC+\xi_c AB^*+\xi_a B^*C+\text{c.c.}, \quad (27)$$

where

$$\xi_b=\frac{|\mathbf{k}+\mathbf{p}|^2(\hat{p}\cdot\hat{k}+\gamma-2)+(\hat{p}\cdot\hat{k}+1)(p+k)^2}{(p-k)^2-|\mathbf{p}-\mathbf{k}|^2}$$
$$=\frac{(k+p)^2(2\hat{p}\cdot\hat{k}+\gamma-2)}{2(pk-\mathbf{p}\cdot\mathbf{k})}-\hat{p}\cdot\hat{k}-\gamma+2, \quad (28)$$

$$\xi_c=\frac{|\mathbf{p}-\mathbf{q}|^2(\hat{p}\cdot\hat{q}+\gamma-2)+(\hat{p}\cdot\hat{q}+1)(p-q)^2}{-(q-p)^2+|\mathbf{q}-\mathbf{p}|^2}$$
$$=\frac{(p+q)^2(2\hat{p}\cdot\hat{q}+\gamma-2)}{-2(pq-\mathbf{p}\cdot\mathbf{q})}-\hat{p}\cdot\hat{q}-\gamma+2, \quad (29)$$

$$\xi_a=\frac{|\mathbf{k}-\mathbf{q}|^2(\hat{k}\cdot\hat{q}+\gamma-2)+(\hat{k}\cdot\hat{q}+1)(k-q)^2}{-(q-k)^2+|\mathbf{q}-\mathbf{k}|^2}$$
$$=\frac{(k-q)^2(2\hat{q}\cdot\hat{k}+\gamma-2)}{-2(qk-\mathbf{q}\cdot\mathbf{k})}-\hat{q}\cdot\hat{k}-\gamma+2, \quad (30)$$

where we used $(p+k)^2-|\mathbf{p}+\mathbf{k}|^2=2(pk-\mathbf{p}\cdot\mathbf{k})$ in Eq. (28), and similar expressions in Eqs. (29) and (30). Substituting $\eta^{(2)}$ and $\eta^{(1)}$ into (21) yields

$$\frac{\partial\mathbf{u}^{(2)}}{\partial\tau}=-i(\xi_c+\gamma-2+\hat{p}\cdot\hat{k})(\mathbf{p}+\mathbf{k})AC$$
$$-i(\xi_b+\gamma-2+\hat{p}\cdot\hat{q})(\mathbf{p}-\mathbf{q})AB^*$$
$$-i(\xi_a+\gamma-2+\hat{k}\cdot\hat{q})(\mathbf{k}-\mathbf{q})CB^*+\text{c.c.} \quad (31)$$

The integration with respect to τ adds the factors $i/(k+p)$, $i/(p-q)$, and $i/(k-q)$ to the terms with AC, AB^*, and CB^*, respectively. Using Eqs. (28)–(30), we get

$$\mathbf{u}^{(2)}=\boldsymbol{\mu}_b AC+\boldsymbol{\mu}_c AB^*+\boldsymbol{\mu}_a B^*C+\text{c.c.}, \quad (32)$$

where

$$\boldsymbol{\mu}_b=\frac{(2\hat{p}\cdot\hat{k}+\gamma-1)(k+p)(\mathbf{p}+\mathbf{k})}{2(kp-\mathbf{p}\cdot\mathbf{k})}, \quad (33)$$

$$\boldsymbol{\mu}_c=\frac{(2\hat{p}\cdot\hat{q}+\gamma-1)(p-q)(\mathbf{p}-\mathbf{q})}{-2(pq-\mathbf{p}\cdot\mathbf{q})}, \quad (34)$$

$$\boldsymbol{\mu}_a=\frac{(2\hat{k}\cdot\hat{q}+\gamma-1)(k-q)(\mathbf{k}-\mathbf{q})}{-2(kq-\mathbf{k}\cdot\mathbf{q})}. \quad (35)$$

