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Superfunctions for Amplifiers

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An amplifier is characterized by its transfer function T, which expresses the dependence of the output signal on the input signal. This signal may be related to power, intensity, energy of a pulse, or its fluence, or any similar physical quantity. The internal structure of the amplified signal (e.g., its spectral content, polarization, temporal behavior, and spatial distribution) is not taken into account. The amplifier is considered to be spatially homogeneous and uniformly pumped. The transfer function is supposed to be known (measured in an experiment). The problem of reconstruction of the behavior of the signal inside the amplifier is formulated. For a given transfer function T, the evolution of the signal inside is interpreted as the superfunction F, satisfying the transfer equation $F(z+1) = T(F(z))$, where z is of coordinate along the propagation direction, while the length of the amplifier is used as a unit of measurement. (For simplicity, distances are measured in units of the length of the amplifier.) Two examples of realistic transfer function T are considered; they correspond to amplification of continuous wave and to amplification of pulses. In these examples, the transfer function and the distribution of the signal along the amplifier can be expressed in terms of special functions. The iterative procedure is suggested as a general method of reconstructing the signal along the amplifier, if neither the transfer function T, nor the superfunction F can be expressed with a simple combination of special functions. The examples show that the iterations converge to a physically meaningful solution. This method is expected to be useful for the characterization of laser materials from the measurement of the transfer function of a bulk sample.

Keywords: theoretical tool, laser materials, transfer function, superfunction

1. Introduction

For optimization of lasers and prediction of new effects, the properties of the laser materials should be characterized. In particular, the gain as function of intensity (or fluence) of the signal should be measured. This article suggests a way to improve precision of these measurements.

The precise characterization of laser materials usually refers to a result within very few decimal significant figures. This is very far from the limits due to statistical uncertainty (due to the discrete nature of photons), or that due to the inhomogeneity of the space-time, that would allow an order of 20 significant figures. Currently, frequency and time can be measured with such a precision; but precision of measurement of nonlinear gain and absorption, yet, is much worse. Usually, of order of two significant figures can be estimated from analysis of the kinetic processes in the gain medium or from the direct measurements. The poor precision of characterization of optical materials may have several origins.

First of all, the laser materials are not so uniform and not so identical, as atoms of the same chemical element and simple molecules of the same chemical are. However, at the development of the purification technology, the fabrication of materials is expected to become a precise technology, where the results of the operation are predictable, unlike in the case of a "kitchen", where the quality of the resulting product depends on the skills of the cook. Then, the properties of the material become more reproducible (not dependent on the manufacturer). The precise characterization of the laser materials becomes more and more important.

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Another reason for the poor precision is the problem regarding the measurement of the gain and absorption of optically thin materials. If the intensity does not change so much along the amplifier, then the measured gain can be attributed to a certain intensity. However, with small variation of the intensity, the precision of the measurement of this variation is poor.

Measurement of the variation of intensity can be precise for the optically thick sample. However, while the intensity varies, it is necessary to know, namely which intensity does the resulting gain (or absorption) corresponds to.

For new laser media, the simulation of the whole experiment, taking into account the finite thickness of the sample, is methodologically correct. However, this implies introducing one additional parameter into the model of the medium. This parameter is length (thickness) of the sample. Consideration of an additional parameter makes the consideration more difficult; thus, it would be desirable to have a tool for calculation of the gain or the absorption at the given intensities of pump and signal from measurements of the transfer function of a thick sample; the transfer function expresses output intensity at the given input intensity. The tool should be universal, i.e., it should not imply any specific model of the laser material.

In the measurement of the gain, in principle, it is possible to keep the pump constant (for example, with the lateral delivery of the pump[1–5]). The variation of signal, contrary, should be significant, in order to measure it at a precision of many decimal digits. The precise measurement of the transfer function is possible, but the variation of the signal inside the sample should be somehow recovered. The same principle can be applied to recover of the absorption of the pump, which depends on pump intensity; in this case, the intensity of the signal should be constant. Using formalism of superfunctions,[6–19] the problem is formulated in general form, without to assuming of any specific model for the gain medium.

