H. Trappmann. D. Kouznetsov

UNIQUENESS OF HOLOMORPHIC SUPERLOGARITHMS

Abstract. We give a simple uniqueness criterion for holomorphic Abel functions with application to superlogarithm and superexponential/tetration.

1. Introduction

There is a lot of discussion about the "true" fractional iterates of the function e^x in the mathematical community. Dating back to 1950 Kneser [1] proved the existence of analytic fractional iterates. However Szekeres (a pioneer in developing the theory of fractional iteration [4]) states in [3]:

"The solution of Kneser does not really solve the problem of 'best' fractional iterates of e^x*. Quite apart from practical difficulties involved in the calculation of Kneser's function on the real axis, there is no indication whatsoever that the function will grow more regularly to infinity than any other solution. There is certainly no uniqueness attached to the solution; in fact if* $g(x)$ *is a real analytic function with period 1 and* $g'(x) + 1 > 0$ *(e.g* $g(x) = \frac{1}{2}\sin(2\pi x)$ *then* $B^*(x) = B(x) + g(B(x))$ *is also an analytic Abel function of* e^x *which in general yields a different solution of the equation."*

By withdrawing our attention from the purely real analytic behaviour of the Abel function to the behaviour in the complex plane we can succeed in giving a simple uniqueness criterion for the above mentioned Abel function (which we call *superlogarithm* here) which in turn determines also the fractional iterates of e^x.

Not only can we show that Kneser's solution is indeed this unique solution, but we have also good means to numerically compute this solution and the corresponding fractional iterates of e^x (also of b^x for $b > e^{1/e}$ in generalization) by a method developed in [2]. Several other methods to

---

2000 Mathematical Subject Classification. 30D05, 39B32, 65Q05, 65R20, 65D15, 41A55, 65D30, .

Key words and phrases. Abel function, superlogarithm, superexponential, superexponentiation, generalized exponential, holmorphism.

numerically compute holomorphic fractional iterates of e^x or the holomorphic superlogarithm have emerged in the past years (for example one is given in [6]). A future research goal would be to put them on a thorough theoretic base (proving convergence and holomorphy) and to verify the here given uniqueness criterion, with the result of a beautiful union of several quite different approaches to the problem.

2. The Superexponential

Instead of using the terms "generalized logarithm" and "generalized exponential" as done by Walker in [5], or "ultra exponential" as done by Hooshmand in [8], we stick to the convention of [7] with the more succinct words "superexponential" and "superlogarithm".

A superexponential to base $b > 0$ is a function f that satisfies

$$f(0) = 1 \tag{1}$$

$$f(z+1) = \exp_b(f(z)) \tag{2}$$

and a superlogarithm is the inverse of a superexponential, or a function g that satisfies:

$$g(1) = 0 \tag{3}$$

$$g(\exp_b(z)) = g(z) + 1 \tag{4}$$

where $\exp_b(z) = b^z = \exp(\ln(b)z)$. For integer values of z any superexponential is already determined to be just the z-times application of $\exp_b$ to 1.

$$f(z) = \text{sexp}_b(z) = {\exp_b}^z(1) \;=\; \underbrace{\exp_b\Big(\exp_b(\ldots\exp_b(1)\ldots)\Big)}_{z \text{ exponentials}}\,. \tag{5}$$

The question is however how to properly define the superexponential to non-integer values of z. A non-analytic solution with a uniqueness criterion was given in [8]. A numerical method to compute the coefficients of the powerseries development at 0 of a superlogarithm was given (though without proof of convergence) in [6]. We use here the robust and fast numerical method given in [2] to compute a/the holomorphic extension. This method is originally described for $b = e$ but can be extended to arbitrary bases $b > \mathrm{e}^{1/\mathrm{e}}$. For real values of the argument, this superexponential is plotted in figure 1 for $b{=}\mathrm{e}$, $b{=}2$ and $b{=}\exp(1/\mathrm{e})$ with thick solid, dashed and thin curves.

