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Holomorphic Extension of the Logistic Sequence[1]

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Abstract—The logistic problem is formulated in terms of the Superfunction and Abelfunction of the quadratic transfer function $H(z) = uz(1 - z)$. The Superfunction F as holomorphic solution of equation $H(F(z)) = F(z + 1)$ generalizes the logistic sequence to the complex values of the argument z. The efficient algorithm for the evaluation of function F and its inverse function, id est, the Abelfunction G are suggested; $F(G(z)) = z$. The halfiteration $h(z) = F(1/2 + G(z))$ is constructed; in wide range of values z, the relation $h(h(z)) = H(z)$ holds. For the special case $u = 4$, the Superfunction F and the Abelfunction G are expressed in terms of elementary functions.

Key words: Logistic operator, Logistic sequence, Holomorphic extension, Superfunction, Abelfunction, Pomeau–Manneville scenario.

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1. INTRODUCTION

The logistic sequence F can be defined with the recurrent formula

$$H(F(z)) = F(z + 1) \tag{1}$$

and the initial condition $F(0)$ for the quadratic transfer function

$$H(z) = uz(1 - z), \tag{2}$$

where u is a positive parameter; let $0 < F(0) < 1$. In the publications about the logistic sequences [1–5], the argument z is assumed to be integer; given $F(0)$, the Eq. (1) determines $F(1)$, $F(2)$, $F(3)$.

In this paper, using the formalism of superfunctions [6–12], the holomorphic extension of function F is constructed. With this extension and the inverse function $G = F^{-1}$, the non-integer iterations of the transfer function can be evaluated

$$H^c(z) = F(c + G(z)). \tag{3}$$

At $c = 1/2$, this gives the half-iteration of the logistic transfer function, id est, function $h = \sqrt{H} = H^{1/2}$ such that

$$hhz = h(h(z)) = H(z). \tag{4}$$

The halfiterations for the transfer functions exp and Factorial are considered in papers [6, 10]. For the quadratic transfer function (2), the graphic of the halfiteration is plotted in Fig. 1 with thick lines for $u = 3$, left; for $u = 4$, central; and for $u = 5$, right. Other curves represent the 0.2th iteration, i.e., $H^{1/5}$, the 0.8th iteration, i.e., $H^{4/5}$, and the 1st iteration, i.e., $H^1 = H$, for the same values of u. The zeroth iteration would correspond to H^0, which is identity function, is not plotted.

The following sections describe the evaluation of functions F and G and discuss the range of validity of relation (4).

2. SUPERFUNCTION

One should work in the complex plane, in order to make a holomorphic extension. The quadratic function H by (2) is presented at the upper graphics in Fig. 2. Function $f = H(z)$ in shown in the complex z-plane at $u = 3$, left; at $u = 4$, center and at $u = 5$, right. Levels $p = \mathrm{Re}(f) = \mathrm{const}$ and levels $q = \mathrm{Im}(f) = \mathrm{const}$ are shown; thick lines correspond to the integer values.

The second row of pictures in Fig. 2 shows, in the same notations, the halfiteration h by (3) at $c = 1/2$. For the evaluation of the halfiteration, the Superfunction F and the Abelgunction G are used. These functions are plotted in the last two rows of Fig. 2 for the same values of parameter u. In the construction of Superfunction F, the crucial is question about the fixed points of the Transferfunction H, which are solutions of equation

$$H(z) = z. \tag{5}$$

For the quadratic H by (2), the Eq. (5) has exactly two solutions, $z = 0$ and $z = 1 - 1/u$. The first one does not depend on u. Below, it is used to develop the Superfunction. For the real transfer function (2) with the real Fixedpoint, the formalism [8, 10] indicates the following asymptoric expansion for the Superfunction F:

[1] The article was translated by the author.

