Holomorphic extension of the logistic sequence

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Moscow University Physics Bulletin, 2010, N.2, in press. The preprint compiled 27 декабря 2009 г.

The logistic problem is formulated in terms of the Superfunction and Abelfunction of the quadratic transfer function $H(z) = uz(1-z)$. The Superfunction F as holomorphic solution of equation $H(F(z)) = F(z+1)$ generalizes the logistic sequence to the complex values of the argument z. The efficient algorithm for the evaluation of function F and its inverse function, id est, the Abelfunction G are suggested; $F(G(z)) = z$. The halfiteration $h(z) = F(1/2 + G(z))$ is constructed; in wide range of values z, the relation $h(h(z)) = H(z)$ holds. For the special case $u=4$, the Superfunction F and the Abelfunction G are expressed in terms of elementary functions.

Keywords: Logistic operator, Logistic sequence, Holomorphic extension, SuperFunction
PACS: 02.30.Ks, 02.30.Zz, 02.30.Gp, 02.30.Sa,

1. INTRODUCTION

The logistic sequence F can be defined with the recurrent formula

$$H(F(z)) = F(z+1) \qquad (1)$$

and the initial condition $F(0)$ for the quadratic transfer function

$$H(z) = u\, z\, (1-z)\ , \qquad (2)$$

where u is a positive parameter. It is assumed that $0<F(0)<1$. In the publications about the logistic equation, the parameter z is assumed to be integer [1, 2, 3, 4, 5]; given $F(0)$, the equation (1) determines $F(1)$, $F(2)$, $F(3)$,...

In this paper, using the formalism of superfunctions [6, 7, 8, 9, 10, 11, 12], the holomorphic extension of function F is constructed. For this extension, the inverse function $G=F^{-1}$ is constructed; then, the non-integer iterations H^c of the transfer function can be evaluated:

$$H^c(z) = F(c + G(z))\ . \qquad (3)$$

At $c=1/2$, this allows to evaluate the half-iteration of the logistic transfer function, id est, function $h=\sqrt{H}=H^{1/2}$ such that

$$hhz = h(h(z)) = H(z)\ , \qquad (4)$$

at least for some range of values of z. Such halfiterations for the transfer functions exp and Factorial are considered in papers [6, 10]. For the quadratic transfer function (2), the graphic of the half-iteration is plotted in figure 1 with thick lines for $u=3$, left; for $u=4$, central; and for $u=5$, right. Other curves represent the 0.2th iteration, i.e., $H^{1/5}$, the 0.8th iteration, i.e., $H^{4/5}$, and the 1st iteration, i.e., $H^1 = H$, for the same values of u. The zeroth iteration would correspond to H^0, which is identity function, is not plotted.

The following sections describe the evaluation of functions F and G and discuss the range of validity of relation (4).

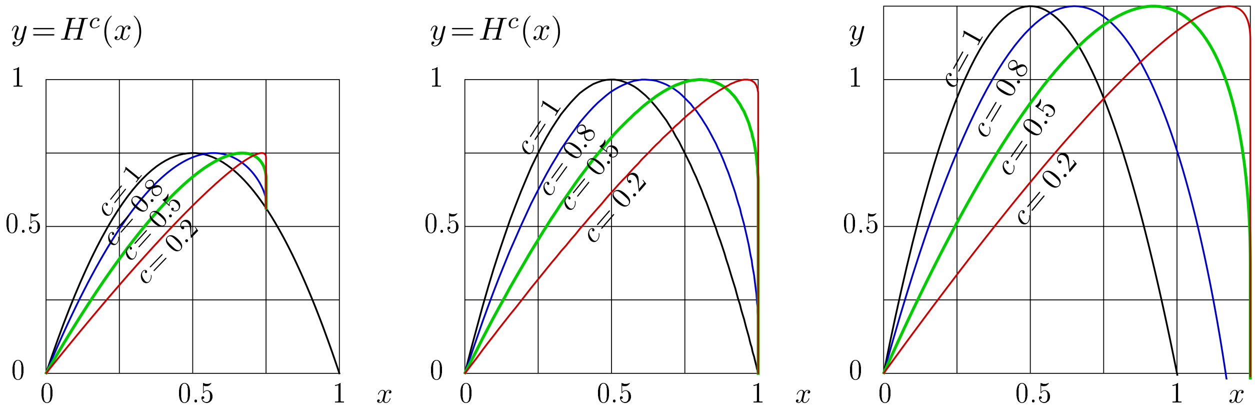

Рис. 1: Various iterations $H^c(x)$ versus real x for $u=3$, left; for $u=4$, center; for $u=5$, right; curves for $c=1$, $c=0.8$, $c=0.5$, $c=0.2$ are drawn.

2. SUPERFUNCTION

One should work in the complex plane, in order to make a holomorphic extension. The quadratic function H by(2) is presented at the upper graphics in figure 2. Function $f = H(z)$ in shown in the complex z-plane at $u=3$, left; at $u=4$, center and at $u=5$, right. Levels $p = \Re(f) = \mathrm{const}$ and levels $q=\Im(f)=\mathrm{const}$ are shown; thick lines correspond to the integer values.