A holomorphic superexponential is expected to have a singularity or branchpoint at integers ≤ -2 at least on some branch, because from $f(z{+}1) = \exp_b(f(z))$ one would conclude that $f(z{-}1) = \log_b(f(z))$ on

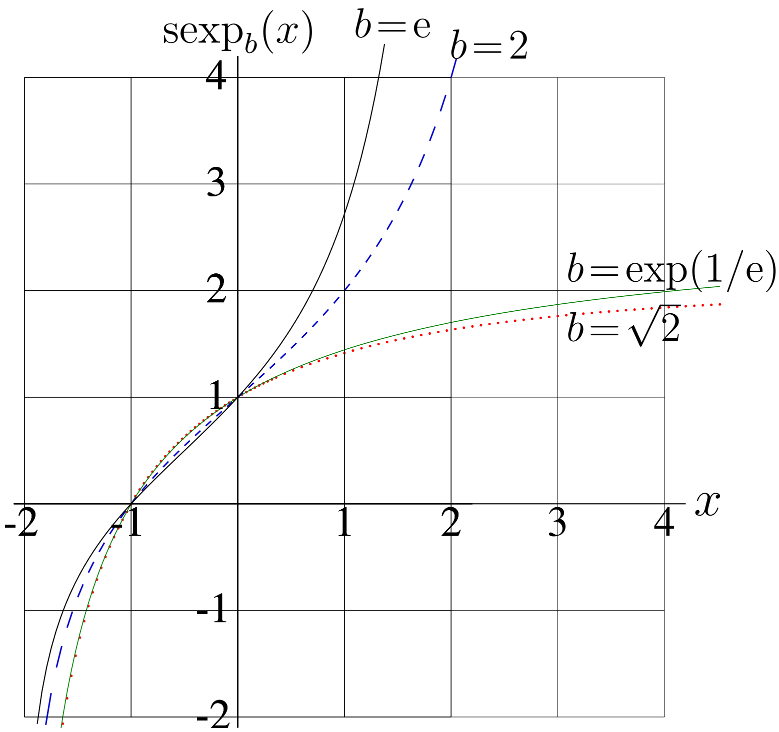

FIGURE 1. Holomorphic superexponential at base $b = \mathrm{e}$ (thick solid), $b=2$ (dashed), $b=\exp(1/\mathrm{e})$ (thin solid) and $b=\sqrt{2}$ (dotted) on the real axis.

some branch and by $f(0) = 1$ is then $f(-1) = 0$ and $f(-2) = \log_b(0)$. To exclude branching we restrict superexponentials to

$$C = \mathbb{C}\setminus\{x \in \mathbb{R} : x \leq -2\}\ . \tag{6}$$

3. Uniqueness

Conventions: Usually one regards a holomorphic function only on an open and connected set (domain). Saying that f is (bi)holomorphic on (an arbitrary set) G here means that f is (bi)holomorphic on some domain $D \supseteq G$.

We call a function h holomorphic on H a $d \mapsto c$ *Abel function of* F iff

$$h(d) = c \tag{7}$$

$$h(F(z)) = h(z) + 1 \tag{8}$$

for all z such that $z, F(z) \in H$.

We call a region H an *initial region of* F iff $F(z) \notin H$ for all $z \in H$ and the boundary ∂H without the fixed points of F make up two disjoint paths $\partial_1 H$ and $\partial_2 H$ such that F maps $\partial_1 H$ bijectively to $\partial_2 H$. For simplicity we additionally demand that $\partial_1 H \subseteq H$ and $(\partial_2 H) \cap H = \emptyset$. We call the initial region *simple* iff $\partial_1 H$ is homeomorphic to $(0,1)$, i.e. if there is a bijective curve $\gamma\colon (0,1) \leftrightarrow \partial_1 H$.

For example, the function $\log(z)$ holomorphic on $H = \mathbb{C} \setminus \{x \in \mathbb{R}\colon x \leq 0\}$ is a $1 \mapsto 0$ (or $e \mapsto 1$) Abel function of $F(z) = ez$ and the annulus with outer radius e (excluding) and inner radius 1 (including) is an initial region of F that is not simple. Another example is the function $z \mapsto z/b$, $b \neq 0$, holomorphic on $H = \mathbb{C}$ is a $b \mapsto 1$ Abel function of $F(z) = z + b$. The strip $\{bz\colon 0 \leq \Re(z) < 1\}$ is an initial region of F that is simple.

Theorem 1. *For each holomorphic function F and each simple initial region H_I of F with $d \in H_I$ there is at most one $d \mapsto c$ Abel function h that maps H_I biholomorphically to a region with above and below unbounded imaginary part.*