Using the above results we may calculate the resonant contribution to $\eta^{(3)}$ from (23). Direct substitution of $\eta^{(2)}$ and $\mathbf{u}^{(2)}$ on the right-hand side of (23) also results in many terms. We keep only those with AB^*C, and rewrite (23) in the following form:

$$\nabla^2\eta^{(3)}-\frac{\partial^2\eta^{(3)}}{\partial\tau^2}=Fr^2AB^*C+\text{c.c.}$$
$$=Fab^*cr^2\exp(i\mathbf{r}\mathbf{x}-ir\tau)+\text{c.c.}, \quad (36)$$

where

$$F=\xi_b(\hat{q}\cdot\hat{r}+\gamma-2)+\xi_c(\hat{k}\cdot\hat{r}+\gamma-2)+\xi_a(\hat{p}\cdot\hat{r}+\gamma-2)$$
$$+\boldsymbol{\mu}_b\cdot(\hat{q}+\hat{r})+\boldsymbol{\mu}_c\cdot(\hat{k}+\hat{r})+\boldsymbol{\mu}_a\cdot(\hat{p}+\hat{r})$$
$$+(\gamma-2)(\gamma-3). \quad (37)$$

While we treate a, b, and c as constants, the product Fab^*cr^2 does not depend on coordinates, so, (37) has the solution

$$\eta^{(3)}=(\mathbf{r}\cdot\mathbf{x})F\frac{ab^*c}{2i}\exp(i\mathbf{r}\mathbf{x}-ir\tau)+\text{c.c.} \tag{38}$$

This solution corresponds to the case in which the nonlinear interaction generates the wave with wave vector $\mathbf{r}$, and its amplitude is proportional to the length of interaction. We assume that this third wave is absent at the origin of coordinates. The origin of coordinates corresponds to the beginning of the region of interaction, if we approximate it with an abrupt function. For more accurate calculus, one should define the smooth spatial structure of a, b, and c and construct the paraxial approximation of Eq. (36) with given $F=\text{const}$. We write a closed expression for this F below.

The phase-matching conditions enable us to simplify the scalar products with the $\boldsymbol{\mu}$'s that appear upon substitution of Eqs. (33)–(35) into Eq. (37). Since $\mathbf{p}+\mathbf{k}=\mathbf{q}+\mathbf{r}$ and $p+k=q+r$, we have for $\boldsymbol{\mu}_b\cdot(\mathbf{q}+\mathbf{r})$ that $(\mathbf{p}+\mathbf{k})\cdot(\hat{q}+\hat{r})=(\mathbf{q}+\mathbf{r})\cdot(\hat{q}+\hat{r})=q+r+q\hat{q}\cdot\hat{r}+r\hat{p}\cdot\hat{r}=(q+r)(1+\hat{p}\cdot r)$, with similar expressions for $\boldsymbol{\mu}_c\cdot(\mathbf{k}+\mathbf{r})$ and $\boldsymbol{\mu}_a\cdot(\mathbf{p}+\mathbf{r})$. Substituting the ξ's from (28)–(30) and some algebra, we get

$$\begin{aligned}F=&\frac{(p+k)^2(2\hat{p}\cdot\hat{k}+\gamma-1)(2\hat{q}\cdot\hat{r}+\gamma-1)}{2(pk-\mathbf{p}\cdot\mathbf{k})}\\&-\frac{(p-q)^2(2\hat{p}\cdot\hat{q}+\gamma-1)(2\hat{k}\cdot\hat{r}+\gamma-1)}{2(pq-\mathbf{p}\cdot\mathbf{q})}\\&-\frac{(k-q)^2(2\hat{k}\cdot\hat{q}+\gamma-1)(2\hat{p}\cdot\hat{r}+\gamma-1)}{2(kq-\mathbf{k}\cdot\mathbf{q})}\\&-(\hat{p}\cdot\hat{k})(\hat{q}\cdot\hat{r})-(\hat{k}\cdot\hat{q})(\hat{k}\cdot\hat{r})-(\hat{q}\cdot\hat{p})(\hat{p}\cdot\hat{r})\\&-(\hat{p}\cdot\hat{k}+\hat{q}\cdot\hat{r}+\hat{k}\cdot\hat{q}+\hat{k}\cdot\hat{r}+\hat{q}\cdot\hat{p}+\hat{p}\cdot\hat{r})(\gamma-2)\\&+(\gamma-2)(3-2\gamma).\end{aligned} \tag{39}$$

