Ackermann functions of complex argument

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Abstract

Existence of analytic extension of the fourth Ackermann function $A(4, z)$ to the complex z plane is supposed. This extension is assumed to remain finite at the imaginary axis. On the base of this assumption, the algorithm is suggested for evaluation of this function. The numerical implementation with double-precision arithmetics leads to residual at the level of rounding errors. Application of the algorithm to more general cases of the Abel equation is discussed.

Key words: Ackermann functions, tetration, ultra-exponential, super-exponential, generalized exponential, Abel equation
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1 Introduction

For integer non-negative values of arguments, the Ackermann function A can be defined [1,2] as follows:

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$$A(m,n) = \begin{cases} n+1 & \text{, if } \ m\!=\!0 \\ A(m\!-\!1,\ 1\ ) & \text{, if } m\!>\!0 \text{ and } n\!=\!0 \\ A\Big(m\!-\!1, A(m,n\!-\!1)\Big) & \text{, if } m\!>\!0 \text{ and } n\!>\!0 \end{cases} \quad . \tag{1}$$

At relatively moderated values of m, functions $A(m,z)$ grow quickly at $z \to +\infty$, at least while z is integer. This makes them useful for the representation of huge numbers.

Powers of 2 are actually used in the most of computers for the "floating-point" variables. A power of 2 is related with the Ackermann functions through the identity

$$2^z = \exp_2(z) = A(3, z-3) + 3 \quad . \tag{2}$$

In this sense, the third of the Ackermann functions is already applied in mathematics of computation.

It is recognized, that the range of numbers that are distinguishable from infinity at the numerical representation, could be greatly extended [3,4], using the representation through a rapidly growing function. The ultra-exponential

$$\exp_2^z(1) = A(4, z-3) + 3 \ . \tag{3}$$

is considered as a candidate for such a function.

The 4-th Ackermann function can be written as follows:

$$A(4, z) = F(z+3) - 3 \ , \tag{4}$$

where F is solution the Abel equation

$$\varphi(z+1) = H\Big(\varphi(z)\Big) \ , \tag{5}$$

for the special case $H = \exp_2$. Some values of $A(4,z)$ are printed in the last three liners of the first column of the Table 1. Symbol "inf" in this table denotes $A(4,2) = 2^{65536} - 3$, that cannot be stored as a floating-point number in a double precision variable. Other values in Table 1 are calculated using the algorithm, described in the following sections.

Usually, the rapidly-growing functions are defined for integers [1,5]; even for the real z, the analytic extension of solution of eq. (5) is not trivial [4,3,6,7].

For the case $H = \exp_a$, eq. (5) can be written as follows:

Table 1
Approximations of Ackermann function $A(4, x+\mathrm{i}y)$ for integer x and y.

x	$A(4,x)$	$A(4,x+\mathrm{i})$	$A(4,x+2\mathrm{i})$
-5	nan	$-2.93341314027949+1.77525045768931\,\mathrm{i}$	$-2.40837342500080+1.60602120134867\,\mathrm{i}$
-4	-3	$-2.65047742724573+0.98718671450859\,\mathrm{i}$	$-2.33425457910192+1.35191185960330\,\mathrm{i}$
-3	-2	$-2.01269088663886+0.80538852548727\,\mathrm{i}$	$-2.06062289136790+1.27835701837457\,\mathrm{i}$
-2	-1	$-1.31849305466899+1.05013171018512\,\mathrm{i}$	$-1.78715845191886+1.48545918347369\,\mathrm{i}$
-1	1	$-0.60526187964136+2.13403567589826\,\mathrm{i}$	$-1.80597055463682+1.98673779218581\,\mathrm{i}$
0	13	$-2.51898925867497+5.23677174007073\,\mathrm{i}$	$-2.55961021720427+2.24512422678756\,\mathrm{i}$
1	65533	$-4.23263111512603-0.65471984727531\,\mathrm{i}$	$-2.98019598556513+1.35682637967319\,\mathrm{i}$
2	inf	$-2.61753248714842-0.18655683639318\,\mathrm{i}$	$-2.40245344004963+0.81900710696401\,\mathrm{i}$

