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COMPUTATION OF THE TWO REGULAR SUPER-EXPONENTIALS TO BASE EXP(1/E)

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ABSTRACT. The two regular super-exponentials to base exp(1/e) are constructed. An efficient algorithm for the evaluation of these super-exponentials and their inverse functions is suggested and compared to the already published results.

1. INTRODUCTION

We call a holomorphic function F a superfunction [9] of some base function f if it is a solution of the equation

$$F(z+1) = f(F(z)) \qquad \text{[f]} \tag{1}$$

In the case $f = \exp_b$, i.e. for the exponential base function $f(z) = b^z$, we call F *super-exponential* to base b. In addition, if the super-exponential F satisfies the equation

$$F(0) = 1 \qquad \text{[F01]} \tag{2}$$

we call it *tetrational*; for integer values of the argument z, equation (1) and (2) implies F to be the z times application of the exponential $\exp_b$ to unity

$$F(z) = \underbrace{\exp_b\Big(\exp_b\big(...\exp_b}_{z\times}(1)...\big)\Big). \qquad \text{[integerz]} \tag{3}$$

Conversely a function G is called Abel function of some base function f if it satisfies

$$G(f(z)) = G(z)+1 \qquad \text{[abel0]} \tag{4}$$

For $f(z) = b^z$ we call G *super-logarithm* to base b. The inverse of a super-exponential is a super-logarithm. (In some ranges of values of z, the relations $F(G(z)) = z$ and $G(F(z)) = z$ hold.)

We have constructed super-exponentials and efficient algorithms of their numerical evaluation in [7] for $1 < b < \mathrm{e}^{1/\mathrm{e}}$, and in [8] for $b > \mathrm{e}^{1/\mathrm{e}}$. However, neither method used in these publications is applicable to base $b = \mathrm{e}^{1/\mathrm{e}}$. Especially this case is analyzed by Walker in [16]; he evaluates the two Abel functions at several points in the complex plane. Here we show that his constructions are equal to the two regular Abel functions (*regular* in the sense of Szekeres [13]) and suggest

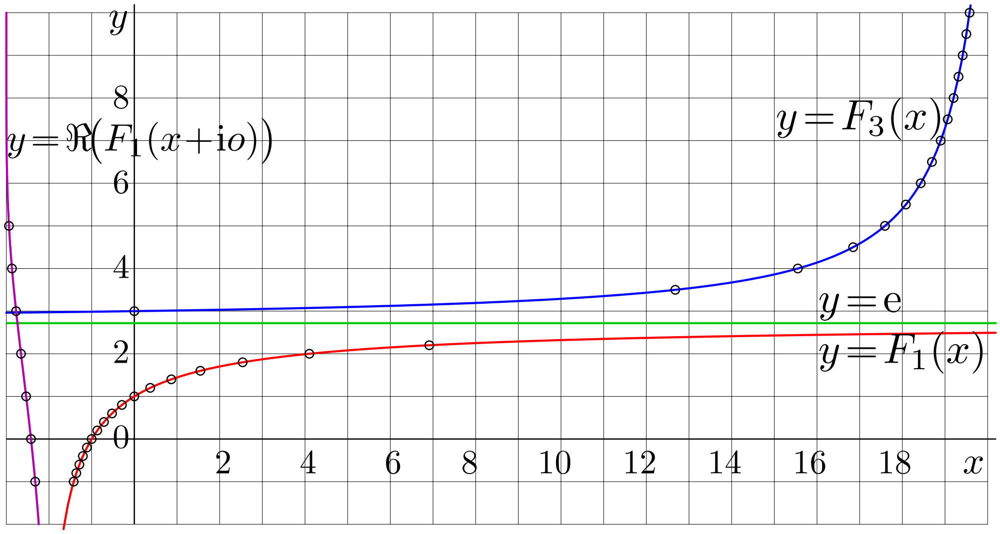

FIGURE 1. Super-exponentials to base $b = \exp(1/\mathrm{e})$ versus real argument; the asymptotic $y = \mathrm{e}$; circles represent the data from [16]. [fig1e1fre]

a faster/more precise alternative algorithm (which goes back to Écalle) that allows to plot the complex maps in real time.

The two super-exponentials F_1 and F_3 along the real axis are shown in figure 1. The circles represent the data from tables 1 and 3 by [16]. The behavior of these functions and their inverses in the complex plane is shown in figure 2.

As in [8], the subscript of the super-exponential (here 1 or 3) indicates the value at 0 of the chosen representative of the class of all super-exponentials obtained by argument shift $F(z+c)$, $c \in \mathbb{C}$ (which we often identify as one super-exponential). We consider two classes of super-exponentials represented by F_1 with $F_1(0)=1$ and by F_3 with $F_3(0)=3$, respectively. According to the definition, F_1 is a tetrational. For the other (growing) super-exponential, the smallest integer from the range of values along the real axis was chosen as value at zero.

2. Four methods of calculating the regular iteration with multiplier 1

In the theory of regular iteration (see e.g. [13] or [10]) there are several algorithms available to compute the regular fractional/continuous iteration and the Abel function of an analytic function at the fixed point 0. Functions h with multiplier 1, e.g. $h'(0) = 1$, are treated differently from functions h with $|h'(0)| \neq 0, 1$.