This is our main result: The efficiency of the phase-matched four-wave interaction is expressed in terms of the wave vectors and the adiabatic constant γ.

Note the symmetry of (39). Under the phase-matching conditions, F is invariant with respect to each of following transformations:

$$\begin{aligned}&\text{(A)}\ \ \hat{p}\leftrightarrows\hat{k},\ \ p\leftrightarrows k,\\&\text{(B)}\ \ \hat{p}\leftrightarrows\hat{q},\ \ p\leftrightarrows -q,\\&\text{(C)}\ \ \hat{p}\leftrightarrows\hat{r},\ \ p\leftrightarrows -r,\\&\text{(D)}\ \ \hat{q}\leftrightarrows\hat{k},\ \ q\leftrightarrows -k,\\&\text{(E)}\ \ \hat{q}\leftrightarrows\hat{r},\ \ q\leftrightarrows r,\\&\text{(F)}\ \ \hat{k}\leftrightarrows\hat{r},\ \ k\leftrightarrows -r.\end{aligned} \tag{40}$$

To see the invariance of F with respect to (C), (E), and (F), note that the phase-matching condition leads to $pk-\mathbf{p}\cdot\mathbf{k}=qr-\mathbf{q}\cdot\mathbf{r}$, and similar expressions for the other denominators in (39).

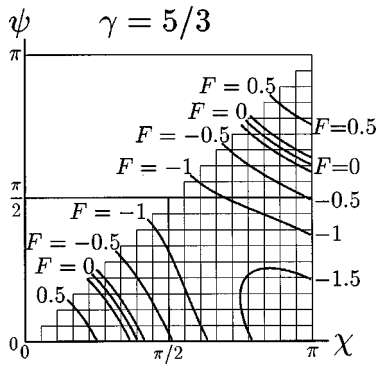

FIG. 5. Efficiency F of backscattering as function of the angles χ and ψ [formula (39)] for a monoatomic gas, $\gamma=5/3$.

Now let us return to our original problem on the backscattering of the wave C: Let $\mathbf{k}$ and $\mathbf{r}$ be anticollinear. Then the efficiency F depends only on the angles χ and ψ, defined in Sec. I. So, we treat it as $F=F(\chi,\psi)$. This function is represented in Fig. 5 for a monoatomic gas, in Fig. 6 for diatomic gas (case of the atmosphere), and in Fig. 7 for a multiatomic gas.

Note the symmetry: $F(\chi,\psi)=F(\pi-\psi,\pi-\psi)$. It follows from the symmetries (40) and the symmetry of the wave numbers as functions of χ, ψ, discussed in Sec. I.

The level $F=0$ corresponds to the absence of backscattering. To make it noticeable we plot also levels $F=\pm0.1$; so, in Figs. 5 and 6 the level $F=0$ appears as a triple line. The disappearance of backscattered waves at some values of angles χ, ψ can be interpreted as the destructive interference of waves reflected by the lattices produced by each pair of incident waves.

The efficiency of the backscattering depends on the constant γ. Thus, the four-wave mixing gives also an extravagant acoustical method to measure the mean number of atoms in the molecules of the gas: location of the $F=0$ line is very different in Figs. 5–7.