$$F(z+1) = \exp_a\Big(F(z)\Big) \ , \tag{6}$$

or

$$F(z-1) = \log_a\Big(F(z)\Big) \ . \tag{7}$$

Usually, the condition

$$F(0) = 1 \tag{8}$$

is assumed. Then, the argument of function F can be interpreted as number of exponentiations in the representaiton

$$F(z) \;=\; \exp_a^z(1) \;=\; \underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a^1}}}}}}_{z \text{ times}} \ , \tag{9}$$

This relation can be considered as definition of tertration and therefore, the Ackermann function $A(4, z)$ for positive integer z.

Due to relation (3), the analytic extension of Ackermann function implies also the analytic extension of the ultra-exponential (at least, for $a = 2$) and vice versa. Various names and notations are used for this operation: "generalized exponential function"by [3], "ultraexponentiation" by [4], "Superexponentiation" by [8], "tetration" by [9]. Such variety of names indicates, that the analysis of properties of this function does not yet bring it to the level of Special functions [10,11].

For the representation of huge numbers, the analytic properties of the the rapidly-grown function F are important; at least to implement the arithmetic operations with huge numbers, stored in such a form. These operations should be implemented (approximated) without to convert the numbers to the conventional (floating-point) representation. Such implementation should be based on analytic properties of the function F rather than on the conversion to the floating-point form.

The analytic properties of a function, used for the representation of numbers, become essential, if the huge numbers appear in the integrand of a contour integral. Then, the asymptotical analysis (for example, integration along the path of stationary phase) can simplify the numerical computation. The function, used for representation of huge numbers, should be analytic.

The analytic extension of the Ackermann functions and ultra-exponential in eq.(3) to complex z plane is important for the application. In order to deal with function F, at the first step, the algorithm for the efficient evaluation (i.e., representation of its value in the conventional floating-point form) is necessary. Such representation is mater of this paper.

The existence of unique analytic extension of solution F of eq.(7) is not obvious. In particular, [4] presents the proof, that analytic extension of the ultra-exponential F to the whole complex plane is not possible; the piecewice function with jumps of the second derivative is suggested instead. Such a proof uses the assumption, that at the real axis, the first derivative of F is non-decreasing function. For a function with rapid growth, there is nothing wrong in having minimum at some morerate value of the argument. For example, the minimum of derivative of Γ function [10,11] does not prohibit to consider $\Gamma(z{+}1)$ as analytic extension of $z!$ for non-integer values of z and to use it in mathematical analysis [12]. One may expect, that a the real axis, the derivative of analytic extension of tetration has minimum between -1 and 0.

The real-analytic extension of tetration is considered by [3] in terms of functional sequences for the case $a = \mathrm{e}$. However, the representation suggested does not offer robust algorithm for the evaluation. Such an algorithm is important for the use. For applications, the robust algorithm for evaluations is even more important than the mathematical proof of existence and uniqueness of the analytic extension.

In this work, I postulate the existence of the analytic extension, and I suggest the algorithm for the evaluation.

2 Assumption

Assume that there exist solution $F(z)$ of equations (7),(8), analytic in the whole z-plane except $z \le -2$, that satisfies condition

$$\lim_{y\to+\infty} F(x+\mathrm{i}y) = L \quad \text{for all fixed real } x \ , \tag{10}$$

where L is eigenvalue of logarithm: $\Re(L)>0$, $\Im(L)>0$ and

$$L = \log_a(L) \ . \tag{11}$$

In the following sections, on the base of the assumption above, the integral equation for values of F at the imaginary axis can be written. Values of $F(z)$ can be expressed through the Cauchi integral [13–16];

$$F(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\Omega} \frac{F(t)\ \mathrm{d}t}{t-z} \ , \tag{12}$$

where contour Ω evolves the point z just once. Such representation allows to approximate function F. The smallness of the residual confirms the assumption above, although cannot be considered as its proof.