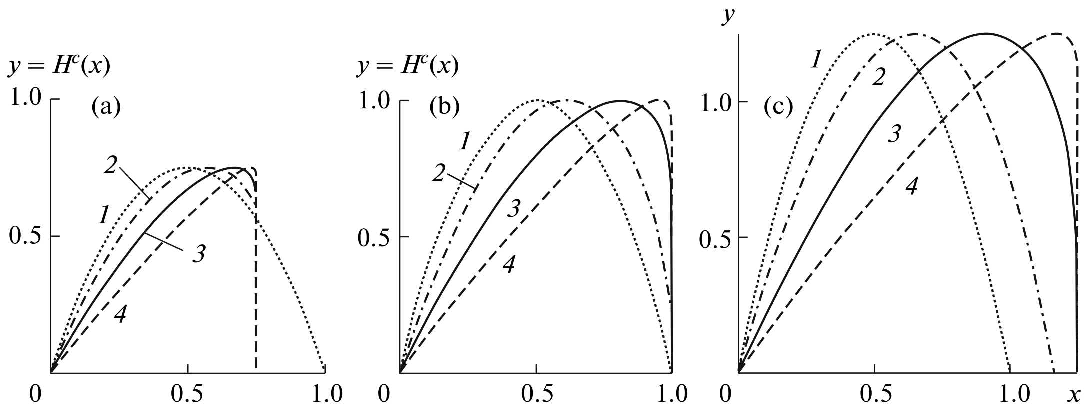

Fig. 1. Various iterations $H^c(x)$ versus real x for $u = 3$, left; for $u = 4$, center; for $u = 5$, right; curves for $c = 1$, $c = 0.8$, $c = 0.5$, $c = 0.2$ are drawn.

$$F(z) = \sum_{n=1}^{N-1} c_n u^{nz} + \mathcal{O}(u^{Nz}). \tag{6}$$

The substitution of (6) into (1) gives the chain of equations for the coefficients c. One can set $c_1 = 1$; variation of this coefficient corresponds to translations of the solution along the real axis. Then

$$c_2 = \frac{1}{1-u},$$
$$c_3 = \frac{2}{(1-u)(1-u^2)}, \tag{7}$$
$$c_4 = \frac{5+u}{(1-u)(1-u^2)(1-u^3)}.$$

The expression (6) gives a way to evaluate the superfunction F at large negative values of the real part of the argument. For other values the recurrence

$$F(z) = H(F(z-1)) \tag{8}$$

can be used, giving the fast and precise implementation. The map of function F in the complex plane is shown in the third row in Fig. 2 for $u = 3$, $u = 4$, $u = 5$. The superfunction F is entire periodic function. For real u, the period

$$T = 2\pi i/\ln(u) \tag{9}$$

is pure imaginary. In vicinity of the half-line $\mathrm{In}(z) = \mathrm{In}(T/2)$, $\mathrm{Re}(z) \to +\infty$, superfunction $F(z)$ has huge values and huge derivatives so, the plotter could not draw the levels and these regions look "empty".

Along the real axis, the superfunction F is smooth and bounded; it approaches zero at $-\infty$ and oscillates at positive values of the argument; at least for $3 \le u \le 4$, the function $F(x)$ remains bounded for all real x. For larger values of the parameter u, the intervals with large negative values appear.

The periodicity with imaginary period is typical for the real regular superfunctions constructed at the real fixed points of the transfer function [8, 10]. (The exponential, as super-function of a linear transfer function, is a particular case of such a rule.)

3. ABELFUNCTION

For construction of the halfitertion declared in the Introduction, the inverse of the Super-function F is required. Such inverse function, id est, $G = F^{-1}$, can be called Abelfunction, because it satisfies the Abel equation [8–10]

$$G(H(z)) = G(z) + 1. \tag{10}$$

Its asymptotic expansion can be obtained by the straightforward inversion of the series (6):

$$G(z) = \log_u\left(\sum_{n=1}^{N-1} s_n z^n + \mathcal{O}(z^N)\right), \tag{11}$$

where s are constant coefficients;

$$s_1 = 1,$$
$$s_2 = \frac{1}{u-1},$$
$$s_3 = \frac{2u}{(u-1)(u^2-1)}, \tag{12}$$
$$s_4 = \frac{(u^2-5)u}{(u-1)(u^2-1)(u^3-1)}.$$

The equations for s can be obtained also at the substitution of the expansion (11) into the Abel equation (10).