The second row of pictures in Figure 2 shows, in the same notations, the halfiteration h by (3) at $c=1/2$. For the evaluation of the halfiteration, the Superfunction F and the Abelgunction G are used. These functions are plotted in the last two rows of figure 2 for the same values of parameter u.

In the construction of Superfunction F, the crucial is question about the fixed points of the Transferfunction H, which are solutions of equation

$$H(z) = z \tag{5}$$

For the quadratic H by (2), the equation (5) has exactly two solutions, $z=0$ and $z=1–1/u$. The first one has does not depend on u. Below, it is used to develop the Superfunction.

For the real transfer function (2) with the real Fixedpoint, the formalism [8, 10] indicates the following asymptoric expansion for the Superfunction F:

$$F(z) = \sum_{n=1}^{N-1} c_n u^{nz} + \mathcal{O}(u^{Nz}) \tag{6}$$

The substitution of (6) into (1) gives the chain of equations coefficiens c. One can set $c_1=1$; variation of this coefficient corresponds to translations of the solution along the real axis. Then

$$\begin{aligned} c_2 &= \frac{1}{1–u} \\ c_3 &= \frac{2}{(1–u)(1–u^2)} \\ c_4 &= \frac{5+u}{(1–u)(1–u^2)(1–u^3)} \end{aligned} \tag{7}$$

The expression (6) gives a way to evaluate the the superfunction F at large negative value of the real part of the argument. For other values the recurrences of the recurrent expression

$$F(z) = H(F(z-1)) \tag{8}$$

can be used, giving the fast and precise implementation. The map of function F in the complex plane is shown in the third row in figure 2 for $u=3$, $u=4$, $u=5$. The superfunction F is entire periodic function. For real u, the period

$$T = 2\pi \mathrm{i}/\ln(u) \tag{9}$$

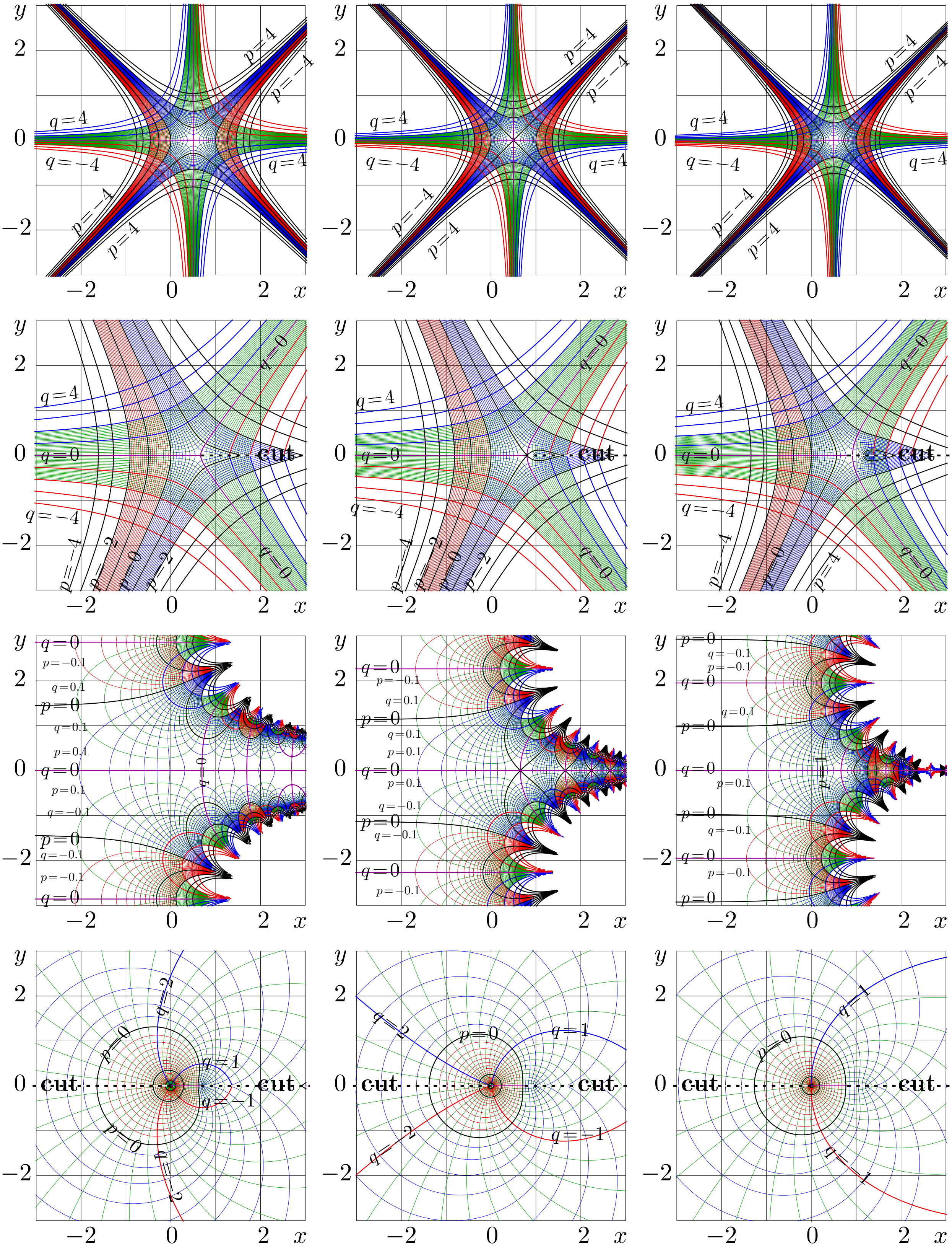

Рис. 2: Maps of the transferfunction H (upper row), its halfiteration (second row), Superfunction F (third row) and the Abelfunction G (bottom) in the complex $z=x+\mathrm{i}y$ plane for $u=3$, left; $u=4$, center; and $u=5$, right.