3 Eigenvalues of logarithm and quasiperiod

The approaching to constant, postulated in the previous section, is attractive property which simplifies the consideration. In this section I analyze this approaching.

The straightforward iteration of eq. (11) with initial value with positive real and imaginary parts allows the numerical approximation. In particular, for $a=\mathrm{e}$,

$$L \approx 0.31813150520476413531265 4 + 1.33723570143068940890116\,\mathrm{i} \ , \tag{13}$$

and, at $a=2$,

$$L \approx 0.82467854614207422231406 5 + 1.56743212384964786105857\,\mathrm{i} \ . \tag{14}$$

Few hundred iterations are necessary to get the error smaller than the last decimal digit in the approximations above. The convergence is exponential; for $a = \mathrm{e}$ and for $a = 2$, the decrement is of order of 1/5. Due to eq. (7), the same decrement characterizes the convergence of the analytical extension $F(z)$ to the limiting values at $\Re(z) \to -\infty$. In order to check this, consider solution of (7) as small perturbation of the eigenvalue of logarithm. Let

$$F(z) = L + \exp(Qz + r) + \varepsilon(z) \ , \tag{15}$$

where Q and r are fixed complex numbers and $\varepsilon(z) \to 0$ at $\Re(z) \to -\infty$ and at $\Im(z) \to +\infty$; let

$$\varepsilon(z) = o\Big(\exp(Qz + r)\Big) \ . \tag{16}$$

Substitution of expression (15) into the eq. (7) gives

$$\ln\Big(L + \exp(Q + Qz + r) + \varepsilon(z + 1)\Big) = \ell(L + \exp(Qz + r) + \varepsilon(z) \ , \tag{17}$$

where

$$\ell = \ln(a) \ . \tag{18}$$

Expanding the left hand side of eq. (17), I write

$$\ln(L) + \ln\!\left(1 + \frac{\mathrm{e}^{Q+Qz+r} + \varepsilon(z+1)}{L}\right) = \ell L + \ell \exp(Qz+r) + \ell\varepsilon(z) \ , \tag{19}$$

Using equations (11) and (18) gives

$$\ln\!\left(1 + \frac{\mathrm{e}^{Q+Qz+r} + \varepsilon(z+1)}{L} \right) = \ell \exp(Qz+r) + \ell\varepsilon(z) \ , \tag{20}$$

While $\varepsilon(z)$ decays faster than e^{Q+Qz+r}, the expansion of (20) gives the relation

$$\frac{\mathrm{e}^{Q+Qz+r}}{L} = \ell \exp(Qz + r) \ , \tag{21}$$

which leads to

$$\exp(Q) = \ell L \ , \tag{22}$$

than means that

$$Q = \ln(\ell L) = \ln(\ell) + \ln(L) = \ln(\ell) + L\ell \ . \tag{23}$$

In particular, at $a = \mathrm{e}$, we have $\ell = 1$ and $Q = L$. Then, quasi-period of the asymptotic solution

$$T = \frac{2\pi \mathrm{i}}{Q} \approx 4.44695072006700782711227 + 1.05793999115693918376341\,\mathrm{i}. \tag{24}$$

At $a = 2$, the evaluation gives

$$Q \approx 0.20511068854498918322452 5 + 1.08646115736547042446528\ \mathrm{i} \tag{25}$$

$$T = \frac{2\pi \mathrm{i}}{Q} \approx 5.58414243554338946020010 + 1.05421836033693734654000\,\mathrm{i}\ . \tag{26}$$

In the upper half-plane, while far from the real axis, the analytic extension F of tetration should show quasi-periodic behavior with period T. In the lower half-plane, quasi-period should be T^*.

