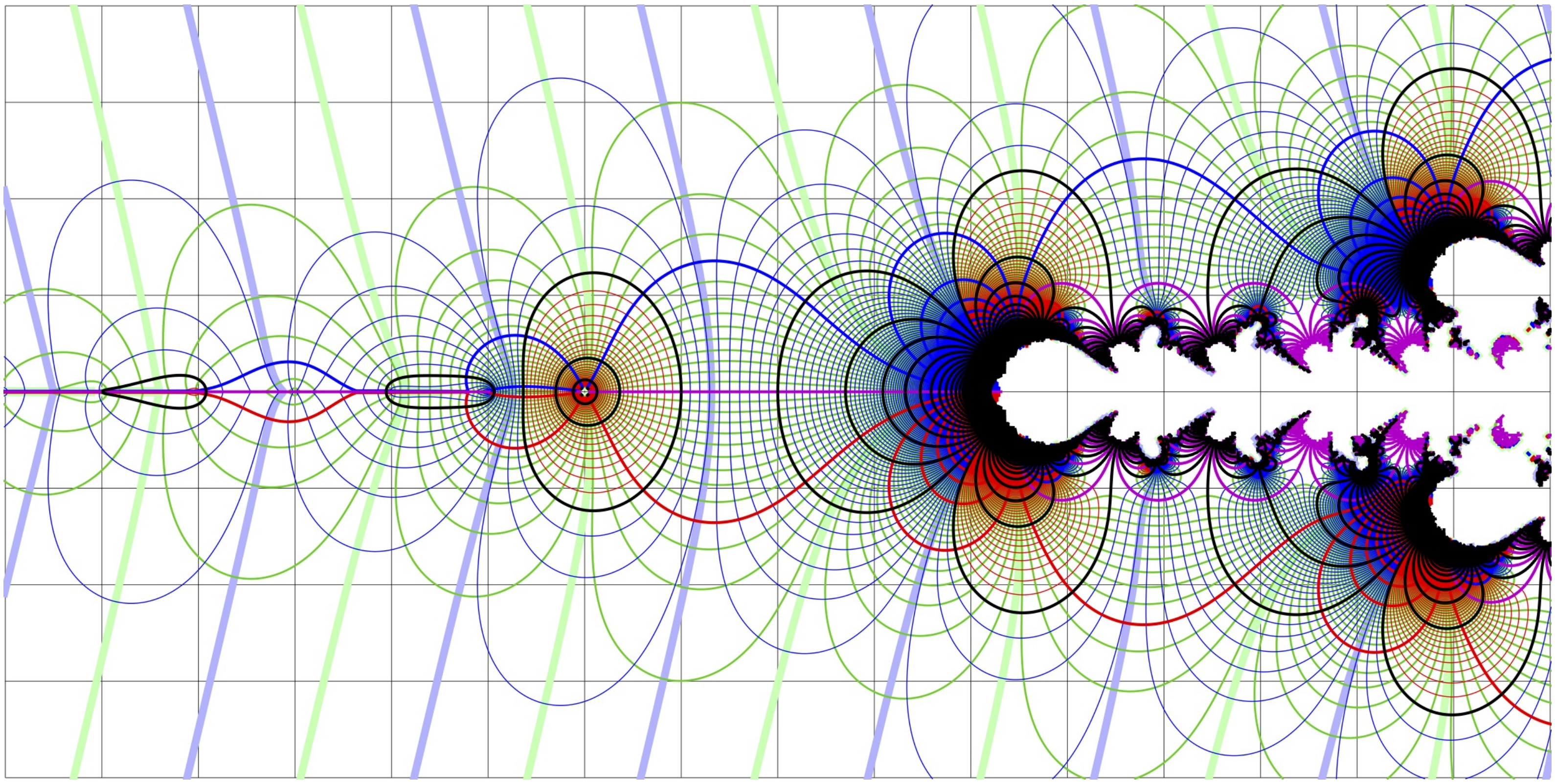


Dmitrii Kouznetsov

Superfunctions

Non-integer iterates of holomorphic functions. Tetration and other superfunctions. Formulas, algorithms, tables, graphics and complex maps.