is pure imaginary. In vicinity of the half-line $\Im(z) = \Im(T/2)$, $\Re(z) \to +\infty$, superfunction $F(z)$ has huge values and huge derivatives so, the plotter could not draw the levels and these regions look "empty".

Along the real axis, the superfunction F is smooth and bounded; it approaches zero at $-\infty$ and oscillates at positive values of the argument; if $u \le 4$, the function is bounded between 0 and unity along all the real axis.

The periodicity with imaginary period is typical for the real regular superfunctions constructed at the real fixed points of the transfer function [8, 10]. The exponential, as superfunction of a linear transfer function, is a particular case of such a rule; the holomorphic extension of the exponential behaves in the simiular way.

3. ABELFUNCTION

For construction of the halfitertion declared in the Introduction, the inverse of the Superfunction F is required. Such inverse function, id est, $G = F^{-1}$, can be called Abelfunction, because it satisfies the Abel equation [8, 9, 10]

$$G(H(z)) = G(z) + 1 \ . \tag{10}$$

Its asymptotic expansion can be obtained by the straightforward inversion of the series (6):

$$G(z) = \log_u \left(\sum_{n=1}^{N-1} s_n z^n + \mathcal{O}\big(z^N\big) \right) \tag{11}$$

The chain of equations for the coefficients s can be found also substituting the expansion (11) into the Abel equation (10). In particular,

$$\begin{aligned} s_1 &= 1 \\ s_2 &= \frac{1}{u-1} \\ s_3 &= \frac{2\,u}{(u-1)(u^2-1)} \\ s_4 &= \frac{(u^2+5)\,u}{(u-1)(u^2-1)(u^3-1)} \end{aligned} \tag{12}$$

In order to extend such an approximation to the large values of the argument, the recurrent formula can be used

$$G(z) = G(H^{-1}(z)) + 1 \ , \tag{13}$$

where

$$H^{-1}(z) = 1/2 - \sqrt{1/4 - z/u} \ . \tag{14}$$

The representation through (11),(13) provides the fast and precise evaluation. Such an algorithm is used to plot the last row in figure 2.

While F, G are already chosen and implemnted, then, for any complex number c, the c-th iteration H^c of the transfer function can be defined with (3). Such iterations satisfy relation

$$H^c(H^d(z)) = H^{c+d}(z) \ ; \tag{15}$$

at least for some range of values of z. In particular, at integer c, the iteration means just sequential application of the function c times,

$$H^c(z) = \underbrace{H\Big(H\big(...H(z)..\big)\Big)}_{c} \tag{16}$$

At $c = 1/2$, the halfiteration $H^{1/2}$ is plotted in the second raw of figure 1. This function has cut, that begins between 1/2 and unity and goes along the real axis to infinity. In particular, this cut limits the range of validity of equation (4).

4. RANGE OF VALIDITY OF $h\,h\!=\!H$

In general, the inverse function of an entire function has branchpoints and cutlines; the only exception is a fractional linear function. Therefore, the relation $G\big(H(z)\big) = z$ should have some limited range of validity; it limits the range of equation (15).

Behavior of function $f = H^{1/2}\big(H^{1/2}(z)\big)$ is shown in figure 3 for $u = 3, 4, 5$. In the left hand side of the complex plane, the pictures