Proof. Assume there are two such Abel functions f and g holomorphic on H_I. Let $H \supset H_I$ be a domain such that $f\colon H \leftrightarrow T_f$ and $g\colon H \leftrightarrow T_g$ are still biholomorphic on H. For the rest of this proof we write h when referring to f as well as to g. T_h has above and below unbounded imaginary part. The inverse function $h^{-1}\colon T_h \leftrightarrow H$ satisfies:

$$h^{-1}(z+1) = F\left(h^{-1}(z)\right)$$

for all z such that $z, z+1 \in T_h$. So we have two biholomorphic functions $\delta_f\colon T_f \leftrightarrow T_g$, $\delta_f := g \circ f^{-1}$ and $\delta_g\colon T_g \leftrightarrow T_f$, $\delta_g := f \circ g^{-1}$ with the property

$$\begin{aligned}\delta_f(z+1) &= g\left(f^{-1}(z+1)\right) = g\left(F\left(f^{-1}(z)\right)\right)\\ &= g\left(f^{-1}(z)\right) + 1 = \delta_f(z) + 1\end{aligned}$$

for each z with $z, z{+}1 \in T_f$; and generally

$$\delta_h(z{+}1) = \delta_h(z) + 1$$

for each z with $z, z{+}1 \in T_h$.

Let $\partial_1 H$ being bijectively parametrized by $\gamma_1 : (0,1) \leftrightarrow \partial_1 H$ and parametrize $\partial_2 H$ by $\gamma_2 = F \circ \gamma_1 : (0,1) \leftrightarrow \partial_2 H$. Without restriction we can assume that h is holomorphic on $\partial_2 H$ too (otherwise we continue it by (8)). Then $h(\gamma_2(t)) = h(\gamma_1(t)) + 1$ for $t \in (0,1)$ by (8). Hence h maps H to a region such that the intersection with a horizontal straight line with imaginary part y is a line of length 1 for each real y. It is closed at the left side and open at the right side.

By the previous property the by integer k translated domains $T_h + k$ cover the whole complex plane. We define $\delta_{h,k}\colon T_h + k \to \mathbb{C}$ by $\delta_{h,k}(z +$

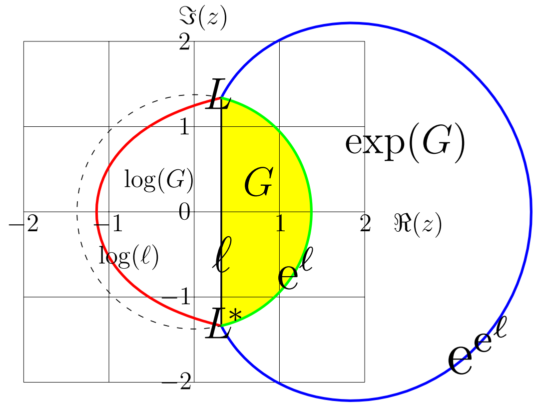

FIGURE 2. Contours $\log_b(\ell)$, ℓ and b^ℓ in the complex z-plane, for base $b=\mathrm{e}$.

$k) = \delta_h(z) + k$. By our property $\delta_h(z+1) = \delta_h(z) + 1$ the function $\delta_{h,k}$ and $\delta_{h,k+1}$ coincide on the open non-empty simply connected set $(T_h + k) \cap (T_h + k + 1)$. In conclusion δ_h can be continued to the whole complex plane. So lets consider δ_h to be an entire functions from here.

Now the sets T_f and T_g have a vicinity $\Delta_c = T_f \cap T_g$ of c in common by (7). By $\delta_f(c) = c = \delta_g(c)$ we get that $\Delta := \Delta_c \cap \delta_f(\Delta_c) \cap \delta_g(\Delta_c)$ is a non-empty open set and δ_f and δ_g are injective there. For $z \in \Delta$ we have:

$$\delta_f\big(\delta_g(z)\big) = g\Big(f^{-1}\big(f(g^{-1}(z))\big)\Big) = g\big(g^{-1}(z)\big) = z$$
$$\delta_g\big(\delta_f(z)\big) = f\Big(g^{-1}\big(g(f^{-1}(z))\big)\Big) = f\big(f^{-1}(z)\big) = z$$

So we see that the entire functions δ_f and δ_g are inverses of each other. But the only entire functions that have an entire inverse are linear functions. By the values $\delta_f(c) = c$ and $\delta_f(c+1) = c+1$ it can only be the identity. So $g(f^{-1}(z)) = z$ for $z \in \Delta$ and hence $g = f$ on H. □

This theorem can not be applied to the uniqueness of the logarithm on the annulus because it is not even holomorphic there (and is anyway not a simple initial region). But this lemma can be applied to the $b \mapsto 1$ Abel function of $F(z) = z + b$. It is unique under the condition that it maps the strip $\{bz\colon 0 \le \Re(z) < 1\}$ biholomorpically to some region with unbounded imaginary part.

Now the somewhat more interesting application is about the uniqueness of the superlogarithm. We call an Abel function of $\exp_b$ a *superlogarithm to base* b short slog_b.