The article is organized in the following way: Section 2 collects the basic equations for superfunctions. Section 3 suggests the procedure for regular iteration of the transfer function. Section 4 describes the example for the continuous wave amplifier. Section 5 suggests the example for the pulsed operation and compare the two examples. Section 6 proposes the formalism for characterization of optical materials, as the Conclusion.

2. Basic Equations

Let F be the intensity of the pump or that of the signal in the continuous wave experiment. This F also can be interpreted as a fluence in the experiment with a short pulse, when the dependence of gain on fluence is analyzed; for the fluences, the general formulas below remain the same. For simplicity, assume that F is the intensity of light in a medium, and the change to the case of fluence is straightforward.

Let the transfer function T of a sample be precisely measured, expressing the output as a function of the input. Is it possible to reconstruct the evolution of intensity $F(x)$ along the coordinate x of the propagation?

For simplicity, assume, that the coordinate x is measured in units of the thickness (or length) of the sample (which is considered to be uniformly pumped for the propagation of the signal). This assumption refers to units of length and does not lead to the loss of generality of the consideration. Then the question above refers to the transfer equation

$$F(x+1) = T(F(x)). \tag{1}$$

Equation (1) just states that, for the amplifier of length unity, the output is transfer function of the input. If at some coordinate x intensity is $F(x)$, then, after to pass the thickness of the sample, the intensity becomes $F(x+1)$, and it is also $T(F(x))$. For the given transfer function T, the solution F is called "superfunction";[6–13] to date, no other suitable term has been established. With the superfunction, the iteration of the transfer function T can be written as follows:

$$T^n(z) = \underbrace{T(T(\ldots T(z)\ldots))}_{n \text{ evaluations of function } T} = F(n + F^{-1}(z)), \tag{2}$$

where F^{-1} is the minus-first iteration of function F, i.e., the inverse function of F, so that

$$F(F^{-1}(z)) = z. \tag{3}$$

For the uniqueness of the solution of Eq. (1), F should be treated as a function of a complex variable. For the application to optical amplifiers, only the positive values of the argument of function F^{-1} allow simple interpretation; the intensity (and the fluence) is supposed to have nonnegative values; similarly, in the physical interpretation, the argument of F is real and values of F are positive. In such a way, Eq. (3) is supposed to hold at least in some vicinity of the positive part of the real axis, and, in particular, for positive values of z.

On the right hand side of Eq. (2), the number n of iterations has no need to be an integer. The half-iteration of the logistic operator, that of the factorial and that of the exponential can be evaluated in such a way.[6–10,20]

Expression $\exp^{1/2}(z)$ should not be confused with $\sqrt{\exp(z)} = \exp(z/2)$; similarly, $\mathrm{Factorial}^{1/2}(z)$ should not be confused with $\sqrt{\mathrm{Factorial}(z)}$. Here, the upper superscript after the name of the function denotes either the derivative (if it is character "prime", i.e., $'$) or number of iterations of this function (if the expression can be interpreted as a number). In these notations, $\sin^2(x) = \sin(\sin(x))$ but never $\sin(x)^2$. Similarly, in quantum mechanics, $P^2\Psi$ is $P(P(\Psi))$, but never $(P\Psi)^2$.

The solution of the transfer Eq. (1) is not unique, even if we measure the intensity in the units of intensity of saturation, and choose the origin of the system of coordinates in such a way that $F(0) = 1$. If F is a solution, then another solution f can be constructed as

$$f(x) = F(x + \eta(x)), \tag{4}$$

where η is a real periodic function with period unity. For this reason, Eq. (1) had been qualified as a "hopeless"; and the reconstruction of the physically-correct superfunction F from the transfer function T had been considered as impossible.[21–23]

The superfunction F becomes unique, if we add certain additional requirements on its behavior in the complex plane,[6–10] and the efficient algorithms are suggested to evaluate namely this solution. Other solutions can be constructed, using Eq. (4).