2017

http://mizugadro.mydns.jp/t/index.php/File:Tetma.jpg cover map
http://mizugadro.mydns.jp/BOOK/443.pdf
http://mizugadro.mydns.jp/t/index.php/Book

This version is under construction, generated December 21, 2017

Russian version of year 2014 (Fig.1):
https://www.morebooks.de/store/ru/book/Суперфункции/isbn/978-3-659-56202-0
List of misprints in the First edition of the Russian version is at
http://mizugadro.mydns.jp/t/index.php/Superfunctions_in_Russian
I load the corrected version as
http://mizugadro.mydns.jp/BOOK/202.pdf
http://www.ils.uec.ac.jp/~dima/BOOK/202.pdf

List of figures is also available online, I load it as
http://mizugadro.mydns.jp/t/index.php/Category:Book

Figure 1: Distribution of the Russian version of this Book, 2014

Contents

Chapter 1

Prologue

Figure 1.1: Cloud in Trousers

I type this Prologue, respecting my teachers of English, and also those of the Russian Literature. They think, that every scribbling, since W.Shakespeare, should begin with some Prologue[1], and should finish with some Epilogue. In addition, the Soviet teachers had promoted the sovietism[2] and the socialistic realism, that implies, that even a cloud should wear some trousers[3] as it is shown in Fig.1.1.

This prologue appears as a kind of trousers, a wrap, to cover the inside; where, my results are presented in a naked and provocative form.

This Book is about holomorphic (analytic) solutions F of equation

$$F(z+1) = T(F(z)) \quad \text{[prologeq]} \tag{1.1}$$

where T is some given holomorphic (analytic) function. Also, this Book is about applications of the solution F, its inverse function $G=F^{-1}$ and interesting properties of function $T^n(z)=F(n+G(z))$. The given function T is called "transfer function"; the solution F is its "superfunction". The inverse function G is called the "Abelfunction".

The Prologue describes not functions F and G above, but this Book: what is it, why is it, how to use it, what is in it, and what is not. I follow the classical example[4] and explain, why this Book is so important.

Readers, who are interested namely in solutions of the equation above and their physical meaning, may scroll from here directly to Introduction, Chapter 2 at page 20.

[1] W. Shakespear. Romeo and Juliet. Prologue. 932/Sh12, Kenkyusba English Classics, 1926.

[2] http://mizugadro.mydns.jp/t/index.php/Sovietism

[3] `http://www.unlikelystories.org/old/archives/cloudintrousers.html` A Cloud in Trousers by Vladimir Mayakovsky, translated from the Russian by Andrey Kneller. Prologue. Cited by the state for 2014.09.10.

[4] François Rabelais. Gargantua [1] : .. Most noble and illustrious drinkers, and you thrice precious pockified blades (for to you, and none else, do I dedicate my writings),

1 For whom is this Prologue

I type this Prologue not for the Readers. I type this Prologue for those, who will not read this Book:

For Editors, managers, sellers, who deal with thousands books, and who have need within a minute to understand, why namely this book should be placed at the first place in the list of recommended literature.

For the experienced critics, who need to read only two or three pages, in order to write the review: *What oddity is this: Superfunctions? What sort of Superfunctions have we here? And thrust into the world by a laserist! As if we are not satisfied with superconductivity, supersymmetry, superfluidity, supermen and supermarkets! As though threes enough had not been cut for paper, and not enough files have been loaded into internet! As though folks enough of all classes had not tired their fingers with keyboards! The whim must take a laserist to follow their example! Really there is such a lot of paper nowadays that it takes time to think what to wrap in it!* [5]

For librarians, who need to find for this Book a suitable place, in order to make it visible at the background of tons of books about superconductivity, supermen, supermarkets and superinflation.

For colleagues, who may wonder, why the laserist, instead of to promote the optical ceramics (for which the big Grants are assigned) deals with superfunctions (which are yet very far from to get the financial support), and behaves as a simpleton, who does not know, *How to Win Friends and Influence People.*

For relatives and friends, who teach me to live and to promote my results. They seem to know very well, how to *influence people.* This book could be a kind of medicine against this kind of influenza. However, more detailed analysis of relations between friends is published separately [23].

For the categories of non-readers mentioned above, I should specify the genre and topic of this book. The next two sections are dedicated to this. With such a specification, one may form his/her/its own opinion about the book, without to read it.