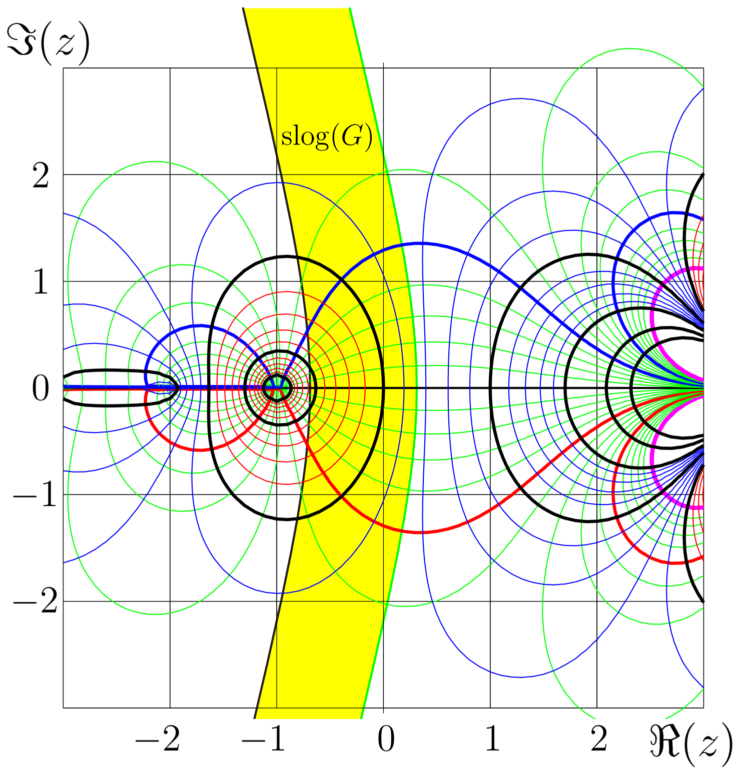

FIGURE 3. Mapping of G with function slog_e. The shaded region is the map of region G, id est, $\mathrm{slog}(G)$. The left hand side of the strip is $\mathrm{slog}(\ell)$; the right hand side is $\mathrm{slog}\big(\exp(\ell)\big)$.

To apply our lemma we first have to find an initial region of $\exp_b$. A straight forward choice G is depicted in figure 2. Here L is the fixed point of $\log_b$ in the upper half plane. The straight line between L and its complex conjugate L^* is given by $\ell(t) = \Re(L) + \mathrm{i}\Im(L)t$ for $-1<t<1$, then the region G bounded by ℓ (inclusive) and b^ℓ (exclusive) is an initial region.

The contour b^ℓ lies on the circle with radius $|L|$, shown with a dashed line. This can be easily derived. By $b^L = L$ we know that $b^{\Re(L)} = |b^L| = |L|$ and hence $b^{\ell(t)} = b^{\Re(L)+\mathrm{i}\Im(L)t} = |L| b^{\mathrm{i}\Im(L)t}$ which is an arc with radius $|L|$ around 0.

The superlogarithm slog_e computed in [2] as inverse of the superexponential sexp_e indeed satisfies (at least graphically) the biholomorphism condition on G, hence it is the only such super logarithm. The image $\mathrm{slog}_e(G)$ is depicted in figure 3. Basically this figure reproduces the part of the figure from [2]; $f = \mathrm{sexp}_e(z)$ is plotted with levels of constant modulus and constant phase in the complex z-plane. Levels $|f| = \mathrm{e}^{-2}, \mathrm{e}^{-1}, 1, \mathrm{e}, \mathrm{e}^2, \mathrm{e}^3, \mathrm{e}^4$ are shown with thick curves. Levels $\arg(f) = \pm 1, \pm 2, \pm\pi$ are shown with thick curves. Intermediate levels are shown with thin lines. The left bound of the region G corresponds to $\Re(f) = \Re(L)$; the right hand side corresponds to $|f| = |L|$.

Moreover we get a uniqueness result regarding the Abel function Ψ of exp constructed by Kneser. His construction uses the Riemann mapping theorem and hence does not directly yield an evaluation algorithm. His region $\mathfrak{H}_0$ in the upper half plane together with its conjugate ${\mathfrak{H}_0}^*$ is just our region G with its boundary, $\overline{G} = \mathfrak{H}_0 \cup {\mathfrak{H}_0}^*$. By Kneser's construction Ψ is biholomorphic on $\mathfrak{H}_0 \setminus \{c\}$ (where c in Kneser's notation is just L in our notation) and can be continued to $\mathfrak{H}_0^*$ on the lower half plane by $\Psi(z^*) = \Psi(z)^*$ (because is real analytic on the real axis). Hence it is biholomorphic on G.