The truncation of the asymptotic series provides the precise-approximation of $G(z)$ at $|z| \ll 1$. In order to extend such an approximation to the large values of the argument, the recurrent formula can be used

$$G(z) = G(H^{-1}(z)) + 1, \tag{13}$$

where

$$H^{-1}(z) = 1/2 - \sqrt{1/4 - z/u}. \tag{14}$$

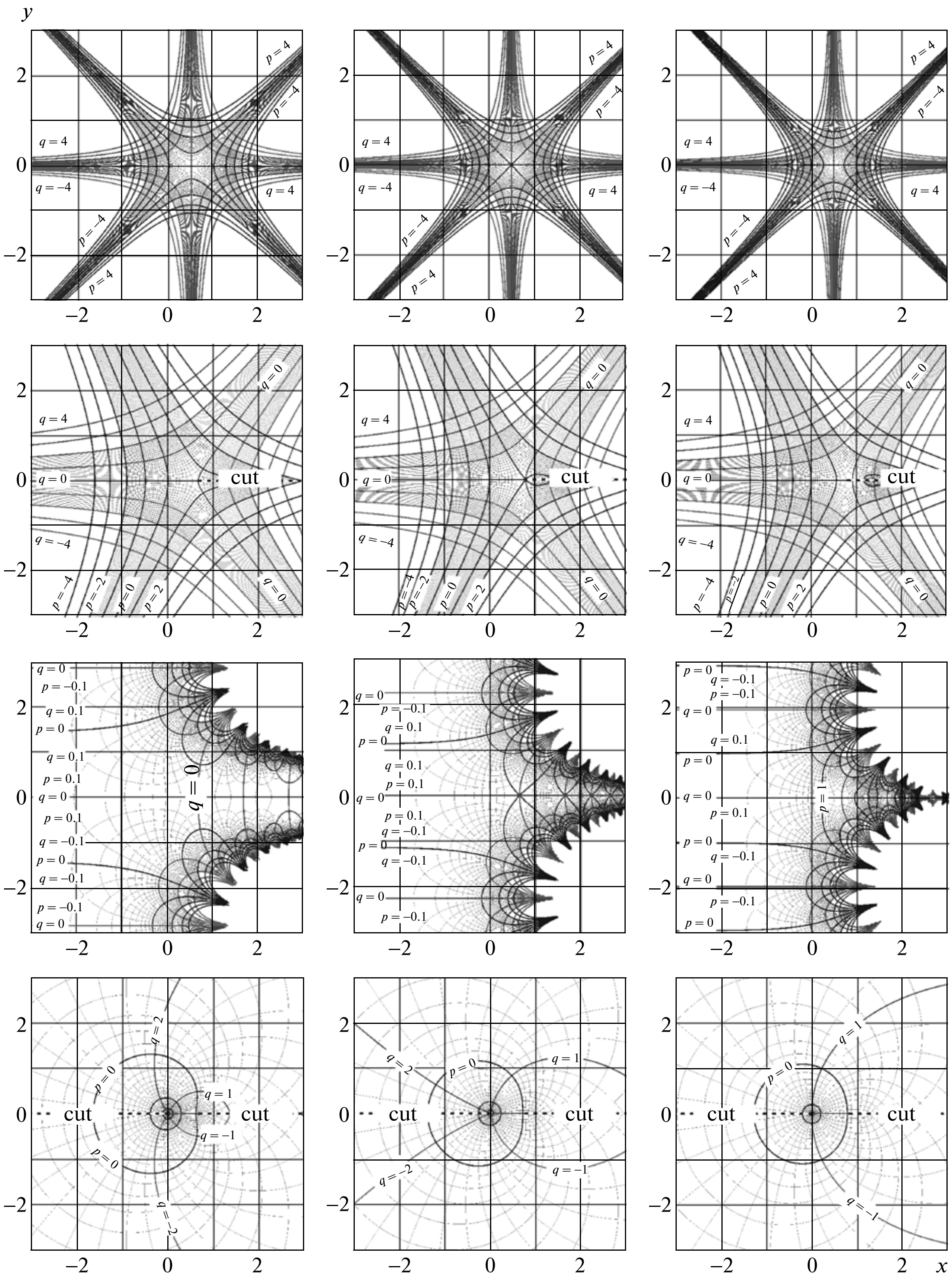

Fig. 2. Maps of the transferfunction $H(z)$ (upper row), its halfiteration (second row). Superfunction $F(z)$ (third row) and the Abelfunction $G(z)$ (bottom) in the complex $z = x + iy$ plane for $u = 3$, left; $u = 4$, center; and $u = 5$, right.