4 Recovery of recursive functions

Various analytic functions can be recovered from the recursive equation of type similar to (5), usind the extension to the complex plane and the Cauchi formula (12). However, the regular hebavior at $\pm\infty$ simlifies the evaluation a lot. In this section I describe the precise evaluation of function $F(z)$, assuming that it satisfies the Abel equation (5) and approaches its limiting values L and L^* at $\pm \mathrm{i}\infty$ exponentially, according to (15).

Let B be a large positive parameter. Let $|\Re(z)| < 1$. Consider the contour Ω of integration, consisting of 4 parts:

A. integration along the line $\Re(t) = 1$ from $t = -\mathrm{i}B$ to $t = \mathrm{i}B$.
B. integration from point $t = 1 + \mathrm{iB}$ to $z = -1 + \mathrm{i}B$, passing above point z
C. integration along the line $\Re(t) = -1$ from $t = \mathrm{i}B$ to $t = -\mathrm{i}B$.
D. integration from point $t = -1 - \mathrm{i}B$ to $t = 1 - \mathrm{i}B$, passing below point z.

Value of B should be large enough, that the $|\exp(BQ - r)|$ is smaller, than desired upper limit of the error of the resulting approximation. Then, approximating the integration over the upper and lower parts (B and C) of the contour Ω, values of F can be substituted to its limiting values L and L^*. Both real and imaginary parts of r happen to be of order of unity, at least for $a=\mathrm{e}$ and $a=2$. Then, in order to get approximation with 14 significant figures, it is sufficient to take $B = 32$.

While $|\Re(z)| < 1$, function $F(z)$ can be expressed with four integrals:

$$
\begin{aligned}
F(z) = & \frac{1}{2\pi} \int_{-B}^{B} \frac{F(1+\mathrm{i}p)\,\mathrm{d}p}{1+\mathrm{i}p-z} - \frac{1}{2\pi} \int_{-B}^{B} \frac{F(-1+\mathrm{i}p)\,\mathrm{d}p}{-1+\mathrm{i}p-z} \\
& - \frac{f_{\mathrm{up}}}{2\pi\mathrm{i}} \int_{-1+\mathrm{i}B}^{1+\mathrm{i}B} \frac{\mathrm{d}t}{t-z} + \frac{f_{\mathrm{down}}}{2\pi\mathrm{i}} \int_{-1-\mathrm{i}B}^{1-\mathrm{i}B} \frac{\mathrm{d}t}{t-z} \quad ,
\end{aligned}
\tag{27}
$$

where f_{up} is some intermediate value of function F at the upper segment Ω_{B} of the contour Ω;

$$
\begin{aligned}
\min_{z\in\Omega_{\mathrm{B}}} \Re\big(F(z)\big) \le \Re(f_{\mathrm{up}}) \le \max_{z\in\Omega_{\mathrm{B}}} \Re\big(F(z)\big) \\
\min_{z\in\Omega_{\mathrm{B}}} \Im\big(F(z)\big) \le \Im(f_{\mathrm{up}}) \le \max_{z\in\Omega_{\mathrm{B}}} \Im\big(F(z)\big)
\end{aligned}
\tag{28}
$$

and f_{down} is some intermediate value of function F at the lower segment Ω_{D} of the contour Ω;

$$
\begin{aligned}
\min_{z\in\Omega_{\mathrm{D}}} \Re\big(F(z)\big) \le \Re(f_{\mathrm{down}}) \le \max_{z\in\Omega_{\mathrm{D}}} \Re\big(F(z)\big) \\
\min_{z\in\Omega_{\mathrm{D}}} \Im\big(F(z)\big) \le \Im(f_{\mathrm{down}}) \le \max_{z\in\Omega_{\mathrm{D}}} \Im\big(F(z)\big)
\end{aligned}
\tag{29}
$$