Further $\mathfrak{H}_0$ is mapped to the upper halpflane such that all translations by integer k cover the whole upper halfplane. Hence $\Psi(\mathfrak{H}_0)$ must have above unbounded imaginary part. The proof of Kneser similarly succeeds for the functions $\exp_b$, $b > e^{1/e}$, (that have no real fixed point) instead of exp.

Corollary 1. *There is one and only one holomorphic superlogarithm to base $b > e^{1/e}$ that maps G (or $\log_b(G)$ or $\exp_b(G)$) biholomorphically to some region with above and below unbounded imaginary part. This is the Abel function Ψ constructed in* [2].

Corollary 2. *There is one and only one holomorphic superexponential to base $b > e^{1/e}$ that maps some region with above and below unbounded imaginary part biholomorphically to G.*

4. The Fractional Iterates of the Exponential

The combination of sexp and slog allows to define the fractional power of the exponential (this is the usual way how to derive fractional iterates via an Abel function, it is described for example in [4]).

$$\exp_b^c(z) = \mathrm{sexp}_b\big(c + \mathrm{slog}_b(z)\big)\ , \quad b > \mathrm{e}^{1/\mathrm{e}}\ ,\ z \in \mathbb{C}\ ,\ c + \mathrm{slog}_b(z) \in C \tag{9}$$

For $b = \mathrm{e}$ and several real c, we plot $\exp^c(x)$ versus x in figure 4. For $c = 1$, this is indeed the usual exponential; and, for $c = 0$, this is the identity function.

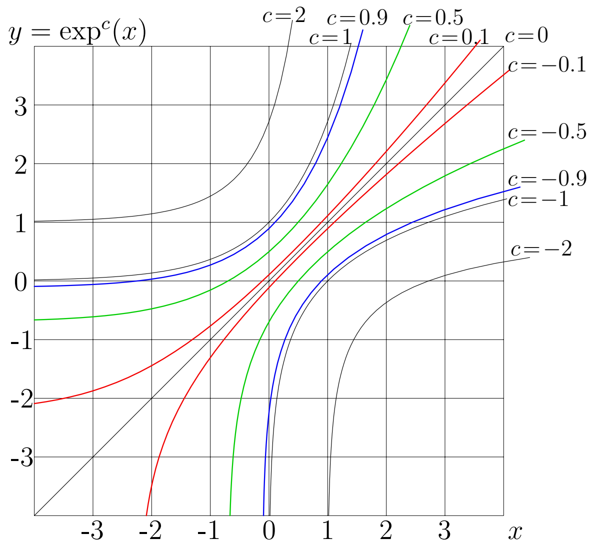

FIGURE 4. Function $y = \exp^c(x)$ calculated by equation (9) for $c = 0, \pm 0.1, \pm 0.5, \pm 0.9, \pm 1, \pm 2$ versus x.

Acknowledgment

Authors are grateful to participants of the mathematical forum [9], citizendium [10] and wikipedia [11] for the important discussion. Several additional figures related to the holomorphic tetration can be found there. D.K. is grateful to colleagues and administration of ILS and especially to Mark Sadgrove for his help and discussions.

References

[1] H. Kneser. "Reelle analytische Lösungen der Gleichung $\varphi(\varphi(x)) = \mathrm{e}^x$ und verwandter Funktionalgleichungen". Journal für die reine und angewandte Mathematik, **187** (1950), 56-67.

[2] D. Kouznetsov. Solution of $F(z{+}1) = \exp(F(z))$ in complex z-plane. Mathematics of Computation, in press.

[3] G. Szekeres. Fractional iteration of exponentially growing function. J. Austral. Math. Soc., **2** (1961/1962), 301–320.

[4] G. Szekeres. Regular iteration of real and complex functions. Acta Math. **100** (1958), 203–258

[5] P. Walker. Infinitely differentiable generalized logarithmic and exponential functions. Mathematics of computation, **196** (1991), 723-733.

[6] P. Walker. On the solutions of an Abelian functional equation. Journal of mathematical analysis and applications **155** No. 1 (1991), 93-110

[7] N. Bromer. Superexponentiation. Mathematics Magazine, **60** No. 3 (1987), 169-174

[8] M. H. Hooshmand. "Ultra power and ultra exponential functions". Integral Transforms and Special Functions **17** (8), 549-558 (2006)
[9] http://math.eretrandre.org/tetrationforum/
[10] Tetration at Citizendium, http://en.citizendium.org/wiki/Tetration
[11] http://en.wikipedia.org/wiki/Tetration