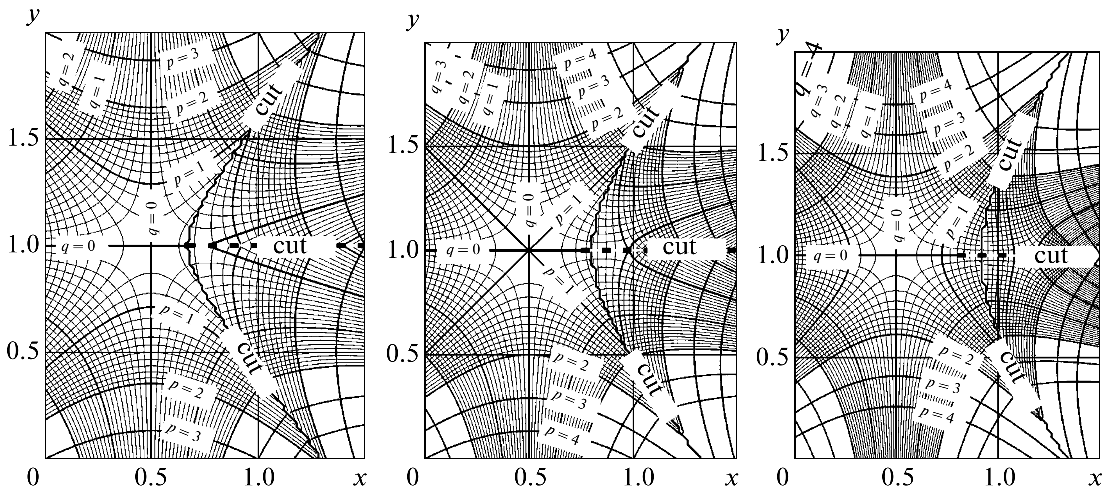

Fig. 3. Function $f = H^{0.5}(H^{0.5}(z))$ in the complex $z = x + iy$ plane for $u = 3$, left; $u = 4$, center; $u = 5$, right.

The representation through (11), (13) leads to the efficient algorithm; it is used to plot the last row in Fig. 2.

While F, G are already chosen and implemented, then, for any complex number c, the c-th iteration H^c of the transfer function can be defined with (3). Such iterations satisfy relation

$$H^c(H^d(z)) = H^{c+d}(z); \tag{15}$$

at least for some range of values of z. In particular, at integer c, the iteration means just sequential application of the function c times,

$$H^c(z) = \underbrace{H(H(\ldots H}_{c}(z)\ldots)). \tag{16}$$

At $c = 1/2$, the halfiteration $H^{1/2}$ is plotted in the second raw of Fig. 1. This function has cut, that begins between 1/2 and unity and goes along the real axis to infinity. In particular, this cut limits the range of validity of Eq. (4).

4. RANGE OF VALIDITY OF $hh = H$

In general, the inverse function of an entire function has branchpoints and cutlines; the only exception is a linear function. Therefore, the relation $G(H(z)) = z$ should have some limited range of validity; it limits the range of Eq. (15).

Behavior of function $f = H^{1/2}(H^{1/2}(z))$ is shown in Fig. 3 for $u = 3, 4, 5$. In the left hand side of the complex plane, the pictures look as the zoomed-in central parts of the top row pictures in Fig. 2. The scratched line shows the margin of the range of validity of the relation $H(z) = H^{1/2}(H^{1/2}(z))$. The cuts along the real axis are marked with dashed lines.

Relation (4) holds for the most of the complex plane. However, it cannot hold for the whole complex plane, because the information, at which oscillation does the function F take some fixed value, is lost at the first step of evaluation by (3). Similar restrictions of the range of validity of Eq. (15) should take place for other transfer functions too; in particular, for functions $\sqrt{\exp}$ and $\sqrt{\text{Factorial}}$ analyzed in the complex plane [6–8, 10, 11].

The monotonic behavior of function $H = \exp$ allows the relation (15) to hold along the real axis. The monotonic behavior of function $H = \text{Factorial}$ allows the relation (15) to hold for $z > 1$. In the similar way, in the case of the logistic operator H by (2), the relation (15) holds at least for $\text{Re}(z) \le 1/2$.

It is common that the Abelfunction, developed at some fixed point, is irregular at another fixed point. Then, the non-integer iteration of the transfer function may have the same irregularities, namely, the branchpoints. The corresponding cutlines limit the range of applicability of the Eq. (15) and, in particular, Eq. (4). However, the Abeflunction (and then, the non-integer iteration of the transfer function) can be irregular also at both fixed points, as the real $\sqrt{\exp}$ does [10, 11].