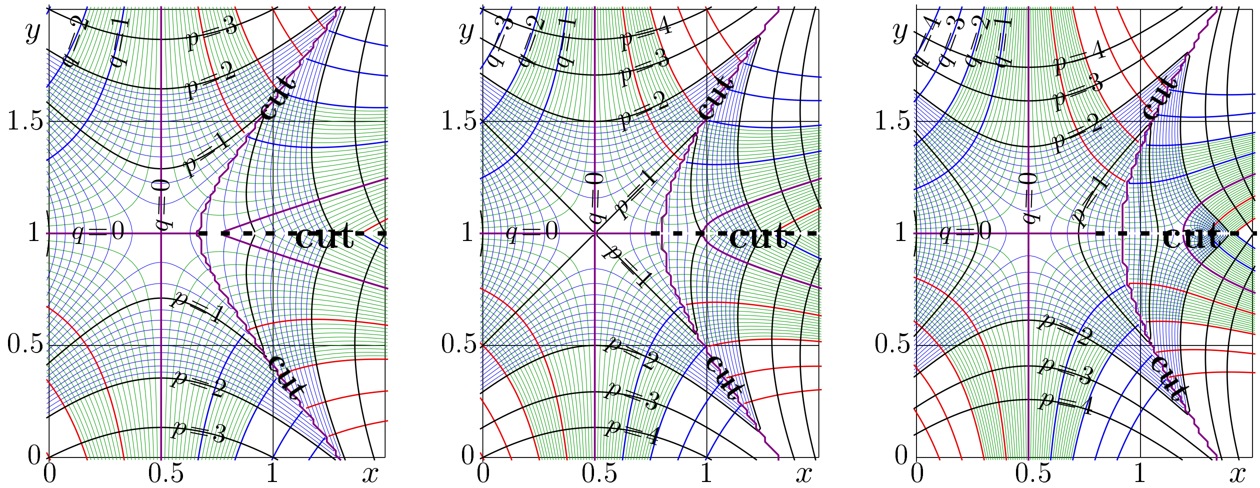

Рис. 3: Function $f = H^{0.5}(H^{0.5}(z))$ in the comlex $z\!=\!x\!+\!\mathrm{i}y$ plane for $u\!=\!3$, left; $u\!=\!4$, center; $u\!=\!5$, right

are just zoom-in of the central parts of the top row in figure 2. The scratched line shows the margin of the range of validity of the relation $H(z) = H^{1/2}\big(H^{1/2}(z)\big)$. The cuts along the real axis are marked with dashed lines.

Relation (4) holds for the most of the complex plane. However, it cannot hold for the whole complex plane, because the information, at which oscillation does the function F take some fixed value, is lost at the first step of evaluation by (3). Similar restrictions of the range of validity of equation (15) should take place for other transfer functions too; in particular, for functions $\sqrt{\exp}$ and $\sqrt{\mathrm{Factorial}}$ analyzed in the compelx plane [6, 7, 8, 10, 11].

The monotonic behavior of function $H = \exp$ allows the relation (15) to hold along the real axis. The monotonic behavior of function $H\!=\!\mathrm{Factorial}$ allows the relation (15) to hold for $z\!>\!1$. In the similar way, in the case of the logistic operator H by (2), the relation (15) holds at least for $\Re(z)\!\le\!1/2$.

It is common that the Abelfunction, developed at some fixed point, is irregular at another fixed point. Then, the non-integer iteration of the transfer function may have the same irregularities, namely, the branchpoints. The corresponding cutlines limit the range of applicability of the equation (15) and, in particular, equation (4). However, the Abeflunction (and then, the non-integer iteration of the transferfunction) can be irregular also at both fixed points, as the real $\sqrt{\exp}$ does [10, 11].

New modifications of the Abelfunctions (and corresponding non-integer iterations of the transfer function) can be generated moving the cutlines, as it is done for the Abelexponential (sometimes called also “superlogarithm”, although it is not a Superfunction of logarithm) and $\exp^c$ in [12]. Usually, such modification have more complicated structure.

5. CASES $u\!=\!3.5699$ AND $u\!=\!3.8284$

In this sections, the two special cases are considered, $u\!=\!3.5699$ and $u\!=\!3.8284$. While z is interpreted as a discrete variable, these values to be margins between regular and irregular behavior [14, 15] in the Pomeau-Manneville scenario [5, 17, 18]. In figure 4 the maps of function F are shown for these cases. Figure 5 shows the behavior of these functions along the real axis. In general, these functions behave in a way, one could expect

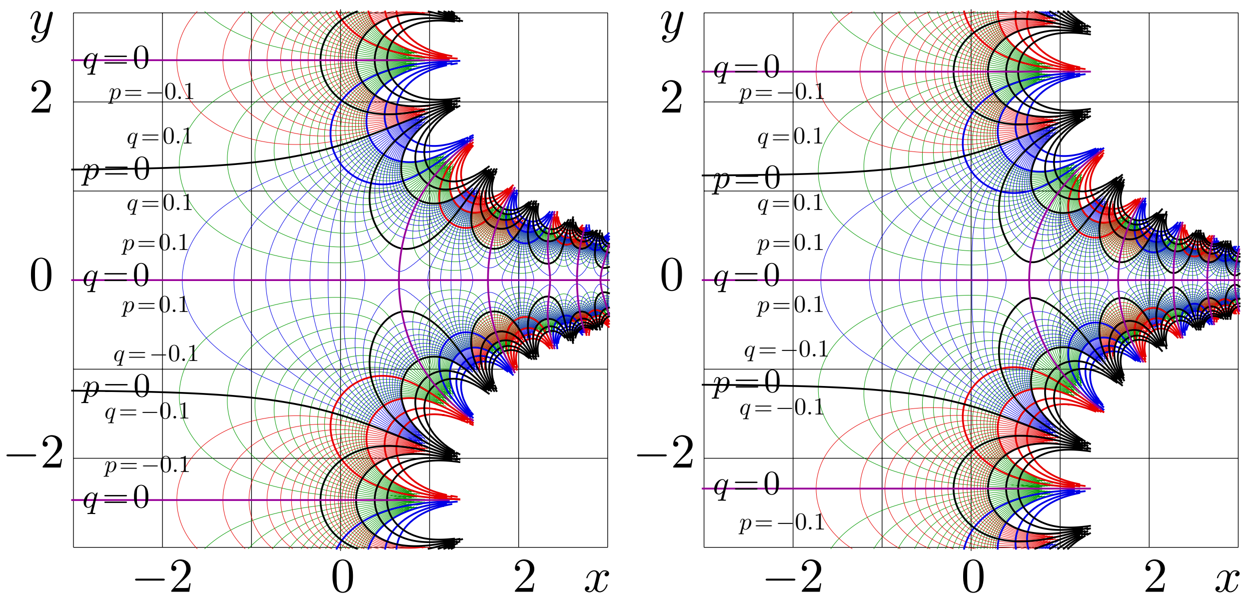

Рис. 4: Maps of superfunction F for $u\!=\!3.5699$ and $u\!=\!3.8284$ in the same notations as in figure 2.