Then, values $F(z)$ at z inside Ω can be approximated with

$$
\tilde{F}(z) = \frac{1}{2\pi} \int_{-B}^{B} \frac{H\big(E(p)\big)\,\mathrm{d}p}{1+\mathrm{i}p-z} - \frac{1}{2\pi} \int_{-B}^{B} \frac{H^{-1}\big(E(p)\big)\,\mathrm{d}p}{-1+\mathrm{i}p-z} + \mathcal{K}(z) \tag{30}
$$

where E is solution of equation

$$
E(y) = \frac{1}{2\pi} \int_{-B}^{B} \frac{H\big(E(p)\big)\,\mathrm{d}p}{1+\mathrm{i}p-\mathrm{i}y} - \frac{1}{2\pi} \int_{-B}^{B} \frac{H^{-1}\big(E(p)\big)\,\mathrm{d}p}{-1+\mathrm{i}p-\mathrm{i}y} + \mathcal{K}(\mathrm{i}y) \tag{31}
$$

for real values of y, and

$$\mathcal{K}(z) = L\left(\frac{1}{2} - \frac{1}{2\pi\mathrm{i}} \ln \frac{1-\mathrm{i}B+z}{1+\mathrm{i}B-z}\right) + L^*\left(\frac{1}{2} - \frac{1}{2\pi\mathrm{i}} \ln \frac{1-\mathrm{i}B-z}{1+\mathrm{i}B+z}\right) \tag{32}$$

is elementary function, which depends also on parameter B. Parameter L is determined by the function H in (5). In particular, at $H=\exp$, the asymptotic value can be approximated with expression (13), and, at $H=\exp_2$, the asymptotic value can be approximated with expression (14).

Integrals in equations (31) were evaluated using the Legendre-Gaussian quadrature formula [10] with 2048 nodes. In the algorithm by [17] of the evaluation of nodes and weights, the float variables were converted to long double; but the following analysis was performed using double and complex double variables.

The discrete analogy of the integral eq. (31) was iterated with initial probe function

$$E_0(y) = \begin{cases} L & \text{at} \quad y > 16 \\ 1 & \text{at} \ -16 \le y \le 16 \\ L^* & \text{at} \quad y < -16 \end{cases} \tag{33}$$

interpreting equality in the discrete analogy of eq. (31) as operator of assignment; fast convergence (within 64 iterations) takes place, while value of E at the grid point number $N{-}1{-}n$ is updated automatically each time when point n is updated; at the symmetric mesh, this forces the resulting approximation $\tilde F$ to be real at the real axis.

The convergence can be even boosted at the irregular order of evaluation of E in the nodes of the mesh. I used to update E at each third node at the first iteration and at each second node at the following iterations.

For real values of z, for $a=2$, the primary approximation $\tilde F(z)$ is shown in figure 1 with dashed line. This function does not satisfy condition $\tilde F(0)=1$. In order take into account (8), the approximation

$$F(z) \approx \tilde F(z{+}z_0) \tag{34}$$

is used, where z_0 is solution of equation

$$\tilde F(z_0) = 0 \ . \tag{35}$$

The corrected function $\tilde F(x{+}z_0)$ is shown in Figure 1 with thin solid line.

Due to simultaneous update of E at the symmetric modes of the mesh, the constant z_0 is real; in the example in Figure 1, $z_0 = 0.0262474248816494$, but