New modifications of the Abelfunctions (and corresponding non-integer iterations of the transfer function) can be generated moving the cutlines, as it is done for the Abelexponential (sometimes called also "superlogarithm", although it is not a Superfunction of logarithm) and expc in [12]. Usually, such modifications have more complicated structure, than the initial Abelfunctions.

5. CASES $u = 3.5699$ AND $u = 3.8284$

In this sections, the two special cases are considered, $u = 3.5699$ and $u = 3.8284$. While z is interpreted as a discrete variable, these values corresponds to be margins between regular and irregular behavior in the Pomeau–Manneville scenario [14–18].

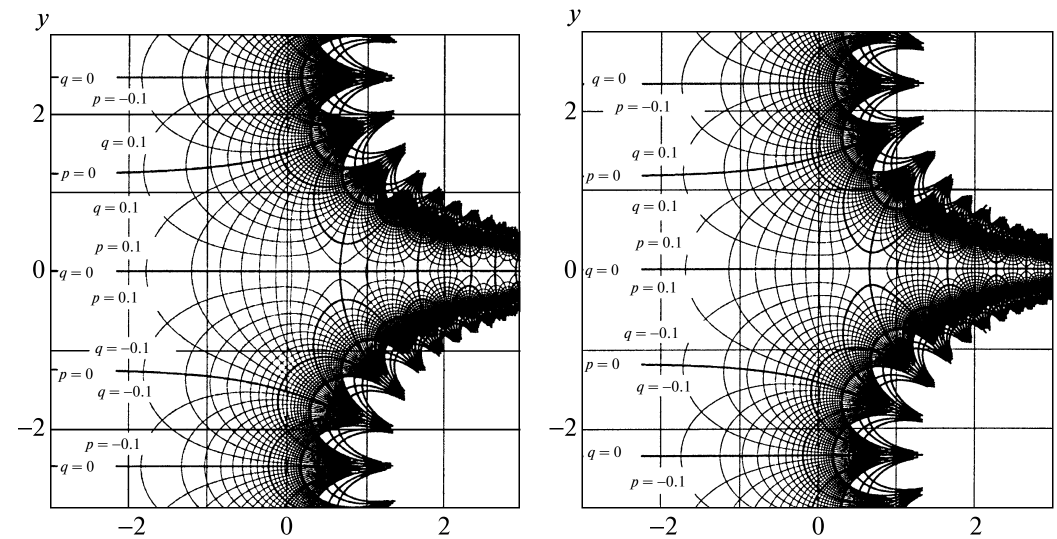

Fig. 4. Maps of superfunction F for $u = 3.5699$ and $u = 3.8284$ in the same notations as in Fig. 2.

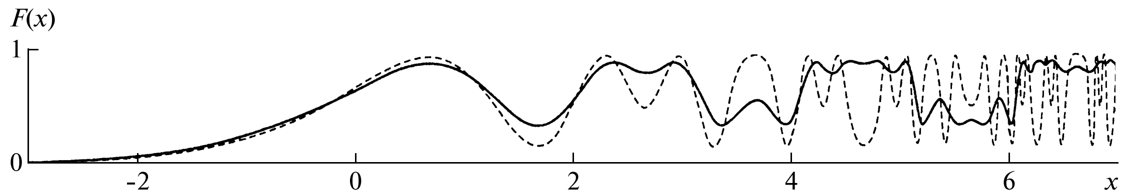

Fig. 5. Graphics of SuperFunction F versus real argument for $u = 3.5699$ (solid) and $u = 3.8284$ (dashed).

In Figure 4 the maps of function F are shown for these cases. Figure 5 shows the behavior of these functions along the real axis. In general, these functions behave in a way, one could expect from the consideration of the discrete values of the argument [14, 15]. In particular, at $u = 3.5699$, visually, one can trace some "sinusoidal" trend with period 2. No qualitative change of the structure is seen at the maps in the complex plane (Fig. 4).