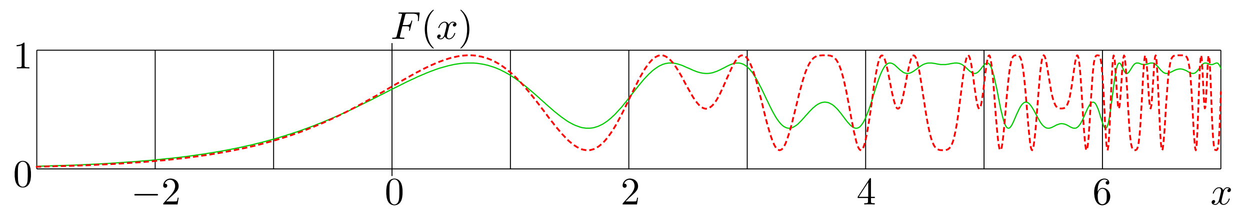

Рис. 5: Graphics of SuperFunction F versus real argument for $u=3.5699$ (solid) and $u=3.8284$ (dashed).

from the consideration of the discrete values of the argument [14, 15]. In particular, at $u = 3.5699$, visually, one can trace some periiodic trend with period 2. No qualitative change of the structure is seen at the maps in the complex plane (fig. (4)).

6. SPECIAL CASE $u\!=\!4$

In the special case $u\!=\!4$, the Superfunction and the Abelfunction can be expressed through the elementary functions. Such an expression can be found from the table of Superfunctions. The raw 8 of Table 1 from [10] corresponds to the transfer function

$$H_0(z) = 2z^2 - 1 \qquad (17)$$

with Superfunction

$$F_0(z) = \cos(2^z) \qquad (18)$$

and Abelfunction

$$G_0(z) = \log_2(\arccos(z))\ . \qquad (19)$$

Then the transform from the last row of the same table at the linear functions

$$P(z) = (1\!-\!z)/2 \qquad (20)$$

and

$$Q(z) = 1 - 2z \qquad (21)$$

gives the new transfer function

$$H_1(z) = P\Big(H\big(Q(z)\big)\Big) = 4z(1\!-\!z) \qquad (22)$$

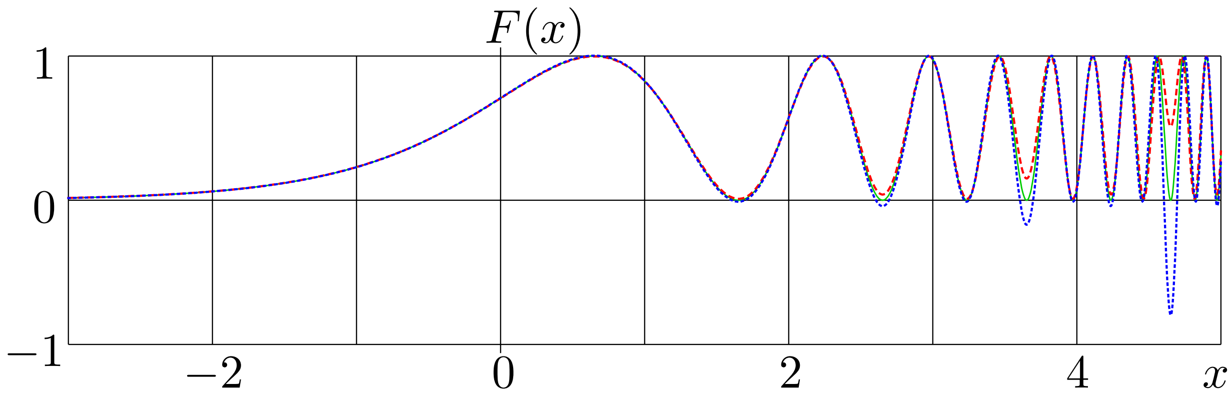

Рис. 6: $F(x)$ as function of real x for $u=3.99$, dashed curve; $u=4$, solid curve; and $u=4.01$, dotted curve.

that coincides with the transfer function H by (2) at $u=4$, and the Superfunction

$$F_1(z) = P(F_0(z)) = \frac{1}{2}\big(1-\cos(2^z)\big) \qquad (23)$$

and the Abelfunction

$$G_1(z) = G_0\big(Q(z)\big) = \log_2\big(\arccos(1-2z)\big) \qquad (24)$$

Functions F_1 and G_1 can be related with F and G plotted in the central column of figure 2 with simple translation:

$$F(z) = F_1(z+1) \ \ , \ \ G(z)=G_1(z)-1 \ .$$

Superfunction F versus real argument is shown in figure 6 for $u=3.99$, dashed curve; $u=4$, solid curve; $u=4.01$, dotted curve. As one could expect, the solid curve looks pretty regular.