In our case we have the base function $f(z) = \mathrm{e}^{z/\mathrm{e}}$ with fixed point e and $f'(\mathrm{e}) = 1$. As the whole theory of regular iteration assumes the fixed point to be at 0, we

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TABLE 1. Computing the Abel function with Lévy's formula. [table:levy]

n	y_n	n	y_n	n	y_n	n	y_n
100	−1.4560	1,000	−1.425788	10,000	−1.4226982	100,000	−1.42241848
101	−1.4557	1,001	−1.425785	10,001	−1.4226982	100,001	−1.42241893
102	−1.4553	1,002	−1.425781	10,002	−1.4226981	100,002	−1.42241823
103	−1.4550	1,003	−1.425778	10,003	−1.4226981	100,003	−1.42241951
104	−1.4547	1,004	−1.425775	10,004	−1.4226981	100,004	−1.42241880
105	−1.4544	1,005	−1.425771	10,005	−1.4226980	100,005	−1.42241891
106	−1.4541	1,006	−1.425768	10,006	−1.4226980	100,006	−1.42241937
107	−1.4538	1,007	−1.425764	10,007	−1.4226980	100,007	−1.42241983
108	−1.4535	1,008	−1.425761	10,008	−1.4226979	100,008	−1.42241913
109	−1.4533	1,009	−1.425758	10,009	−1.4226979	100,009	−1.42241958

move the fixed point to 0 via a conjugation with the linear transformation τ: Let $\tau(z) = \mathrm{e}\,(z+1)$ then $\tau^{-1}(z) = z/\mathrm{e} - 1$ and

$$\tau^{-1} \circ f \circ \tau(z) = \mathrm{e}^z - 1 =: h(z)$$
$$f = \tau \circ h \circ \tau^{-1}$$

The regular iterates $f^{[t]}$ at the fixed point e are then given by $f^{[t]} = \tau \circ h^{[t]} \circ \tau^{-1}$, where the regular iterates of h can be obtained in one of the later described ways. The regular Abel function α of h (up to an additive constant determined by u) is defined by the inverse of $\sigma_u(t) = h^{[t]}(u)$. We call this σ_u the regular superfunction of h with $\sigma_u(0) = u$.

$$\sigma_u(t) := h^{[t]}(u) \qquad \alpha_u := \sigma_u^{-1} \qquad h^{[t]}(z) = \sigma_u(t + \alpha_u(z)) \tag{5}$$

The regular Abel function A_u of f at e with $A_u(u) = 0$ and the regular superfunction F_u of f at e with $F_u(0) = u$ can be obtained by

$$A_u = \alpha_{\tau^{-1}(u)} \circ \tau^{-1} \qquad F_u = \tau \circ \sigma_{\tau^{-1}(u)}. \tag{6}$$

The classic limit formula of Lévy [11] (see also Kuczma [10] theorem 3.5.6) for the regular Abel functions of h with multiplier 1 is:

$$\alpha_u(z) = \lim_{n\to\infty} \frac{h^{[n]}(z) - h^{[n]}(u)}{h^{[n+1]}(u) - h^{[n]}(u)} \tag{7}$$

$$A_u(z) = \lim_{n\to\infty} \frac{f^{[n]}(z) - f^{[n]}(u)}{f^{[n+1]}(u) - f^{[n]}(u)} \tag{8}$$

One can verify that in our case of $h(z) = \mathrm{e}^z - 1$, Lévy's formula converges just too slowly; it is difficult to reach sufficient precision to make any camera-ready plot of the Abel function. In table 1 we display

$$y_n = \frac{f^{[n]}(-1) - f^{[n]}(1)}{f^{[n+1]}(1) - f^{[n]}(1)} \to A_1(-1)$$

There is another interesting possibility to compute the regular superfunction, which we call here *Newton limit formula* (probably first mentioned by Écalle in [2])

TABLE 2. Computing the Abel function with Fatou's formula. [table:fatou]

n	a_n	n	a_n	n	a_n
1,000	-1.4224939	10,000	-1.422367740	100,000	-1.42235507550
1,001	-1.4224938	10,001	-1.422367738	100,001	-1.42235507549
1,002	-1.4224936	10,002	-1.422367737	100,002	-1.42235507548
1,003	-1.4224935	10,003	-1.422367736	100,003	-1.42235507546
1,004	-1.4224934	10,004	-1.422367734	100,004	-1.42235507545
1,005	-1.4224932	10,005	-1.422367733	100,005	-1.42235507543

because of its similarity to the Newton binomial series of $x^t = (1 + (x - 1))^t$:

$$\sigma_u(t) = h^{[t]}(u) = \sum_{n=0}^{\infty} \binom{t}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} h^{[m]}(u) \tag{9}$$

$$F_u(t) = f^{[t]}(u) = \sum_{n=0}^{\infty} \binom{t}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} f^{[m]}(u). \tag{10}$$

However also this method has a depressing slow convergence, moreover we need a bigger internal precision caused by the involved summation. For example for 1000 summands and 2000 bits precision with $u = 1$ and $t = -1.4223536677333$ we get $F_u(t) \approx -0.9875$ while we would expect a value very close to -1 (see table 1).