In particular, the algorithm described in Refs. 6–8 allows to construct the non-integer iteration of the transfer function T^n, that is regular in vicinity of its fixed point (regular iteration). This algorithm had been established and verified for the quadratic transfer function,[7] for $T = \mathrm{Factorial}$,[6] and for $T = \exp_b$ at $1 < b < \exp^2(-1)$.[8] In principle, these transfer functions may have applications in various branches of physics, but it is difficult to realize, for example, the optical amplifier with factorial transfer function. Realistic transfer functions of optical amplifiers do not raise their derivatives at large values of the argument, but, in contrast, show saturation, describing reduction of the amplification coefficient for the strong input signal. The fast growth of superfunctions considered in the literature[6,8,10] makes these examples difficult to explain to colleagues, who work with laser science. For application in the Laser Science, two more realistic transfer functions T are suggested as examples:

$$T(z) = \mathrm{Doya}(z) = \mathrm{LambertW}(z\mathrm{e}^{1+z}), \tag{5}$$

$$T(z) = \mathrm{Keller}(z) = z + \ln(\mathrm{e} - \mathrm{e}^{-z}(\mathrm{e} - 1)). \tag{6}$$

In Eq. (5), LambertW refers to the special function that is recognized in many programming languages; $f = \mathrm{LambertW}(z)$ is the solution of equation $f\mathrm{e}^f = z$. The names Doya and Keller are introduced to simplify the identification of these functions with databases. The Keller function is described in various publications. The name of Keller function is chosen after the last name of one of authors of publication in Applied Physics in year 2004.[24] These names are supposed to simplify the finding of the efficient implementations and call of the corresponding codes.[14–17)]

For the transfer functions T by Eqs. (5) and (6), the superfunctions $F(x) = T^x(1)$ can be expressed analytically:

$$F(x) = \mathrm{Tania}(x) = \mathrm{Doya}^x(1) = \mathrm{WrightOmega}(x+1) = \mathrm{LambertW}(\mathrm{e}^{x+1}), \quad (7)$$

$$F(x) = \mathrm{Shoka}(x) = \mathrm{Keller}^x(1) = x + \ln(\exp(x) + \mathrm{e} - 1). \quad (8)$$

Names Tania and Shoka (as well as Keller and Doya above) are introduced to simplify the use the same notations in various applications. Efficient algorithms for the evaluation of the functions abote functions are presented,[14–17,19)] their properties are known; so, these Tania and Shoka also should be considered as special functions.

Below, the general method of evaluation of non integer iterates of the transfer function T of general kind is described. Then, functions $T = \mathrm{Doya}$ and $T = \mathrm{Keller}$ by Eqs. (5) and (6) are considered as examples, to show the efficiency of the method: the approximations of F are constructed with the general method and compared with the exact solutions.

The goal of this paper is to convince colleagues to perform precise measurements of the transfer functions for uniformly pumped bulk samples. The precision of the reconstruction of the properties of the material with the formalism of superfunctions should be compared with the precision of the direct measurements of the same properties with optically thin samples. This formalism should help in the correct treatment of experimental data for optically thick samples, at least to avoid errors due to the reabsorption; such errors may lead to the publication of results contradicting the Second law of thermodynamics.[25–28)]

Following the general methodology,[29–33)] I expect, that the intelligent fitting of accurate experimental data will allow the extrapolation and the prediction of new phenomena, similar to the Bisson effect (switching of emissivity and photoconductivity at high pumping of the Yb-doped ceramics).[34)] This way seems to be more logical, more civilized, than treating of all data with the same simple model[35,36)] or searching for new effects in an extensive series of measurements of properties of optical media with one or two decimal digits (which can be extracted from the curves presented in publications describing experiments in laser science).

3. Regular Iteration

In this section, the method of construction of the regular iterations[6–8)] is repeated for the case when zero is a fixed point of the transfer function, i.e., here and throughout, $T(0) = 0$. This description neglects the amplified spontaneous emission, and the zero input leads to the zero output.