At $u=4$, the comparisons of the "exact" expressions of F and G through the elementary functions (23) and (24) to the numerical implementations through the asymptotic expressions (6), (11) and the recurrent formulas (8), (13) confirm the high precision of the numerical implementations. Of order of 14 correct digits can be achieved with the complex⟨double⟩ variables.

7. FIXED POINT $1-1/u$

The fixed point $z=1-1/u$ also can be used as an asymptotic of the superfunction of the transfer function (2). Such superfunction can be expresed asymptotically

$$\begin{aligned} F(z) \ = \ & \frac{u-1}{u} \\ & + \sum_{n=1}^{N-1} d_n\Big((u-2)^z\cos(\pi z+\phi)\Big)^n \\ & + \mathcal{O}\Big((u-2)^z\cos(\pi z+\phi)\Big)^N \end{aligned} \qquad (25)$$

where phase ϕ and coefficients d are constants. The subsitution into equation (1) gives the chain of equations for the coefficients. In particular, we may set $d_1=1$; then

$$\begin{aligned} d_2 &= \frac{-u}{(u-1)(u-2)} \\ d_3 &= \frac{-u^2}{(u-1)(u-2)(u-3)} \\ d_4 &= \frac{-(u-7)^3\, u^3}{(u-2)(u-3)(u^3-8u^2+22u-21)} \end{aligned} \qquad (26)$$

Such languages as Mathematica allows to calculate exactly a ten of such coefficients, giving an approximation valid while the effective parameter of expansion, id est, $(u-2)^z\cos(\pi z+\phi)$, is small. The truncted sum gives several correct decimal digits at

$$\pi|\Im(z)| + \ln(u-2)\,\Re(z) < -4 \qquad (27)$$

The extension with (8) gives the fast and precise algorithm; it is used to plot figure 7. The figure corresponds to $\phi=0$. At the top, the map of superfunction F is shown for $u=4$;

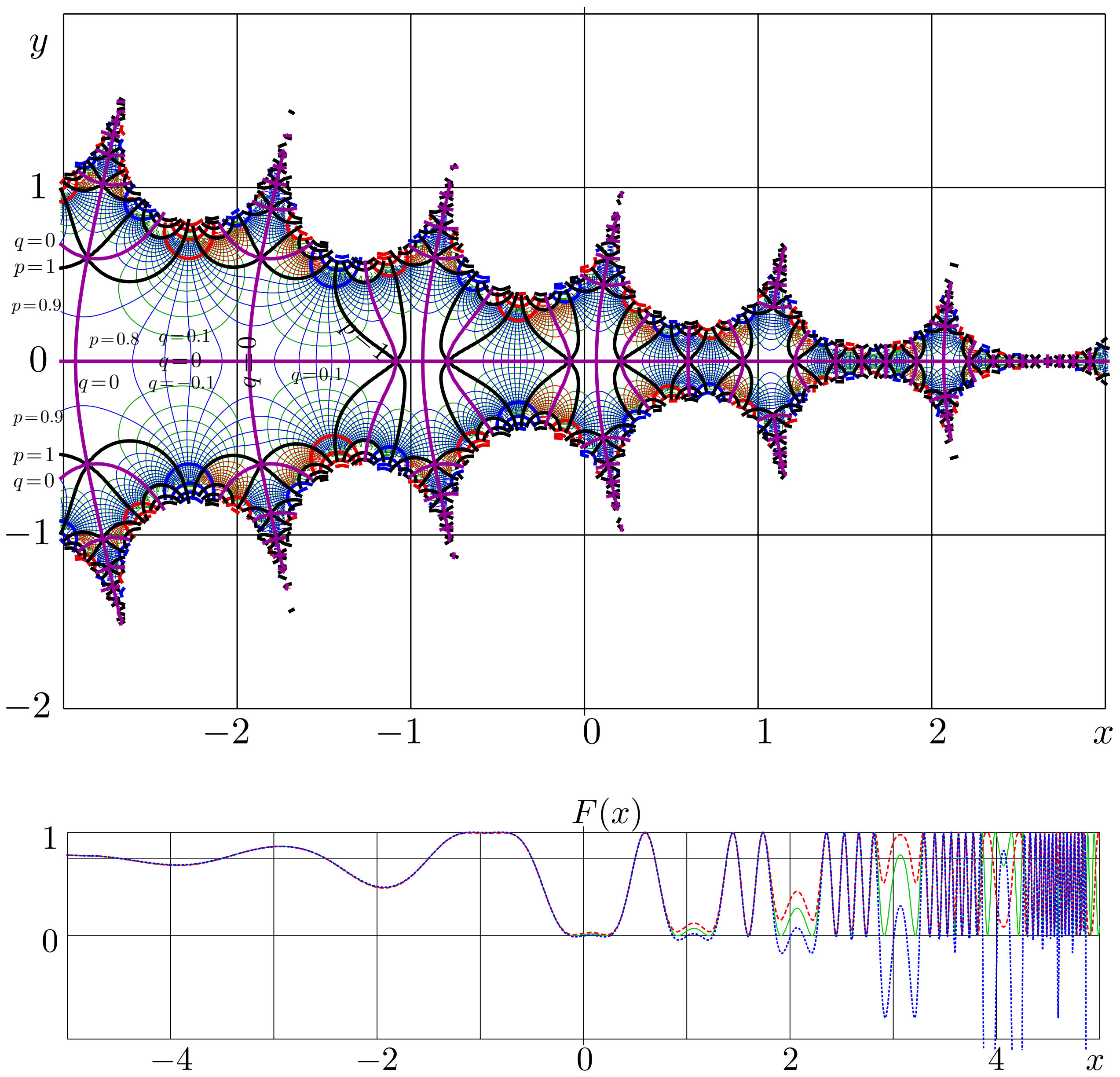

Рис. 7: Map of Superfunction F by (25),(8) for $u\!=\!4$, top, and $F(x)$ versus real x for $u\!=\!3.99, 4, 4.01$, bottom.