III. NUMERICAL ESTIMATIONS

The fundamental result of the previous section expresses the amplitude of the phase-matched reflected wave in terms of the wave numbers of the incident waves and the angles between their wave vectors; it increases with the wave number. So, the efficiency of this effect is limited by the maximal

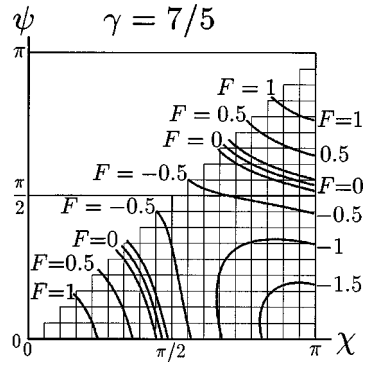

FIG. 6. Efficiency F of backscattering as function of the angles χ and ψ [formula (39)] for diatomic gas, $\gamma=7/5$.

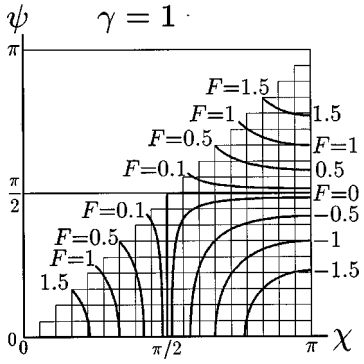

FIG. 7. Efficiency F of backscattering as function of the angles χ and ψ [formula (39)] for extremally multiatomic gas, $\gamma=1$.

wave number that can propagate without significant absorption. For air at 20 °C and a water fraction of 5×10^{-3}, the absorption coefficient reaches $3\times10^{-4}\,\mathrm{m}^{-1}$ at $k=10\,\mathrm{m}^{-1}$ (see, for example, Ref. 12); so, the natural limit for the wave numbers where we still can neglect absorption at a few hundred meters is $k\approx10\,\mathrm{m}^{-1}$.

Another important concern is regarding the nonlinear degradation of plane waves as they travel to the region of interaction. This process begins with the phase-matched second-harmonic generation, discussed in Sec. I. To estimate this effect, consider the case of one-dimensional propagation.

In analogy with Ref. 10, we rewrite Eqs. (14) and (15) as

$$\dot{\eta}+u'+(\eta u)'=0, \tag{41}$$

$$\dot{u}+\eta'+uu'+(\gamma-2)\eta\eta'=0, \tag{42}$$

where $\eta=\eta(x,\tau)$ and $u=u(x,\tau)$; the prime indicates the derivative with respect to the first argument, and the dot denotes the derivative with respect to the last one. Note that we define neither prime as the derivative with respect to x, nor the dot as the derivative with respect to τ, because below, we use this notation also for the case when the first argument is not simply x.

Here we keep only the quadratic terms with respect to the amplitude of the wave: the second-order contribution is largest. We search the solution of Eqs. (41) and (42) as the sum of counter-propagating waves H and J:

$$\eta=H(x-\tau,\tau)+J(x+\tau,\tau), \tag{43}$$

$$u=H(x-\tau,\tau)-J(x+\tau,\tau). \tag{44}$$

Substituting Eqs. (43) and (44) into (41) and (42), we get

$$\dot{H}+\dot{J}+2HH'+2JJ'=0, \tag{45}$$

$$\dot{H}-\dot{J}+HH'+JJ'-H'J-HJ'$$
$$+(\gamma-2)(HH'+JJ'+H'J+HJ')=0. \tag{46}$$

Adding these two equations and neglecting the nonlinear interaction with the counter-propagating wave J, we get

$$2\dot{H}+(\gamma+1)HH'=0. \tag{47}$$

The solution of this equation can be written as[13]

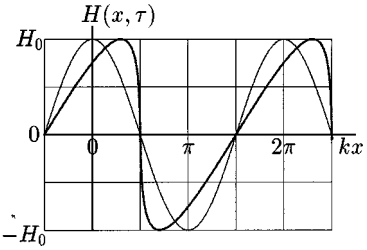

FIG. 8. Deformation of the initial waveform of the monochromatic acoustical wave due to higher-harmonics generation: initial waveform $H(x,0)=H_0\cos(kx)$ (thin line) and deformed waveform $H(x,\tau)$ at $\tau=0.95\tau_{\mathrm{dis}}$ [thick line, formula (48)].