Another formula to compute an Abel function of $\mathrm{e}^x - 1$ is given in Walker's text [16]. He computes an Abel function g_1 of $a(z) = 1 - \mathrm{e}^{-z}$ and an Abel function g_2 of h with a formula which goes back to Fatou [4]:

$$g_1(z) = \lim_{n\to\infty} -\frac{1}{3}\log(n) + \frac{2}{a^{[n]}(z)} - n, \quad z < 0. \tag{11}$$

We derive the corresponding Abel function of h by knowing that $a(z) = -h(-z)$.

$$\alpha^1_{\mathrm{W}}(z) = g_1(-z) = \lim_{n\to\infty} -\frac{1}{3}\log(n) - \frac{2}{h^{[n]}(z)} - n, \quad z < 0 \quad [\mathrm{eq:walker}] \tag{12}$$

$$\alpha^2_{\mathrm{W}}(z) = g_2(z) = \lim_{n\to\infty} -\frac{1}{3}\log(n) - \frac{2}{h^{[-n]}(z)} + n, \quad z \ge 0 \quad [\mathrm{eq:walker2}] \tag{13}$$

The convergence of this formula is better than that of Lévy but still rather slow (which Walker notices too and that's why he introduces a slightly accelerated version which we omit here for brevity). To have an impression of the convergence of Fatou's/Walker's formula, we display

$$\begin{aligned} y_n &:= -\frac{2}{h^{[n]}(-\mathrm{e}^{-1} - 1)} + \frac{2}{h^{[n]}(-1)} - 1 \\ &\longrightarrow \alpha^1_{\mathrm{W}}(\tau^{-1}(-1)) - \alpha^1_{\mathrm{W}}(\tau^{-1}(0)) - 1 = A_1(-1) \end{aligned}$$

in table 2.

Before we give the fourth method and showing that Walker's formula is equivalent to it, we start with some formal background about regular iteration.

Definition 1 (regular iteration)**.** For every formal powerseries

$$h(z) = z + \sum_{n=m}^{\infty} h_n z^n, \quad m \ge 2, h_m \ne 0 \tag{14}$$

and each $t \in \mathbb{C}$ there is exactly one formal powerseries $h^{[t]}(z) = z + \sum_{n=m}^{\infty} h^{[t]}{}_n z^n$, such that $h^{[t]}{}_m = t \cdot h_m$ and $h^{[t]} \circ h = h \circ h^{[t]}$. We call $h^{[t]}$ the regular iteration of h. It satisfies $h^{[1]} = h$ and $h^{[s+t]} = h^{[s]} \circ h^{[t]}$ and is given by the formula:

$$h^{[t]}{}_N = \sum_{n=0}^{N-1} \binom{t}{n} \sum_{m=0}^{n} \binom{n}{m} (-1)^{n-m} h^{[m]}{}_N \tag{15}$$

$$= \sum_{m=0}^{N-1} (-1)^{N-1-m} \binom{t}{m} \binom{t-1-m}{N-1-i} h^{[m]}{}_N \tag{16}$$

where (16) can already be found as formula (2.19) in [5].

The formal powerseries $h^{[t]}$ is not necessarily convergent even if h is. We call a function which has the powerseries $h^{[t]}$ as asymptotic expansion at 0 a regular iteration of h.

If $z \mapsto h^{[t]}(z)$ is an analytic function in some domain, we call the function $\sigma(w) = h^{[t]}(z_0)$ a regular superfunction of h for any z_0 in the domain, and we call its inverse a regular Abel function of h. Usually we identify Abel functions that only differ by a constant and we identify superfunctions that are translations of each other (i.e. $\tilde{\sigma(x)} = \sigma(x + c)$ is identified with σ).

Definition 2. Let h be a formal powerseries of the form $h(x) = x + \sum_{k=m}^{\infty} h_k x^k$, $h_m \neq 0$. Its iterative logarithm is the unique formal powerseries j of form $j(x) = \sum_{k=m}^{\infty} j_k x^k$ with $j_m = h_m$ that satisfies the Julia equation

$$j \circ h = h' \cdot j. \tag{17}$$

One obtains the Julia equation when differentiating the Abel equation and then substituting $\alpha' = 1/j$. For reference we give the first few coefficients of the iterative logarithm j of $h(x) = \mathrm{e}^x - 1$:

$$j(x) = \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{1}{48}x^4 - \frac{1}{180}x^5 + \frac{11}{8640}x^6 - \frac{1}{6720}x^7 + \dots$$

From this iterative logarithm one can get a description of the regular Abel function by $\alpha = \int \frac{1}{j}$.

$$\alpha'(x) = \frac{1}{j(x)} = 2x^{-2} + \frac{1}{3}x^{-1} - \frac{1}{36} + \frac{1}{270}x + \frac{1}{2592}x^2 - \frac{71}{108864}x^3 + \dots$$

If we integrate this to get α the term x^{-1} becomes $\log|x|$ for real x:

$$\alpha(x) = \underbrace{-2x^{-1} + \frac{1}{3}\log|x|}_{s(x)} \underbrace{- \frac{1}{36}x + \frac{1}{540}x^2 + \frac{1}{7776}x^3 - \frac{71}{435456}x^4 + \dots}_{v(x)} \tag{18}$$

However this formula can not be used to get arbitrary precision, because $v(x)$ is not convergent as we show now. (However with a suitable truncation the divergent series can be used to obtain good precision.)