6. SPECIAL CASE $u = 4$

In the special case $u = 4$, the Superfunction and the Abelfunction can be expressed through the elementary functions. Such an expression can be found from the table of Superfunctions. The raw 8 of Table 1 from [10] corresponds to the transfer function

$$H_0(z) = 2z^2 - 1 \tag{17}$$

with Superfunction

$$F_0(z) = \cos(2^z) \tag{18}$$

and Abelfunction

$$G_0(z) = \log_2(\arccos(z)). \tag{19}$$

Then the transform from the last row of the same table at the linear functions

$$P(z) = (1 - z)/2 \tag{20}$$

and

$$Q(z) = 1 - 2z \tag{21}$$

gives the new transfer function

$$H_1(z) = P(H_0(Q(z))) = 4z(1 - z),) \tag{22}$$

that coincides with the transfer function H by (2) at $u = 4$, and the Superfunction

$$F_1(z) = P(F_0(z)) = 1/2(1 - \cos(2^z)) \tag{23}$$

and the Abelfunction

$$G_1(z) = G_0(Q(z)) = \log_2(\arccos(1 - 2^z)). \tag{24}$$

Functions F_1 and G_1 can be related with F and G plotted in the central column of Fig. 2 with the simple translation:

$$F(z) = F_1(z + 1), \quad G(z) = F_1(z - 1).$$

Superfunction F versus real argument is shown in Fig. 6 for $u = 3.99$, dashed curve; $u = 4$, solid curve; $u = 4.01$, dotted curve. As one could expect, the solid curve looks pretty regular.

I have compared the "exact" expressions of F and G through the elementary functions (23) and (24) to the numerical implementations through the asymptotic expressions (6), (11) and the recurrent formulas (8), (13). The comparison confirms the high precision of the numerical implementations. Of order of 14 correct digits can be achieved with the complex<double> variables.

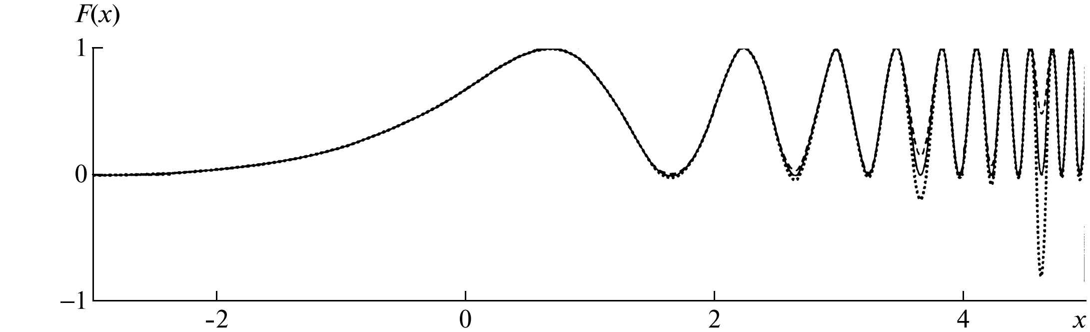

Fig. 6. $F(x)$ as function of real x for $u = 3.99$, dashed curve; $u = 4$, solid curve; and $u = 4.01$, dotted curve.