The exact solution of Eq. (1) is constructed as the fundamental sequence of the approximations. The primary approximation $\tilde{F}$ of the solution F of the transfer Eq. (1) is constructed as the expansion with exponentials

$$\tilde{F}(x) = \varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + \cdots, \quad (9)$$

where parameter $\varepsilon = \mathrm{e}^{kx}$ and coefficients $k, a_2, a_3, \ldots$ are constants. Parameter e^k has sense of the coefficient of amplification a weak signal: at weak input, the only leading term in expansion (9) is important; after to pass the unit length of the amplifier, the signal becomes e^k of the input signal. Value of positive parameter ε is supposed to be small. Parameters k and a are easy to evaluate. Then, any truncation of the series (9) gives the primary approximation $\tilde{F}$, i.e., the first element in the sequence mentioned.

For function F with displaced argument in the left hand side of the transfer Eq. (1), we have expression

$$\tilde{F}(x+1) = \mathrm{e}^k\varepsilon + a_2\mathrm{e}^{2k}\varepsilon^2 + a_3\mathrm{e}^{3k}\varepsilon^3 + \cdots. \quad (10)$$

Substituting $\tilde{F}$ instead of F into the transfer Eq. (1), the expansion of the right hand side can be written as follows:

$$T(\tilde{F}(x)) = T'\varepsilon + T'a_2\varepsilon^2 + T'a_3\varepsilon^3 + \cdots + \frac{T''}{2}(\varepsilon + a_2\varepsilon^2 + \cdots)^2 + \frac{T'''}{6}(\varepsilon + \cdots)^3 + \cdots, \quad (11)$$

where $T' = T'(0)$, $T'' = T''(0)$, and $T''' = T'''(0)$ are derivatives of the transfer function T at zero. Then, from the transfer Eq. (1) we obtain the relations

$$\mathrm{e}^k = T', \quad (12)$$

$$\mathrm{e}^{2k}a_2 = T'a_2 + T''/2, \quad (13)$$

$$\mathrm{e}^{3k}a_3 = T'a_3 + T''a_2 + T'''/6, \quad (14)$$

and so on, determining the parameters in the expansion (9); in particular,

$$k = \ln(T'), \quad (15)$$

$$a_2 = \frac{T''/2}{(T'-1)T'}, \quad (16)$$

$$a_3 = \frac{T''a_2 + T'''/6}{((T')^2 - 1)T'}. \quad (17)$$

The truncated series with only one or a few terms in expansion (9) gives a good approximation $\tilde{F}$ of F for small values of ε. At a positive k, the approximation refers to the large negative x. For moderate values of x, the transfer function can be approximated with

$$F(x) \approx F_n(x) = T^n(\tilde{F}(x-n)), \quad (18)$$

giving a way to approximate the superfunction F with any precision required, id est, to evaluate it. For given x, the sequence $\{F_n(x), n = 0, 1, 2, \ldots\}$ is fundamental; hence, it determines the exact value of the solution $F(x)$. For this reason, the sign of approximative equality, i.e., $\approx$, in Eq. (18) should not be interpreted as an indication of the approximate character of the formalism suggested. The exact expression for the superfunction can be written in the following form:

$$F(x) = \lim_{n\to\infty} T^n(\tilde{F}(x-n)). \quad (19)$$

In this sense, the regular iteration is exact, so exact, as exact is, for example, function sin, defined with its Taylor expansion as the corresponding limit.

The inverse function $G = F^{-1}$ can be expressed as the solution of the Abel equation

$$G(T(z)) = G(z) + 1. \quad (20)$$

For the case $T'(0) \neq 1$, i.e., the gain is not zero, the expansion for G can be written in a way, similar to (9):

$$G(z) = \ln\left(\frac{1}{T'}\right) + \ln(z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \cdots). \quad (21)$$

The coefficients b can be found, inverting expansion (9), for example, with the Mathematica "InverseSeries" function; in particular.

$$b_2 = -a_2 = \frac{-T''/2}{(T'-1)T'}, \quad (22)$$

$$b_3 = 2a_2^2 - a_3. \quad (23)$$

The same coefficients also can be found by substituting the expansion (21) of the Abel function G into the Abel equation (20) and equalizing the coefficients at equal power of z in the left and right hand sides. If function T is expressed through the special functions (i.e., there exist simple expressions for derivatives of T), then, typically, some tens of coefficients b can be calculated analytically, i.e., exactly, with some Mathematica or Maple software.