At the bottom, the function $F(x)$ is ploted versuw real x for $u = 3.99, 4, 4.01$, id est, the same values as in figure 6.

The superfunction constructed in such a way is asymptotically-periodic; quasi-period

$$T = \frac{2\,\pi\,\mathrm{i}}{\ln(u\!-\!2) - \pi\,\mathrm{i}} \tag{28}$$

in the upper halfplane and T^* in the lower half-plane. In particular, at $u = 4$, equation (28) gives $T \!\approx\! -1.907159353\!+\!0.4207872484\,\mathrm{i}$; the quasiperiodicity is seen in at the top part of 7. This quasi-periodicity is determined by the leading term in the expansion (25). The quasi-periodic behavior is also typical for the superfunctions [6, 7, 8, 10].

Various superfunctions of the logistic operator can be constructed, assuming that they approach a fixed point while the real part of the argument goes to $-\infty$. Some of them can be expressed through the elementary functions, but I do not yet count with such a representation for any non-trivial superfunction that approaches the fixed point $1-1/u$.

8. BOUNDARIES OF THE TIME DERIVATIVE

Variable z in (6), (8) may have sense of time. Then the Superfunction F can be interpreted as some smooth, infinitely differentiable physical process. Being measured at integer values of time, this process generates the logistic sequence. At least for $u=4$, the representation (23) gives the time derivative of such process:

$$\begin{aligned} F(z) &= F_1(z+1) = \tfrac{1}{2}\Big(1-\cos\left(2^{z+1}\right)\Big) \\ F'(z) &= \qquad \ln(2)\, 2^z \, \sin\left(2^{z+1}\right) \end{aligned} \qquad (29)$$

The upper bound for the modulus of the derivative grows exponentially:

$$|F'(z)| \le \ln(2)\, 2^z \ , \qquad (30)$$

at least for real values of time z. The same bound seems to be valid also for $u<4$. However, for $u>4$, the double-exponential growth is allowed;

$$|F(z)| < \exp(2^z) \ \ , \qquad (31)$$

according to the row 5 of the Table of Superfunctions, [10], Table 1; in this case, the quadratic term in the expansion of the logistic transferfunction dominates. Then the derivative can be estimated as

$$|F'(z)| < \ln(2)\, 2^z \, \exp(2^z) \qquad (32)$$

In such a way, the holomorphic extension leads to the estimate of rate of growth of the logistic sequences.

9. MORE SUPERFUNCTIONS

The holomorphic extension F of the logistic sequence is not unique. It can be developed at any of fixed points of the logistic Transferfunction $H(z)$ by (2).

Also, the new superfunctions $\tilde F$ can be expressed through some already constructed superfunction with the periodic modification of the argument:

$$\tilde F(z) = F(z+\varepsilon(z)) \qquad (33)$$

where ε is some 1-periodic function holomorphic at least in some vicinity of the real axis. Such Superfunctions may grow up in the direction of the imaginary axis and also may have additional singularities in the complex plane. The superfunction F by (6),(8) seems to be the only non-trivial periodic superfunction. The observation of the of various extensions of the logistic sequences can be summarized as follows: :

Hypothesis 0. For any $u>2$, any holomorphic extension F of the logistic sequence, id est, solution of $F(z+1) = u\,F(z)\big(1-F(z)\big)$, that cannot be expressed with (6),(8), has at least an exponential growth in the direction of the imaginary axis.

In such a way, the hypothesis 0 declares the uniqieness of the periodic holomorphic extension of the logistic sequence. The proof may be matter for the future research.

10. CONCLUSION

There is nothing especial in the logistic trasnfer function; the superfunction can be constructed following the general procedures [7, 8, 10, 12]. The asymptotic expansion (6) allow the fast and precise evaluation of the superfunction, id est, the holomorphic extension F of the logistic sequence [1, 2, 3, 4], and its inverse function G. Such functions are plotted in figure 2 for various values of parameter u. The logistic sequence can be interpreted as a smooth, infinitely differentiable process $F(z)$, measured at the integer values of time z.

With given Superfunction F and the Abel-function $G = F^{-1}$, the non integer iteration H^c of the transfer function H can be constructed in the standard way through equation (3). At $c = 1/2$, this gives the halfiteration; such a "square root" of the logistic operator is plotted in the second raw in figure 6 for $u = 3, 4, 5$.

The growth of the holomorphic extension in the direction of the imaginary axis allows to formulate the criterion of the uniqueness, although the proof of hypothesis 0 and the application to the realistic physical systems can be matter for the future research.