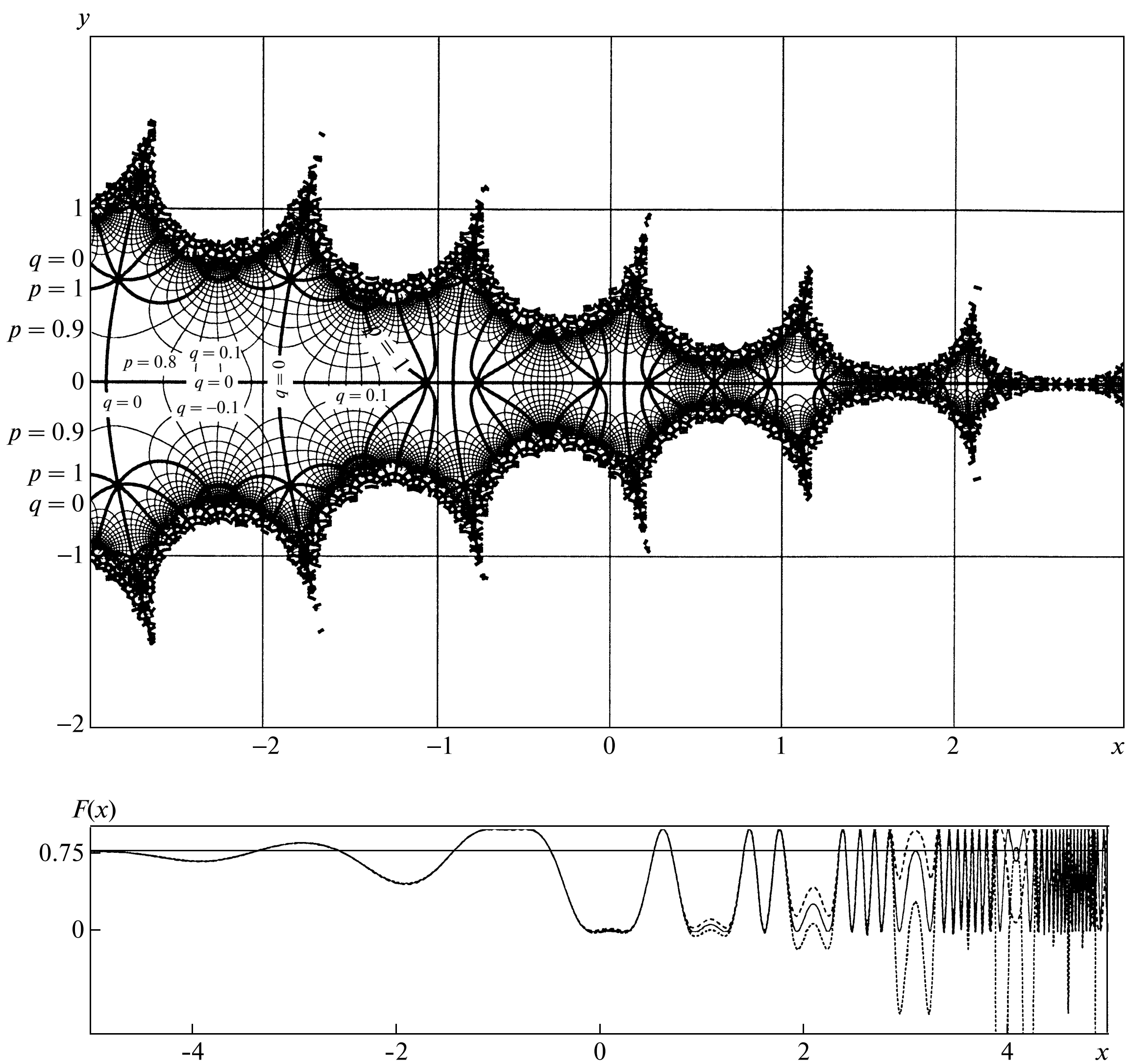

Fig. 7. Map of superfunction F by (25), (8) for $u = 4$, top, and $F(x)$ versus real x for $u = 3.99, 4, 4.01$, bottom.

7. FIXED POINT $1 - 1/u$

The fixed point $z = 1 - 1/u$ also can be used as an asymptotic of the Superfunction of the transfer function (2). It is shown in Fig. 7. Such Superfunction can be expressed asymptotically

$$F(z) = \frac{u-1}{u} + \sum_{n=1}^{N-1} d_n((u-2)^z \cos(\pi z + \phi))^n + \mathbb{O}((u-2)^z \cos(\pi z + \phi))^N, \qquad (25)$$

where phase ϕ and coefficients d are constants. The substitution into Eq. (1) gives the chain of equations for the coefficients. We may set $d_1 = 1$; then

$$d_2 = \frac{-u}{(u-1)(u-2)},$$
$$d_3 = \frac{-u^2}{(u-1)(u-2)(u-3)}, \qquad (26)$$
$$d_4 = \frac{-(u-7)^3 u^3}{(u-2)(u-3)(u^3 - 8u^2 + 22u - 21)}.$$

Such languages as Mathematica allows to calculate exactly a ten of such coefficients. This expansion provides approximation valid while the effective parameter of expansion, id est, $(u - 2)^z\cos(\pi z + \phi)$, is small. The truncted sum gives several correct decimal digits at

$$\pi|\mathrm{Im}(z)| + \ln(u-2)\mathrm{Re}(z) < -4. \qquad (27)$$

The extension with (8) gives the efficient algorithm; it is used to plot Fig. 7. The figure corresponds to $\phi = 0$. At the top, the map of Superfunction F is shown for $u = 4$. At the bottom, the function $F(x)$ is plotted versus real x for $u = 3.99$, 4, 4.01; these are the same values as in Fig. 6.

The Superfunction constructed in such a way is asymptotically-periodic; quasi-period

$$T = \frac{2\pi i}{\ln(u-2) + \pi i} \qquad (28)$$

in the upper halfplane and T^* in the lower half-plane. This quasi-periodicity is determined by the leading term in the expansion (25). The quasi-periodic behavior is also typical for the Superfunctions [6–8, 10]. In the region of the quasi-periodic behavior, the Fig. 3 show the fractal-like behavior. Similar behavior for the tetrational (Superfunction of the exponential) is discussed in [12].