Preliminary 1 (Baker 1958 [1] Satz 17)**.** *The regular iteration $h^{[t]}$ of $h(x) = \mathrm{e}^x - 1$ has non-zero convergence radius exactly if t is an integer.*

Preliminary 2 (Ecalle 1975 [3])**.** *Let h be a formal powerseries with multiplier 1. Its regular iteration powerseries $h^{[t]}$ has a positive radius of convergence for all $t \in \mathbb{C}$ if and only if its iterative logarithm has a positive radius of convergence.*

Theorem 3. *The formal powerseries $v(z)$ in (18) has 0 convergence radius.*

Proof. Suppose that $v(z)$ has non-zero radius of convergence. Then also $v'(z) = \alpha'(z) - s'(z)$ has non-zero radius of convergence. Then $z^2 v(z)$ has non-zero radius of convergence and then $z^2 s'(z) + z^2 v(z)$ is a powerseries with non-zero radius of convergence with non-zero zeroth coefficient. Then

$$j(z) = \frac{1}{\alpha'(z)} = \frac{z^2}{z^2 s'(z) + z^2 v(z)}$$

has non-zero radius of convergence. Then by theorem 2 the regular iteration $h^{[t]}$ of $h(x) = \mathrm{e}^x - 1$ has non-zero radius of convergence for all t. This is in contradiction to theorem 1. Hence, the series diverges. □

Nonetheless we can use the formula (18) in a different way to calculate the regular Abel function of $h(x) = e^x - 1$. If we truncate v to N summands, denoted by v_N and $\alpha_N := s + v_N$, then Écalle showed (see [2], p. 78 ff., 95 ff.) that there are $2 \cdot (m-1)$ different regular Abel functions α^j, $1 \le j \le 2 \cdot (m-1)$ (α^j is defined on the j-th petal — each petal touching the fixed point 0 — of the now called Leau-Fatou flower see [12]) given by:

$$\alpha^j(z) = \lim_{n\to\infty} \alpha_N^j(f^{[(-1)^{j+1}n]}(z)) - (-1)^{j+1} n, \quad N \in \mathbb{N} \tag{19}$$

where $\alpha_N^j(z)$ is $\alpha_N(z)$ with the logarithmic term $\log|z|$ (in (18)) replaced by $\log\left(z e^{-(\theta_0 + j\frac{\pi}{m-1})\mathrm{i}}\right)$ and θ_0 is the unique number in the interval $(-\frac{\pi}{m-1}, \frac{\pi}{m-1}]$ such that $h_m = |h_m| e^{-\mathrm{i}(m-1)\theta_0}$.

This applied to $h(z) = \mathrm{e}^z - 1$ where $m = 2$, $h_m = 1/2$, $\theta_0 = 0$, we get the two regular Abel functions

$$\alpha^1(z) = \lim_{n\to\infty} \frac{1}{3}\log(-h^{[n]}(z)) - \frac{2}{h^{[n]}(z)} + v_N(h^{[n]}(z)) - n, \quad \Re(z) < 0 \tag{20}$$

$$\alpha^2(z) = \lim_{n\to\infty} \frac{1}{3}\log(h^{[-n]}(z)) - \frac{2}{h^{[-n]}(z)} + v_N(h^{[-n]}(z)) + n, \quad \Re(z) > 0 \tag{21}$$

$$A^1(z) = \alpha^1\left(\frac{z}{\mathrm{e}} - 1\right), \Re(z) < \mathrm{e} \tag{22}$$

$$A^2(z) = \alpha^2\left(\frac{z}{\mathrm{e}} - 1\right), \Re(z) > \mathrm{e} \tag{23}$$

From these 4 methods we know that Lévy's formula (7), the Newton formula (9) and Écalle's method (20) and (21) calculate the *regular* iteration/Abel function. We show in the last part of this section that Walker's formula is equal to Écalle's formula and hence also computes the regular Abel function.

The application of theorem 1.3.5 in [10] gives the following:

Preliminary 4 (Thron 1960 [14] Theorem 3.1.)**.** *Let h be analytic at 0 with powerseries expansion of the following form*

$$h(x) = x + h_m x^m + h_{m+1} x^{m+1} + \ldots, \quad h_m < 0, m \ge 2$$

then

$$\lim_{n\to\infty} n^{1/(m-1)} h^{[n]}(x) = (-h_m(m-1))^{-1/(m-1)}.$$

Theorem 5. *The by Walker constructed Abel functions α_{W}^1 and α_{W}^2 given in (12) and (13) are the two regular Abel functions of $\mathrm{e}^x - 1$.*

Proof. We show that the difference of Walker's and Ecalle's limit formulas is a constant. The differences are:

$$\delta_1(z) = \lim_{n\to\infty} \frac{1}{3}\log(-h^{[n]}(z)) + \frac{1}{3}\log(n) = \lim_{n\to\infty} \frac{1}{3}\log\Big(-nh^{[n]}(z)\Big), \quad z<0$$

$$\delta_2(z) = \lim_{n\to\infty} \frac{1}{3}\log(h^{[-n]}(z)) + \frac{1}{3}\log(n) = \lim_{n\to\infty} \frac{1}{3}\log\Big(nh^{[-n]}(z)\Big), \quad z>0$$

As $x \mapsto -h(-x)$ and $x \mapsto h^{-1}(x)$ for $h(x) = \mathrm{e}^x - 1$ is of the form required by preliminary 4 with $m-1=1$ we see that each of $nh^{[n]}(z)$ and $nh^{[-n]}(z)$ converges to a constant independent on z. □

3. Numerics of the two super-logarithms

The truncation of the series v keeping the term of the 15th power was used to build-up an approximation of A^1 that returns at least 15 decimal digits for $|z/\mathrm{e}-1| < \frac{1}{2}$. For larger values, the representation

$$A^1(z) = A^1\big(\exp(z/\mathrm{e})\big) + 1 \quad \text{[gz1]} \tag{24}$$

(equivalent to (20)) is iteratively used. This allows to extend the approximation to a wide domain keeping of order of 14 correct decimal digits. Then

$$A_1(z) = A^1(z) - A^1(1) \tag{25}$$

(26)

is the regular Abel function with the additive constant chosen such that $A_1(1) = 0$. The contour plot of this function is shown in the top, right picture of figure 2. The function is periodic; the period is $T_1 = 2\pi\mathrm{e}\,\mathrm{i} \approx 17.079468445347134131\ \mathrm{i}$. For real values $z>\mathrm{e}$, the representation diverges, indicating natural way to place the cut of the range of holomorphism.