Once the superfunction F and the Abel function $G = F^{-1}$ are constructed, the nth iteration of the transfer function T, regular at zero, can be expressed with

$$T^n(z) = F(n + G(z)). \quad (24)$$

In this expression, the number n of iterations has no need to be an integer; it can be real or even complex. If function G is regular in vicinity of the fixed point 0, and F is regular in vicinity of the real axis, then, the expression (24) can be interpreted as regular iteration; $T^n(z)$ can be expanded to the converging Taylor series with powers of z. Such an expansion can be called "Regular iteration".

In previous works,[6–8] for various transfer functions, using the "double" arithmetics, on the order of 14 correct decimal digits had been achieved, evaluating the superfunctions with the procedure, similar to that described above. The iterations by Eq. (24) were evaluated with similar precision. This indicates the good stability of the algorithm. The precision mentioned greatly exceeds the needs of characterization of the laser materials of 21st century. Measurement of the gain versus intensity with, say, 7 decimal digits should be considered as a good achievement of the laser technology, allowing to reveal the deviations from the simple kinetic models.[35,36]

After observing the smoothness of the iterations constructed by Eq. (24) through the superfunctions,[6–8] one may expect, that the representation will also provide the physically meaningful solution in realistic cases. In particular, this expectation refers to the case of transfer functions with unique fixed point and the construction of the iterations through Eqs. (9)–(19). This can be formulated as the following conjecture:

For a realistic transfer function T with a single real fixed point, Eqs. (9)–(19) give a solution, what corresponds to the physical amplifier.

Such a conjecture follows from the last, sixth of the TORI axioms,[29–33] that declares:

If two concepts satisfying S1–S5 have some common range of validity, then, in this range, the simplest of them has priority. This axiom suggests that the simplest among the non trivial solutions of Eq. (1) should be considered first.

The previous publications about superfunctions[6–10] deal with functions that are difficult to realize as transfer functions of realistic amplifiers with a simple model. To confirm, that this method also works for the transfer functions with saturation, typical for laser media, in the next two sections two such examples are suggested.

4. Realistic Example: Continuous Wave Amplifier

The simple transfer function for the laser amplifier can be expressed through the LambertW function; let T be defined with Eq. (5). Graphic of function T by Eq. (5) is shown on the left hand side of Fig. 1 with the thick line. Properties of function Doya are described at TORI[16] and the efficient (id est, fast and precise) algorithm for the evaluation are supplied there. At small values of the argument,

$$T(z) = \mathrm{e}z - \mathrm{e}(\mathrm{e}-1)z^2 + O(z)^3. \quad (25)$$

The corresponding linear and quadratic approximations of T are shown in the left hand side of Fig. 1 with thin lines. Evaluations by Eq. (15) gives $k = 1$, so, from Eq. (15), $\varepsilon = \mathrm{e}^x$. Then, evaluation by Eq. (16) gives $a_2 = -1$. The primary approximation by the expansion (9) with single term gives $\tilde{F}(x) = \exp(x)$; the primary approximation with two terms give $\tilde{F}(x) = \mathrm{e}^x - \mathrm{e}^{2x}$. These approximations are shown with highest and lowest curves in the right hand side of Fig. 1. For comparison, the iterates

$$F_n(z) = T^n \tilde{F}(z - n) \quad (26)$$

are also shown for $n = 1, 2, 3, 4$. For this example, the superfunction F can be expressed analytically through the function WrightOmega, thich is built-in function in various programming languages (in particular, in Matlab, Maple, and Mathematica):

$$F(x) = \mathrm{Tania}(x-1) = \mathrm{WrightOmega}(x) = \mathrm{LambertW}(\mathrm{e}^x). \quad (27)$$

This function is also shown on the right hand side of Fig. 1. Increasing n, the iterates $F_n(x)$ by Eq. (26) quickly converge to the exact solution $F(x)$.