8. BOUNDARIES OF THE TIME DERIVATIVE

Variable z in (6), (8) may have sense of time. Then the Superfunction F can be interpreted as some smooth, infinitely differentiable physical process. Being measured at integer values of time, this process generates the logistic sequence. At least for $u = 4$, the representation (23) gives the time derivative of such process:

$$F(z) = F_1(z+1) = \frac{1}{2}(1 - \cos(2^{z+1})),$$
$$F'(z) = \ln(2)2^z \sin(2^{z+1}). \qquad (29)$$

The upper bound for the modulus of the derivative grows exponentially:

$$|F'(z)| \le \ln(2)2^z. \qquad (30)$$

at least for real values of time z. The same bound seems to be valid also for $u < 4$. However, for $u > 4$, the double-exponential growth is allowed, but

$$|F(z)| < \exp(2^z), \qquad (31)$$

according to the row 5 of the Table of Superfunctions, [10], Table 1; in this case, the quadratic term in the expansion of the logistic transferfunction dominates. Then the derivative can be estimated as

$$|F'(z)| < \ln(2)2^z \exp(2^z). \qquad (32)$$

In such a way, the holomorphic extension leads to the estimate of rate of growth of the logistic sequences.

9. MORE SUPERFUNCTIONS

The holomorphic extension F of the logistic sequence is not unique. It can be developed at any of fixed points of the logistic Transferfunction $H(z)$ by (2). Also, the new Superfunctions $\tilde{F}$ can be expressed through some already constructed Superfunction with the periodic modification of the argument:

$$\tilde{F}(z) = F(z + \varepsilon(z)), \qquad (33)$$

where ε is some 1-periodic function holomorphic at least in some vicinity of the real axis. Such Superfunctions may grow up in the direction of the imaginary axis and also may have additional singularities in the complex plane. The imaginary direction F by (6), (8) seems to be the only non-trivial holomorphic extension that does not grow in the periodic Superfunction. The observation of various extensions of the logistic sequences can be summarized in the hypothesis below:

Hypothesis 0. For any $u > 2$, any holomorphic extension F of the logistic sequence, id est, solution of $F(z + 1) = uF(z)(1 - F(z))$, that cannot be simply expressed with (6), (8), has at least an exponential growth in the direction of the imaginary axis.

The Hypothesis 0 specifies some kind of uniqueness of the holomorphic extension of the logistic sequence. The proof (and, perhaps, the adjustment) of this hypothesis can be matter for the future research. However, the extension allows the translations and several Superfunctions with exponential growth in the imaginary direction may exist.

10. CONCLUSION

There is nothing especial in the logistic transfer function; the Superfunction can be constructed following the general procedures [7, 8, 10, 12]. The asymptotic expansion (6) allow the fast and precise evaluation of the Superfunction, id est, the holomorphic extension F of the logistic sequence [1–4], and its inverse function G. For the case $u = 4$, the holomorphic extension can be expressed with the elementary function (23). Such a representation allows to interpret the logistic sequence as a smooth, infinitely differentiable process, measured at the integer values of time z.

With given Superfunction F and the Abelfunction $G = F^{-1}$, the non integer iteration H^c of the transfer function H can be constructed in the standard way through Eq. (3). At $c = 1/2$, this gives the halfiteration, plotted in the second raw in Fig. 2. In the case $u = 4$, this halfiteration also can be expressed through the elementary functions (23), (24).

In the similar way, the holomorphic extension can be constructed for other sequences, which are usually interpreted as chaotic and quasi-stochastic [1–4]. One can construct the half-iterations in the way suggested by [6–13].

Being extended to the complex plane, the logistic sequence looks completely regular, so regular, as the sinusoidal with smoothly decreasing period. The case $u = 4$ gives a good example; the logistic sequence is expressed through sinusoidal of the exponentially growing argument.

The hypothesis 0 declares the uniqueness of the periodic holomorphic extension of the logistic sequence with imaginary period. The proof and the application to the realistic physical systems can be matter for the future research. In general, the assumptions about the asymptotic behavior in the complex plane are essential for the uniqueness of the holomorphic extensions of the recurrent sequences; the extension of the logistic sequence is not an exception.

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