As a test of the self-consistency of the implementations of the tetrational F_1 and the Abel function A_1, the central part of figure 2, the agreement

$$D(z) = -\lg\left|\frac{A_1(F_1(z)) - z}{A_1(F_1(z)) + z}\right| \quad \text{[D0]} \tag{27}$$

is shown with level of $D(z) = \text{const}$. Symbol "15" indicates the region where $D(z) > 14$. The picture indicates, that in the central part, the implementations of functions F_1 and $A_1 = F_1^{-1}$ are consistent within at least 14 decimal digits. In the right hand side, the branches of functions F_1 and A_1 do not match, and the agreement is poor. The similar test with the agreement

$$D_{1,\mathrm{a}}(z) = -\lg\left|\frac{F_1(A_1(z)) - z}{F_1(A_1(z)) + z}\right| \quad \text{[Da]} \tag{28}$$

was performed too; values of $D_{1,\mathrm{a}}$ are of order 15 in the wide domain of the complex plane, indicating, that the algorithm works close to the best precision achievable with the complex $<$ double $>$ variables.

The second super-logarithm A^2 has also good approximation for small values of $|z-\mathrm{e}|$. For large values, the extension to a wide range in the complex plane can be similarly realized with

$$A^2(z) = A^2\big(\log(z)\mathrm{e}\big) - 1 \quad \text{[gz2]} \tag{29}$$

The resulting function $A_3(z) = A^2(z) - A^2(3)$ is plotted in the right, bottom part of figure 2. This function is not periodic.