The same function F can be obtained as a solution of the differential equation

$$F'(x) = \frac{F(x)}{1 + F(x)}, \quad (28)$$

which corresponds to the amplification of light in the gain medium with simple kinetics. Then, the solution $F =$ WrightOmega corresponds to the initial condition $F(1) = 1$, and $F = \mathrm{Tania}$ corresponds to initial condition $F(0) = 1$. There are other differences between functions Tania and WrightOmega, that appear in the complex plane and determine amount of the cut lines. Function WrightOmega has many cut lines, while function Tania has only two cutlines.[14,15,19] Behavior in the complex plane can be used to specify the physical solution of Eq. (1); for this

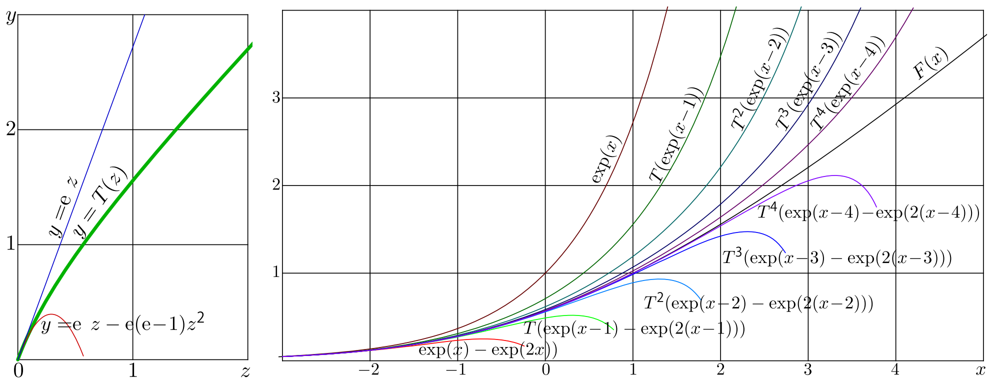

Fig. 1. (Color online) Left: Transfer function T by (5) and its approximations. Right: superfunction F by (7) and its approximations $\tilde{F}$ by the regular iteration (18) for $n = 0, \ldots, 4$.

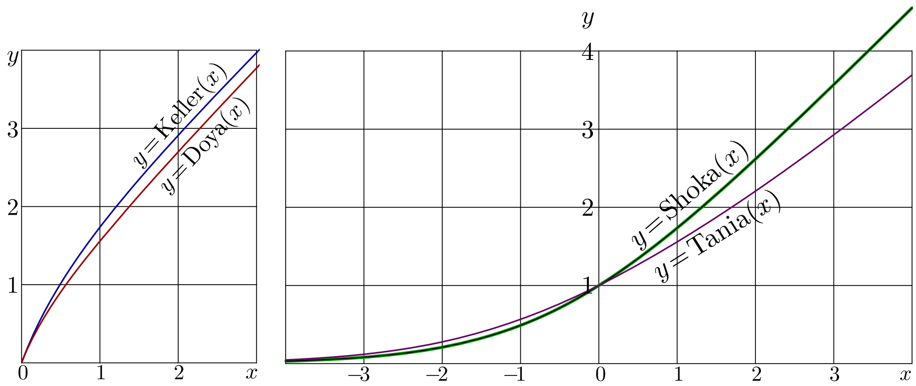

Fig. 2. (Color online) Function Keller by Eq. (6) in comparison with function Doya by Eq. (5), left; function Shoka by Eq. (8) in comparison with function Tania by Eq. (7), right.

reason, it is better to use function Tania than function WrightOmega. In addition, the condition $F(0) = 1$ helps to compare superfunctions for various transfer functions.

Figure 1 shows that the iterations F_n by Eq. (18) quickly converge to the physically meaningful solution F. The same is expected for the cases of other realistic transfer functions T; even if no expressions of those functions through the special functions are available.