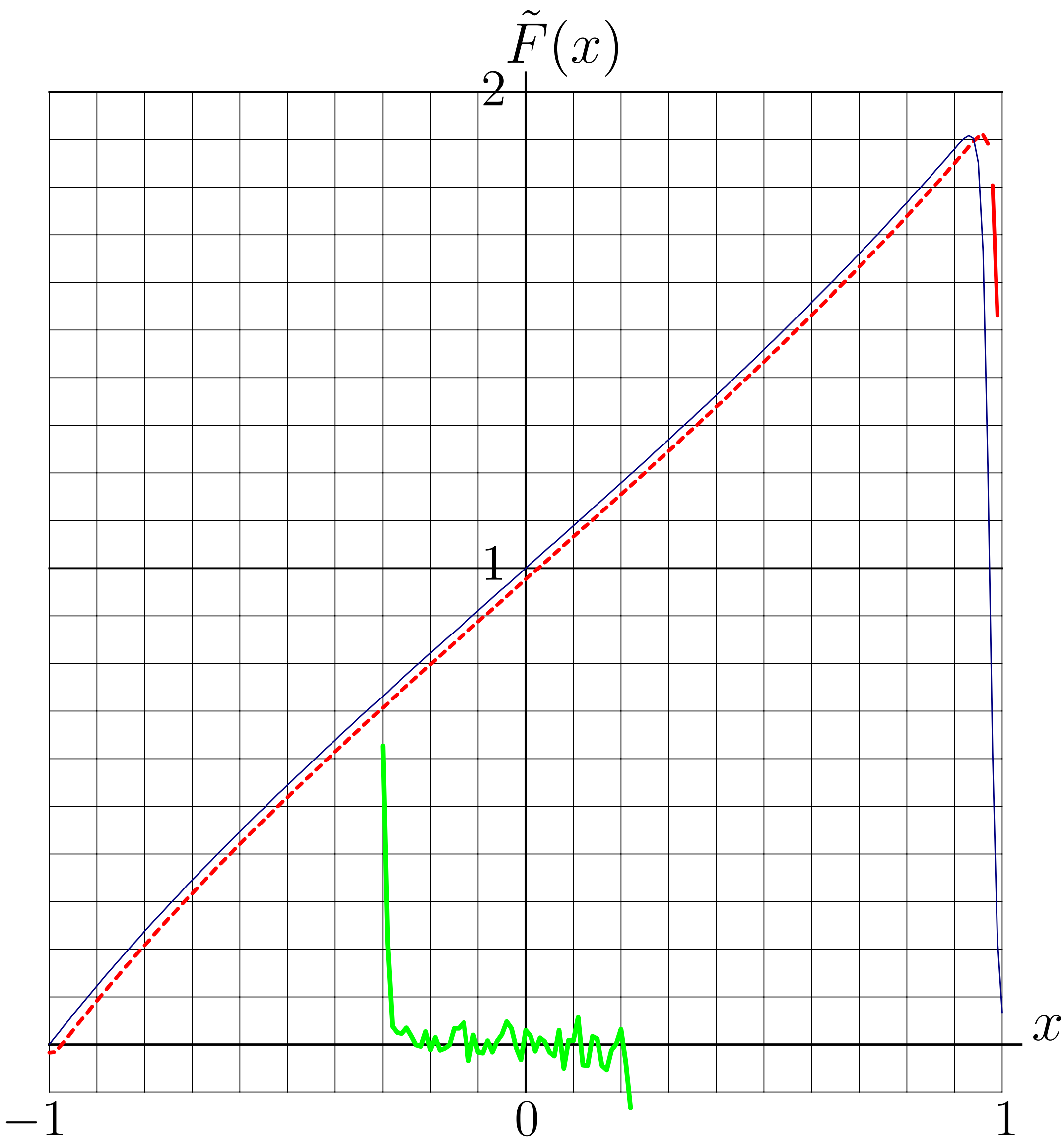

Fig. 1. Approximation $\tilde F(x)$,dashed, and its correction $\tilde F(x+z_0)$, thin solid; residual by eq. (36), scaled with factor 10^{13}.

this value is specific for the chosen $B=32$, number $N=2048$ of nodes of the mesh, initial condition, and on the order of update of values of function E at the iterational solution of the discrete analogy of eq. (31).

The residual

$$\mathrm{residual}(z) = \tilde F(z-0.5) - \log_2\Big(\tilde F(z+0.5)\Big) ~, \qquad (36)$$

scaled with factor 10^{13} is shown a the bottom of figure 1 with thick line.

The resulting approximation of F is analytic by definition, even with the discrete sums instead of the integrals; because the finite sum of analytic functions is also analytic. However, in general case, the iterational procedure has no need to converge, and, even if it converges, it has no need to converge to the solution of eq. (31). As the result, in general, function $\tilde{F}$ has no need to approximate the solution of equation (5); at the replacement of φ to $\tilde{F}$, the residual can be large. Nevertheless, for cases mentioned, the fast convergence and small residuals were observed.