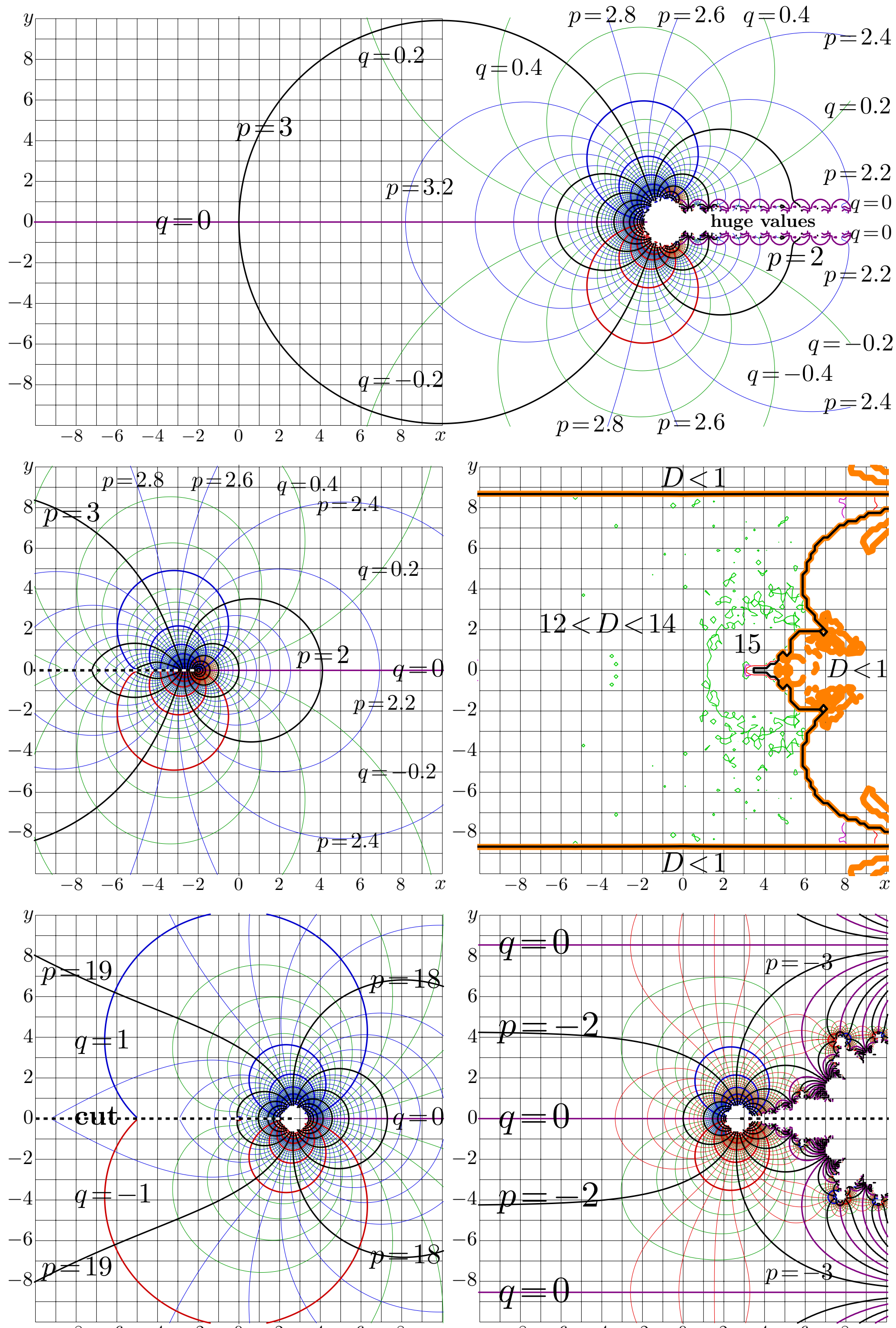

FIGURE 2. Function $f = F_3(z)$ in the complex $z = x + \mathrm{i}y$ plane, top picture; Functions $f = F_1(z)$ and agreement $f = D(z)$ by (27), central row; $f = A_3(z)$, and $f = A_1(z)$, bottom. For functions F and G, levels $p = \Re(f) = \mathrm{const}$ and $q = \Im(f) = \mathrm{const}$ are shown; thick lines correspond to the integer values. For the agreement D, level $f = 1$ is shown with light thick strips; level $f = 2$ is shown with black thick lines, level $f = 14$ is shown with thin lines. [fige1e]

The expansion for the growing super-exponential F_3 can be expressed with (41) through the same expression (1), but we use the upper sign in the argument of the logarithmic function. Then, the super-exponential has cut in the negative direction of the real axis, and is regular (although fast growing) in the positive part of the real axis. This function is shown at the left, bottom corner of figure 2.

4. A new expansion of the super-exponentials

This section describes the evaluation of the two super-exponentials to base $b = \exp(1/\mathrm{e})$. The base function $h(z) = \exp_b(z) = \exp(z/\mathrm{e})$ has the only fixed point $z = \mathrm{e}$. The super-exponential is expected to approach this point asymptotically. Consider the expansion of the super-exponential f in the following form:

$$(30)\quad f(z) = \mathrm{e}\cdot\left(1-\frac{2}{z}\left(1+\sum_{m=1}^{M}\frac{P_m\big(-\ln(\pm z)\big)}{(3z)^m}+\mathcal{O}\left(\frac{|\ln(z)|^{m+1}}{z^{m+1}}\right)\right)\right) \qquad \text{[fo]}$$

where

$$(31)\qquad P_m(t) = \sum_{n=0}^{m} c_{n,m}t^n \qquad \text{[P]}$$

The substitution of (1) into equation

$$(32)\qquad f(z+1) = \exp(f(z)/\mathrm{e}) \qquad \text{[Fexpe]}$$

and the asymptotic analysis with small parameter $|1/z|$ determines the coefficients c in the polynomials (31). This gives the the equations for the coefficients c of the polynomials P. In particular,

$$(33)\qquad P_1(t) = t$$

$$(34)\qquad P_2(t) = t^2+t+1/2$$

$$(35)\qquad P_3(t) = t^3+\frac{5}{2}t^2+\frac{5}{2}t+\frac{7}{10}$$

$$(36)\qquad P_4(t) = t^4+\frac{13}{3}t^3+\frac{45}{6}t^2+\frac{53}{10}t+\frac{67}{60}$$

$$(37)\qquad P_5(t) = t^5+\frac{77}{12}t^4+\frac{101}{6}t^3+\frac{83}{4}t^2+\frac{653}{60}t+\frac{2701}{1680} \qquad \text{[Pc]}$$

The evaluation with 9 functions P gives the approximation of $f(z)$ with 15 decimal digits at $\Re(z) > 4$. For small values of z, the iterations of formula

$$(38)\qquad f(z) = \ln(f(z+1))\ \mathrm{e} \qquad \text{[fzL]}$$

can be used. With complex < double > precision, the resulting approximation returns of order of 14 correct decimal digits in the whole complex plane, except the singularities.

For the case of tetrational, $t = -\ln(-z)$. Then, the tetrational F_1 can be expressed with

$$(39)\qquad F_1(z) = f(z+x_1) \qquad \text{[x1]}$$

where $x_1 \approx 2.798248154231454$ is solution of equation $f(x_1)=1$.

The same expressions (37) can be used also for the growing super-exponential; in this case, $t = -\ln(z)$. Then, the expression