5. Realistic Example: Pulsed Amplifier

For a medium with the simple model of kinetics, at pulsed operation, the transfer function of an amplifier can be expressed with elementary function Keller by Eq. (6). For the transfer function $T = \text{Keller}$ by (6), the superfunction $F = \text{Shoka}$ by Eq. (8) appears to be elementary function,

$$F(z) = \text{Shoka}(z) = z + \ln((\mathrm{e} - 1) + \mathrm{e}^z); \tag{29}$$

and the Abel function $G = \text{ArcShoka}$ can be expressed as follows

$$G(z) = \text{ArcShoka}(z) = z + \ln\left(\frac{1 - \mathrm{e}^{-z}}{\mathrm{e} - 1}\right). \tag{30}$$

In a wide range of values of z, the relation $\text{Shoka}(\text{ArcShoka}(z)) = z$ takes place. Function Shoka by Eq. (8) is shown at the right hand side of Fig. 2 (thick line) in comparison with function Tania by Eq. (7) (thin line).

Function Shoka can be "recovered" from the transfer function $T = \text{Keller}$ numerically, as the superfunction, using the general recipe (9)–(19). The iterations of the primary approximation of the superfunction converge to function, that is displacement of function Shoka,[18] in the way, similar to the convergence of approximations of superfunction of transfer function $T = \text{Doya}$ to function $F = \text{WrightOmega}$, shown at the right hand side of Fig. 1. The additional condition $F(0) = 1$ helps to compare the physically-meaningful superfunctions for the two cases.

For each of functions defined above (Doya, Tania, Keller, Shoka), the representation through the special functions (elementary functions, the WrightOmega function, the LambertW function) is available. Due to the simple representations and the efficient implementations, these functions are good examples for the numerical test for the efficiency of the general algorithm of evaluation of the superfunctions, the Abel functions and the non integer iterates of the transfer functions. The two examples verify, that the general concept of recovery of superfunction by Eqs. (9)–(19) can be applied in realistic cases, typical for the laser science.

I expect, that the same method is efficient in the general case, for arbitrary transfer function T of any homogeneous amplifier, even if no simple analytic representation of the transfer function is available. Such a formalism should be an efficient tool for characterization of nonlinear optical materials (and, in particular, laser media); it allows to work with optically thick samples and this should provide better precision, than the measurements with the optically thin samples. The examples above show, that the method provides the precise recovery of the local behavior of the signal from the transfer function of a bulk sample.

6. Conclusions

New method is suggested for recovery of nonlinear properties of the optical materials from the measured transfer function T of the optically thick samples. This method is based on the solution F of the transfer Eq. (1). The iteration (19) of representation (9) is suggested as general method to find the physically meaningful solution F. This method determines the superfunction F of a transfer function T of the amplifier. This superfunction F may have sense of the distribution of intensity inside the amplifier for the continuous-wave operations, as well as the sense of the fluence for the pulsed operation.

In general, the solution F of Eq. (1) is not unique, even if the additional condition $F(0) = 1$ is applied. From the analysis of the behavior of various superfunctions in the complex plane,[6,8–10] one could expect that the iteration by Eqs. (9)–(19) provides the simplest solution. From the TORI axioms,[29–33] the simplest non-trivial solution is expected to have the physical sense. To verify this expectation, two realistic examples are considered. The transfer functions T are denoted as the Doya function by Eq. (5) and the Keller function by Eq. (6). The corresponding superfunctions are denoted as the Tania function by Eq. (7) and the Shoka function by Eq. (8). All the four functions (Doya, Keller, Tania, Shoka) are expressed through the special functions (elementary functions, LambertW function, WrightOmega function). The quick convergence of the limit in Eq. (19) to the exact solution F confirms the efficiency of the straightforward numerical implementation of Eqs. (9)–(19). For a physically-meaningful transfer function T of an amplifier, Eqs. (9)–(19) provide the physically-meaningful solution F of the transfer Eq. (1).

Solution F obtained by Eq. (19) may have sense of the distribution of a signal (intensity or fluence) in a homogeneously pumped optical amplifier. Such a formalism should be a useful tool for the characterization of laser materials, allowing the calculation of gain as a function of signal (intensity or fluence) from the measurements of the transfer function of the optically thick samples, while the variation of the signal (intensity or fluence) is on the order of the signal. Such a characterization is expected to provide better precision, than the direct measurement of the nonlinear response in optically thin samples.

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