The residual at the substitution of function $\tilde{F}$ into the eq. (7) at $a=\mathrm{e}$ and at $a=2$ was evaluated, using approximations for $F(\pm 0.5+\mathrm{i}y)$ at $-24<y<24$ and those for $F(x)$ at $-0.8 < x < 0.8$. The residual is of order of 10^{-14}, while the real part of the argument of function $\tilde{F}$ does not exceed 0.7 . Evaluating $\tilde{F}(z)$ at larger values of $|\Re(z)|$, the point z approaches the contour of integration; at $|\Re(z)| > 0.7$, the approximation becomes less accurate (see bottom part of Figure 1). At $x \approx 0.95$, the error of $\tilde{F}(x)$ is seen also in its graphic at the upper-right corner of figure 1.

Due to equations (6), (7), the approximation of $F(z)$ within the strip $\Re(z) \leq 1/2$ is sufficient for the evaluation in the whole complex z-plane. Smallness of the residual indicates (although does not yet prove) smallness of the error of the resulting approximation. In particular, the thin curve in the central part of figure 1 shows the precise approximation.

The good approximation of $F(z)$ at $|\Re(z)| \leq 1/2$ allows to evaluate it in the whole complex plane, using equations (6) and (7). Then, using expression (4), the Ackermann function $A(4,z)$ can be approximated for complex z, and the range of the evaluation is limited only by the ability to represent the result as a floating-point number. The example of such evaluation is considered in the next section.

5 Ackermann function

From the approximation (34) of tetration F, the Ackermann function $A(4,z)$ can be evaluated for complex values of z; at least in the cases, when $A(4,z)$ can be stored in the computer with floating points representation of data; I used the standard double precision arithmetics in order to make all my results easy reproducible. Few examples of such evaluation are printed in table 1. Symbol “nan” in the left upper corner of the table means that I am not yet successful to give meaning to $A(4,-5)$. Symbol “inf” at the bottom denotes value $2^{65536}-3$, that cannot be stored in a double-precision variable as a number.

The approximation for the Asckemann function is shown in Figure 2. Levels, corresponding to integer values of the real part, and those for integer values of the imaginary part, are shown. The function $A(4,z) = F(z+3) - 3$ has cut at $z \leq -5$, shows almost (visually) linear growth at $-4<z<-2$, and then rapidly