$$H\left(x+\frac{\gamma+1}{2}H(x,0)\tau,\tau\right)=H(x,0). \tag{48}$$

Note that here x has the sense of the local space coordinate of the wave as it is moving. This solution (and the initial equations) become invalid if H' becomes infinite. The distance traveled as this takes place is defined in nonlinear acoustics as the discontinuity distance. We denote it τ_{dis}. Note that we may interpret τ as the distance of propagation.

To estimate value of τ_{dis}, we take the derivative of (48) with respect to x:

$$H'\left(x+\frac{\gamma+1}{2}H(x,0)\tau,\tau\right)\left(1+\frac{\gamma+1}{2}H'(x,0)\tau\right)=H'(x,0). \tag{49}$$

Hence,

$$H'\left(x+\frac{\gamma+1}{2}H(x,0)\tau,\tau\right)=\frac{H'(x,0)}{1+[(\gamma+1)/2]H'(x,0)\tau}. \tag{50}$$

The denominator becomes zero at $[(\gamma+1)/2]H'(x,0)\tau=-1$. If the initial wave is monochromatic, $H(x,0)=H_0\cos(kx)$, then the shock waves appear at $\tau=\tau_{\mathrm{dis}}=2/[(\gamma+1)kH_0]$. The solution (48) is plotted in Fig. 8 by the thick line at $\tau=0.95\tau_{\mathrm{dis}}$. To compare, we plot in the same graph the initial wave $H(x,0)=H_0\cos(kx)$ with a thin line.

At values of τ larger than τ_{dis} the solution becomes invalid. This gives the natural limit to the amplitude. For example, if the distance of propagation $L\approx100\,\mathrm{m}$, and the wave number $k\approx10\,\mathrm{m}^{-1}$, then the amplitude of waves cannot be greater than $H_{\max}=2/[(\gamma+1)kL]\approx10^{-3}$.

If we make the initial amplitude greater than $H_{\max}$, shock waves appear at the distance τ_{dis}. They consume the energy of the wave before it reaches the region of interaction (Fig. 1). At the given initial amplitude, the length of adiabatic propagation can be doubled, if we take $H(-x,\tau_{\mathrm{dis}})$ as the initial condition.

In what follows we collect the results of previous sections to estimate the amplitude of the signal reflected by the four-wave interaction in the atmosphere for the case of Fig. 1. From here on we retain the orders of magnitude only.

Suppose that all wave numbers p, q, k, and r are of the same order of magnitude. So, we may write k instead of p, q, r. Suppose that all distances of propagation are of the same order of magnitude, L. Then, since $2/(\gamma+1)\approx1$, the initial amplitudes should be about

$$H_{\mathrm{i}} \approx \frac{1}{kL}. \tag{51}$$

At larger amplitudes, shock waves appear: The tangent of the front of acoustical waves becomes infinite; definitely, we are out of our approximation. Of course, physically, the gradient of density remains finite, but so high that the diffusion of molecules of the gas causes strong dissipation, and waves lose their power before they reach the region of interaction.

Suppose that the transversal size of all sources is D. During propagation, each beam becomes larger due to diffraction. At distance L, its size becomes

$$l \approx \frac{L}{kD}, \tag{52}$$

where we assume that $l \ll L$.

Due to the expansion of each beam, the amplitude of each wave becomes D/l times less, and the amplitude of the interacting waves is about

$$H \approx \frac{D}{l} H_{\mathrm{i}} \approx \frac{D^2}{L^2}. \tag{53}$$

The amplitude of the backscattered sound is about

$$H_{\mathrm{r}} \approx F k l H^3 \approx F \frac{D^5}{L^5}, \tag{54}$$

where F is the angular factor calculated in Sec. II.