$$(40)\qquad f(z) = \exp(f(z-1)/\mathrm{e}) \qquad \text{[fzE]}$$

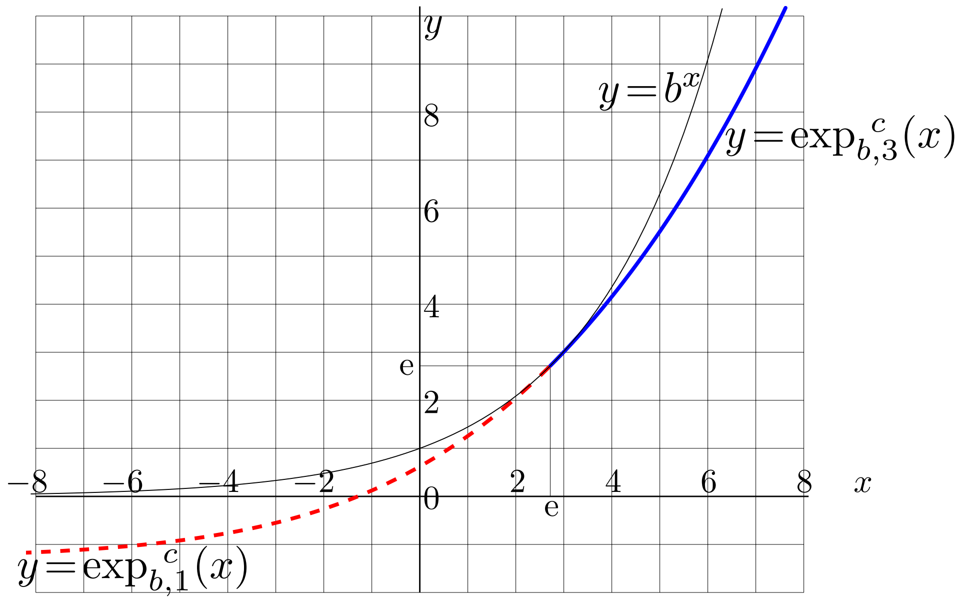

FIGURE 3. Functions $y=\exp^c_{b,1}(x)$ by (42), dashed; $y=\exp^c_{b,3}(x)$ by (43), thick solid; $y=\exp_b(x)=b^x$, thin solid, versus real x for $c=1/2$, $b=\exp(1/\mathrm{e})$. [figq]

allows the evaluation of growing super-exponential at small $|z|$. The specific super-exponential F_3 can be expressed as

$$F_3(z) = f(z + x_3) \qquad \text{[x3]} \tag{41}$$

where $x_3 \approx -20.28740458994004$ is solution of equation $f(x_3) = 3$. Function F_3 by (41),(40),(1) is shown in the top part of figure 1 for real argument and in the top picture of figure 2 for the complex values of its argument; function F_1 by (39),(38),(1) is shown in the bottom part of figure 1 for real argument and in the left hand side of the central row in figure 2 for the complex values of its argument.

In such a way, in this section the two superfunctions of the exponential to base $b=\exp(1/\mathrm{e})$ are constructed, F_1 and F_3. These functions are shown in the first two figures. F_1 is a tetrational. F_3 is entire, and show the fast growth along the real axis. At large values of the $|z|$, the function $F_1(z)$ approaches value e. Function $F_3(z)$ approaches value e for $|z| \to \infty$ except the positive direction of the real axis; in this direction, this function shows "faster than any exponential" growth.

5. Non-integer iteration

Each of the pairs (F_1, A_1) and (F_3, A_3) can be used to construct the regular iteration of the exponential to base $b=\exp(1/\mathrm{e})$:

$$\exp^c_{b,1}(z) = F_1\big(c + A_1(z)\big) \qquad \text{[q1]} \tag{42}$$

$$\exp^c_{b,3}(z) = F_3\big(c + A_3(z)\big) \qquad \text{[q3]} \tag{43}$$

These functions are shown in figure 3 for $c = 1/2$. For comparison, the function $y = \exp_b(z)$ is plotted with thin curve. The fractal iteration provides the smooth

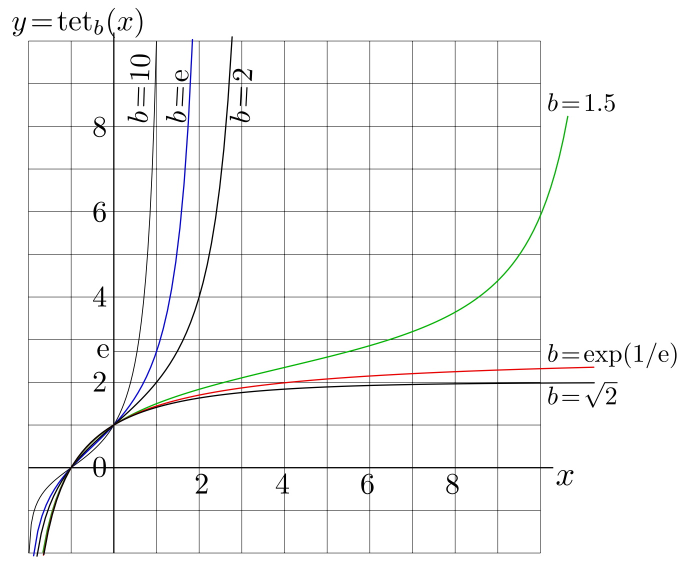

FIGURE 4. Tetrationals to base $b=10$, $b=\mathrm{e}\approx 2.71$, $b=2$, $b=1.5$, $b=\exp(1/\mathrm{e})\approx 1.44$ (the same curve as F_1 in figure 1), and $b=\sqrt{2}\approx 1.41$ versus real argument. [fig10q2]

(holomorphic) transition from the exponential at $c=1$ to the identity function at $c=0$.

Visually, the thick solid curve looks like a continuation of the dashed curve. However, the analytic continuation is not possible because $z = \mathrm{e}$ is branch point of $A_1(z)$. Similar visual effects are discussed in [8] for the case $b = \sqrt{2}$.

The fractional iterations by (42),(42) can be evaluated for complex values of z and even for complex values of c; but only at integer values of c these two functions can be considered as holomorphic extensions of each other.

The non-integer iteration of the exponentials provides a set of functions that grow up faster than any polynomial but slower than any exponential. Such functions may find applications in various areas of physics and technology.

6. Comparison of tetrationals with different bases

In this section, the tetrationals F_1 to base $\exp(1/\mathrm{e})$ is compared to tetrationals to various bases. In fig 4, the tetrational tet_b versus real argument is shown for $b = 10$, $b = \mathrm{e} \approx 2.71$, $b = 2$, $b = 1.5$, $b = \exp(1/\mathrm{e}) \approx 1.44$ and $b = \sqrt{2} \approx 1.41$. The functions for $b > \exp(1/\mathrm{e})$ are evaluated using the Cauchy algorithm described in [7]. For $b<\exp(1/\mathrm{e})$, the regular iteration described in [8] is used. For $b=\exp(1/\mathrm{e})$, the tetrational is just F_1 shown also in figure 1).