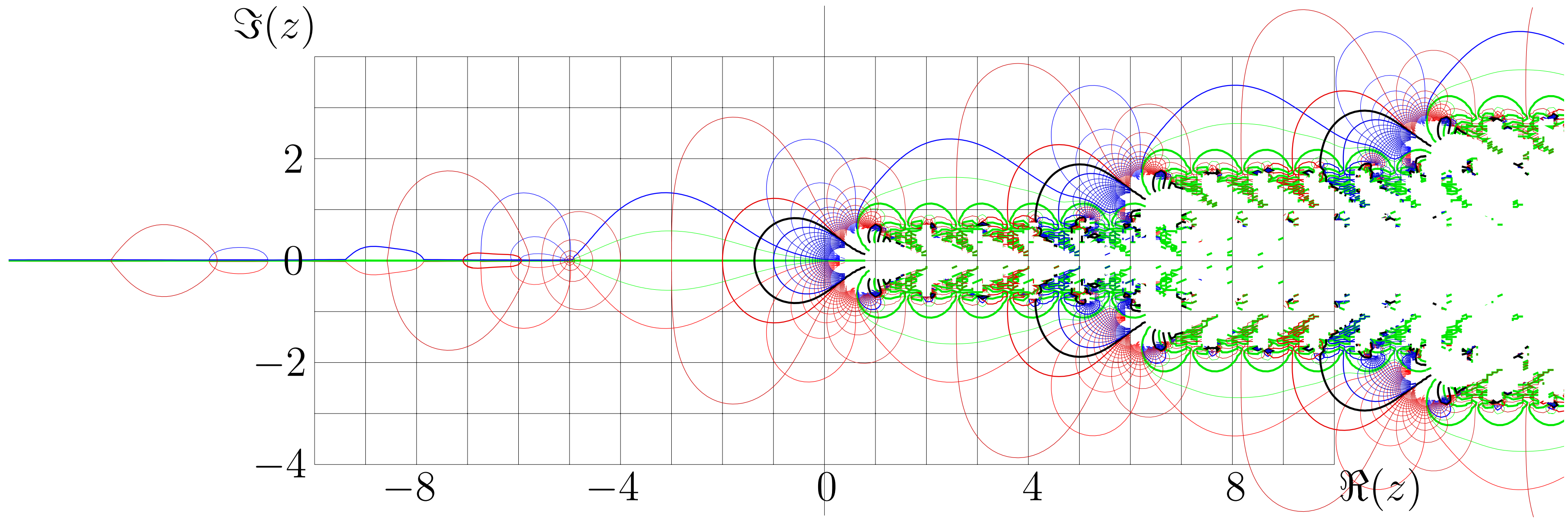

Fig. 2. Portrait of the Ackermann function $f=A(4,z)$ in the complex z-plane. Lines $\Re(f) \in$ integers and $\Im(f) \in$ integers are drawn.

grows up at $z > 0$. Along the imaginary axis, this function varies smoothly, and approaches values $L-3$ and L^*-3 at $\pm \mathrm{i}\infty$.

In the left hand side upper corner of figure 2, $A(4, z) \approx L - 3$; therefore, there are no lines there; similarly, in the left hand side lower corner, $A(4, z) \approx L^* - 3$. In the right hand side of the figure, in vicinity of the real axis, the function passes through the integer values, but the density of the lines is so high, that is not possible to draw them; so, this part of the figure is also left empty.

In the intermediate range of figure 2, at translations with period T or T^*, the same quasi-periodic structure is reproduced. In agreement with estimate (15), the analytic extension of the Ackermann function is asymptotically-periodic; at $\Im(z) > 0$, the pattern in vicinity of point z visually coincides with that at $z+T$. Quasi-period T can be approximated with eq. (26).

The fourth Ackermann function can be approximated for complex values of the second argument. However, the algorithm above and figure 2 cannot substitute the mathematical proof. The proof of existence and uniqueness of the solution E of eq. (31) waits for attention of professional mathematicians. It would be good to prove the existence of the analytic extension of $A(m, z)$ for complex values of z for all integer n; case $m = 4$ can be the first step. (The analytic extension of the Ackermann function for the complex values of the first argument may be also considered.) Then, the Ackermann function A and tetration F should be declared as special functions.

6 Conclusion

The Fourth of the Ackermnann functions can be expressed in terms of tetration F by eq. (4), and approximated through eq. (34), (30) for complex values of the argument. Such approximation indicates the existence of the analytic extension of the Ackermann function for the complex values of the second argument, that exponentially approaches its asymptotic values L and L^* at the imaginary axis.

The range of evaluation suggested is limited only by the ability to store the result as a floating-point number. Examples of values of the Ackermann function $A(4, z)$, approximated in such a way, are printed in Table 1.

The distribution of the analytic extension of $A(4, z)$ in the complex z-plane is shown in figure 2. Up to my knowledge, it is first portrait of the Ackermann function in the complex plane, ever reported.

The algorithm of evaluation of Ackermann function with contour integral is robust; with standard double-precision arithmetics, the residual of order of 10^{-14} is achieved. The iterational solution of eq. (31), required for the evaluation, takes a minute of CPU time; then, figure 2 can be generated in real time.

Existence and uniqueness of the analytic extension to the complex z-plane of the Ackermann function $A(4, z)$, plotted in figure 2, still needs the proof. Application of the same procedure to higher Ackermann functions, as well as to other cases of the Abel eq. (5), can be matter for the future research.

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