If the reflected signal is detected with an antenna of the same size D, an additional factor kD should be used to calculate the amplitude in the focus; so, the amplitude at the receiver should be about

$$H_D \approx F \frac{kD^6}{L^5}. \tag{55}$$

For example, if $F \approx 1$, $k \approx 10\,\mathrm{m}^{-1}$, $D \approx 1$ m, and $L \approx 100$ m, we have that the amplitude of the signal at the detector should be about 10^{-9}. This means that the receiver should be able to detect sound of pressures about $10^{-9} P_0 \approx 10^{-4}\,\mathrm{N/m^2} \approx 10^{-3}\,\mu$m Hg. The resolution of detectors is limited by thermal noise pressure. Pressure resolutions of $10^{-3}\,\mu$mHg have been reported,[14] and it is still far from the theoretical limit.[15]

From Sec. I we know the frequency of the backscattered signal. Thus, a narrow spectral filter can be used to improve the signal to noise ratio. This makes it possible to extend the distance to the region of interaction for a few hundreds meters more for the same size of sources.

In Ref. 5 we presented some additional speculations about the nonplanar geometry of waves, which causes the partial focusing of the backscattered wave and increases the amplitude of registered signal for an order of magnitude.

Finally, let us estimate the sensitivity of such a probe to the wind velocity. The angular deviation ϕ of an acoustical beam by a wind of velocity v follows from the analysis in Chap. 8 of Ref. 11. Roughly, $\phi \approx v/v_{\mathrm{s}}$. At the region of interaction, of size l, such angular deviation causes a drastic dephasement of the interacting waves when $kl\phi \approx \pi$. Taking l from Eq. (52), we find the wind velocity that causes such dephasement, $v \approx \pi v_s/(kl) \approx \pi v_s D/L$. The limit δv of resolution will be better by the signal-to-noise ratio factor ν:

$$\delta v \approx \pi \frac{v_{\mathrm{s}}}{\nu} \frac{D}{L}. \tag{56}$$

For example, for $D \approx 1$ m and $L \approx 10$ m as in the previous example, and taking $v_{\mathrm{s}} \approx 340$ m/s, and $\nu \approx 10$, we have $\delta v \approx 1$ m/s.

Note that wind parallel to the plane of Fig. 1 causes bending of the reflected beam; however, in this case, the beam still lies within the same plane. Such bending can be corrected (and, therefore, measured) by the adjustment of wave numbers p and q. As for the bending caused by the orthogonal component of wind, it cannot be compensated in such a manner, and the detector should be displaced from the location of source 3, giving us the measure of the component of the wind orthogonal to the plane Fig. 1.

IV. CONCLUSIONS

The resonant nonlinear interaction of acoustic waves is analyzed. Wave numbers of two waves which result in the efficient backscattering of the third wave (Fig. 1) are calculated (Figs. 2 and 3) as functions of angles between the wave vectors. The efficiency of this process is calculated and presented graphically for various values of the adiabatic constant γ (Figs. 5–7).

The possible application of the phase-matched four-wave interaction as a remote acoustic probe is suggested. The nonlinear degradation of acoustical waves limits the values of the wave number, initial amplitude and the distance of propagation in such probe. For wave numbers of about 10 m^{-1}, with sources of size of about 1 m and a distance to the intersection of 100 m, the relative pressure (relative to the atmospheric pressure) in the backscattered wave can be about 10^{-10}. The frequency of the backscattered signal is calculated (Fig. 4), so angular and spectral filters can be used.

The qualitative calculation of the distribution of amplitude in the backscattered beam should imply the consideration of the transversal (and, maybe, longitudinal) structure of incident beams. Such calculations are a possible continuation of this work.

The resolution on the measurement of the velocity of wind is estimated to be in the order of 1 m/s. The complete analysis of an acoustical remote probe based on four-wave interaction in moving media can be made using the proper transformation of the wave vectors and also could become a subject for future investigations.

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