In many cases, the requirements about the behavior of the extension in the complex plane are essential for the efficient reconstruction and the uniqueness; the extension of the logistic sequence is not an exception.

Acknowledgenent

The topic of this research was formulated by P.V.Elutin [19] and it is dedicated to the teachers of Quantum Mechanics [19, 20]. I am grateful to A.V.Borisov, H.Trappmann and the participants of his forum [21] for the important discusisons.

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[1] *Tufillaro N., Tyler A., Jeremiah R.* An experimental approach to nonlinear dynamics and chaos. 1992. NY.

[2] *Strogatz S.* Nonlinear Dynamics and Chaos. NY, 2000.

[3] *Sprott J. C.* Chaos and Time-Series Analysis. Oxford 2003.

[4] *Bjerklöv K.* Strange Nonlinear Attractors in the quasi-periodic family. Communications in Mathematical Physics. **286**. 2009. P.137–161 `http://www.springerlink.com/content/d73u67714013j826/`

[5] *Papanyan V. O., Grigoryan Yu. I.*, Scenario of a transition of oscillatory striations to chaos in a steady-state discharge in rare gases and their mixtures used in lasers, QUANTUM ELECTRON, 1994, **24** , 287–291.`http://www.quantum-electron.ru/pdfrus/fullt/1994/4/75.pdf`

[6] *Kneser H.* Reelle analytische Lösungen der Gleichung $\varphi(\varphi(x)) = e^x$ und verwandter Funktionalgleichungen. Journal für die reine und angewandte Mathematik. **187**. (1950). P.56–67. Reprint:`http://www.ils.uec.ac.jp/~dima/Relle.pdf`

[7] *Kouznetsov D.* Solution of the equation $F(z + 1) = \exp(F(z))$ in the complex z-plane. Mathematics of computation. **78**. (2009). P.1647–1670. `http://www.ams.org/mcom/2009-78-267/S0025-5718-09-02188-7/home.html` Preprint: `http://www.ils.uec.ac.jp/~dima/PAPERS/2009analuxpRepri.pdf`

[8] *Kouznetsov D., Trappmann H.* Portrait of the four regular super-exponentials to base sqrt(2). Mathematics of Computation, in press, 2010. Preprint: `http://www.ils.uec.ac.jp/~dima/PAPERS/2009sqrt2.pdf`

[9] *Abel N. H.* Correlative of the functional equation. Crelle's Journal. 1827. N 2. P.389.

[10] *Kouznetsov D., Trappmann H.* Superfunctions and sqrt of factorial. Moscow University Physics Bulletin. 2010. In press; Preprint: `http://www.ils.uec.ac.jp/~dima/PAPERS/2009supefae.pdf`

[11] *Kouznetsov D.* Superexponential as special function. Vladikavkaz Mathematical Journal. 2010. In press. Preprint:`http://www.ils.uec.ac.jp/~dima/PAPERS/2009vladie.pdf`

[12] *Kouznetsov D., Trappmann H.* Cut lines of the inverse of the Cauchi-tetrational. Preprint ILS UEC 2009, `http://www.ils.uec.ac.jp/~dima/`

PAPERS/2009fractae.pdf

[13] *Knoebel R. A.* Exponentials reiterated. American Mathematics Monthly. 1981. **88**. P.235-252.

[14] *Chavas J.-P. and Holt M. T.* On Nonlinear Dynamics: The Case of the Pork Cycle. American Journal of Agricultural Economics. 1991. **73**, No.3, P.819-828.

[15] *Yajnik K. S.* Coexistence of regularity and irregularity in a nonlinear dynamical system. http://www.ias.ac.in/currsci/sep25/articles20.htm

[16] *Gac J. M., Zebrowski J. J.* Giant suppression of the activation rate in dynamical systems exhibiting chaotic transitions. Acta Physica Polonica B, 2008 **39**, P.1019-1033 http://th-www.if.uj.edu.pl/acta/vol39/pdf/v39p1019.pdf

[17] *Pomeau Y., Manneville P.* Intermittent transition to turbulence in dissipative dynamical systems. Commun. Math. Phys. 1980, **74**, P.189-197.

[18] *Myrzakulov P., Kozybakov M. Zh., Sabdenov K. O.* Modeling of the acoustic instability in the camera of the jet motor. Bulletin of the Tomsk Polytechnical Institute, 2006, **309**, P. 109-113.(in Russian)http://www.duskyrobin.com/tpu/2006-06-00028.pdf

[19] *Elutin P. V.* The holomorphic extension of the logistic mapping is ambitious program... The continual generalization of the logistic mapping will worth your time and efforts, if you can reveal the model that gives the law of the evolution $F(t)$ - or determine the characteristics $F(t)$, boundaries of the time derivatives). - Private communication, 2009,Oct 9,5:32pm.

[20] *Saraeva I. M., Romanovsky Yu. M., Borisov A. V. (composers).* Vladimir Dmitrievich Krivchenkov. Physical Department of MSU. 2008. (in Russian). online version, P.81; http://www.phys.msu.ru/rus/about/structure/admin/OTDEL-IZDAT/HISTORY/

[21] *Trappmann H.* Tetration forum, 2009, http://math.eretrandre.org/tetrationforum/index.php