At moderate values of argument or order of unity or smaller, the curves for $b=1.5$, $b=\exp(1/\mathrm{e})$ and $b=\sqrt{2}$ are very close. In order to see the difference, the complex maps are shown in figure 5 for $b=1.5$, left, for $b=\exp(1/\mathrm{e})$, central, and for $b = \sqrt{2}$, right. The central part of figure 5 is just zoom-in from the picture at left hand side from the second row in figure 2. The efficient algorithms of the computation allow to plot all the figures with some reserve of resolution; at the online version, they still can be zoomed-in.

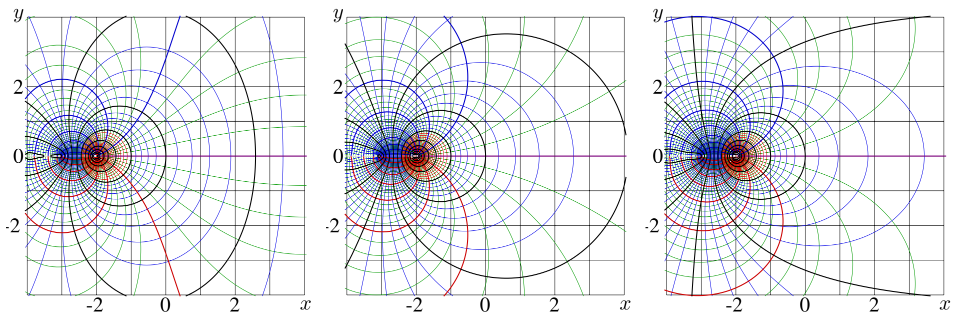

FIGURE 5. Tetrationals $\mathrm{tet}_b(x+\mathrm{i}y)$ for $b=1.5$, $b=\exp(1/\mathrm{e})\approx 1.44$ (the central part of map of F_1 in left picture of second row in figure 2), and $b=\sqrt{2}\approx 1.41$ in the x, y plane. [fig:poltora]

Figure 5 indicates no qualitative change of the tetrational at small variation of base in vicinity of value $b=\exp(1/\mathrm{e})$. In particular, within the loop $\Re\big(\mathrm{tet}_b(z)\big)=1$ (this loop goes through the origin of coordinates in all the three pictures), the zooming-in of the central parts of pictures in figure 5 is necessary to see the difference. One could expect $\mathrm{tet}_b(z)$ to be continuous function of real b at any fixed z from the range of the holomorphism.

From here, many questions occurs. Are the holomorphic tetrationals constructed in [8], here and in [15] (generalization of [6] which is conjectured to be the super-exponential in [7]) — which together cover the base range $(1,\infty)$ — analytic as a function of the base b, particularly in the point $b=\mathrm{e}^{1/\mathrm{e}}$? If so, what is the range of holomorphism? If not: can one obtain an operation $b\mapsto \mathrm{tet}_b(z)$ defined for b in a vicinity of $\mathrm{e}^{1/\mathrm{e}}$ such that for each b the function $z\mapsto \mathrm{tet}_b(z)$ is a real-analytic super-exponential on $(-2,\infty)$ and the function $b\mapsto \mathrm{tet}_b(z)$ is holomorphic for each z?

The questions above require the systematic analysis of $\mathrm{tet}_b(z)$ as function of two complex variables, b and z. Perhaps, the algorithms [7, 8] can be adopted for the case of complex values of b.

7. Conclusion

We present a part of the theory of regular iteration and apply it to the case $f(z)=\mathrm{e}^{z/\mathrm{e}}$. We extract a quite efficient algorithm to calculate the two super-logarithms to base $\mathrm{e}^{1/\mathrm{e}}$ compared with the various other methods that offspring from the theory. We suggest a new non-polynomial approximation for the two super-exponentials to base $\mathrm{e}^{1/\mathrm{e}}$.

One of the super-exponentials, namely $F_1=\mathrm{tet}_{\exp(1/\mathrm{e})}$, is interpreted as tetrational to base $\exp(1/\mathrm{e})$. It is holomorphic in the range $\mathbb{C}\setminus\{x\in\mathbb{R}: x\le -2\}$ and raises along the real axis, asymptotically approaching the limiting value e. The function approaches the same value e also in any other direction, id est, at large values of $|z|$.

Another super-exponential, F_3, is an entire function. Along the real axis it raises from the limiting value e at $-\infty$ to infinity, growing faster than any exponential. Outside the positive part of the real axis, $F_3(z)$ approaches e at $|z|\to\infty$ in the similar way as the tetrational F_1 does.

For bases $b > \exp(1/\mathrm{e})$ and $1 < b < \exp(1/\mathrm{e})$, the efficient algorithms were already reported in [7, 8]; so, all the range $b > 1$ is now covered. The plots of tetrational tet_b for $b = \exp(1/\mathrm{e}) \approx 1.44$ look similar to those for $b = 1.5$ and those for $b = \sqrt{2} \approx 1.41$; one may expect, that at any fixed value of z from some range, the tetrational $\mathrm{tet}_b(z)$ is a continuous function of b at least for $b > 1$.

The consideration of $\mathrm{tet}_b(z)$ as a function of two complex variables b and z requires the future research. Value $b = \exp(1/\mathrm{e})$ is suspected to be a branchpoint; the future analysis of the holomorphic properties in vicinity of this point is expected either to confirm or to negate this suspicion.

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