[5] N.Gogol. Evenings at a Farmhouse near Dikanka. MARCH 27, 2013. [4] : "What oddity is this: Evenings on a Farm near Dikanka? What sort of Evenings have we here? And thrust into the world by a beekeeper! God protect us! As though geese enough had not been plucked for pens and rags turned into paper! As though folks enough of all classes had not covered their fingers with inkstains! The whim must take a beekeeper to follow their example! Really there is such a lot of paper nowadays that it takes time to think what to wrap in it."

2 Russian version and copyleft

This book had been planned as the English version of the Russian book “Суперфункции” [6]. I did my best efforts trying to translate the Russian expressions into English; figures 1.1,1.2 prove my best efforts.

Figure 1.2: Translation may cause confusion

I provide references to the English translations of the classical books cited in the Russian version. However, in many cases, the translation is not possible. Often, it is difficult to find the English equivalents of the references cited. So, I give up trying to translate the Book; I re-write it in English.

I reuse some formulas and figures, that are available at the free sites; in particular, at Mizugadro and Citizendium. The links are mentioned in the Supplement 4, at page 301 . I load the gallery of the images at `http://mizugadro.mydns.jp/t/index.php/Category:Book` ; I hope, this server will last longer, than the previous holder of TORI at http://tori.ils.uec.ac.jp/TORI, that had been attacked 2013.02.27. the access to http://tori.ils.uec.ac.jp/TORI had been disabled. It costed certain efforts to arrange its clone at `http://mizugadro.mydns.jp/t`. The reason of the aggression could be the critical article prepared for “Physics Today” [7]. Unfortunately, namely this way the politicians answer the critics, while they cannot build any logical argumentation. I hope, those politicians pay to administration of our university some significant amount per each day, while TORI is not available at its original URL. I have no other explanation of the attitude or our administration, who had post-factum approved that barbarian attack. I try to make Science, not the criminal investigation, so, I mention only the scientific point of view on the events. However, I hope, one day the professional criminalists carefully investigate the case, as well as the origins of other attacks that happened 2013.02.27, 2014.02.27, 2015.02.27, 2016.02.27, 2017.02.27 (and can be expected for 2018.02.27). [8]

[6] `http://mizugadro.mydns.jp/t/index.php/Superfunctions_in_Russian`
`http://www.ils.uec.ac.jp/~dima/BOOK/202.pdf`
`http://mizugadro.mydns.jp/BOOK/202.pdf` Д.Кузнецов. Суперфункции. Lambert Academic Press, 2014. (In Russian)

[7] `http://mizugadro.mydns.jp/PAPERS/2013physToday.pdf` D.Kouznetsov. Corruption in Russian science. 2013, preprint

[8] `http://mizugadro.mydns.jp/t/index.php/Tori_attacked`
`http://budclub.ru/k/kuznecow_d_j/toriattacked.shtml` D.Kouznetsov. TORI attacked, 2013

I allow the free use of my results, and I try to simplify the use. In particular, I provide generators of images, used in the Book. I would not specify this in details, but one recent observation indicates that the problem is very serious. Since year 2014, the strange phenomenon is observed in the resources of wikimedia: the administrators claim, that publication of my results violate my authors rights. The same, and even worse, happen with results by other authors from the USSR and, especially, texts of the Soviet dissidents. The administrators ignore the permissions provided by the copyright owners, and remove the texts and the images, claiming, that the loading or the texts and images are "Copyright violation". The similar phenomenon could take place in the Nazi Germany. The nazi used to arrest, to kill the opponents, and withdrawn their books from shops and libraries with pretext of defence of the intellectual property. They killed many authors, and, if the author is already dead, the fascists claim, that now the State becomes owner of the copyright property. However, the State, after to kill the authors, prohibit the distribution of their works. May be, the professional historian corrects me, that the German nazi acted in a little bit different way. But everyone may look at wikisource, and, especially, to its Russian version and see, that namely this happens at wikimedia, and, especially, since year 2014. Perhaps, the Russian Soviet veterans decided, that the invasion into Ukraine and plundering of Crimea is not sufficient, and begun to plunder also the wiki resources. I mention this in the special statement [9].

I type the paragraph above in order to stress, that I had issued the permission to distribute my results under the free GNU license; the only attribution is required. The attempt to "defend" my author's right in the way described above, is a fraud: several years ago I had allowed the free distribution, and I provide the links, that confirm this. In particular, the permission refers to the plots, used in this book and their generators. I specify the URL of the image at each picture. If some administrator claims, that use of this image under the "free" license violates my copyright, let me know. I think, such an administrator should be qualified as liar, impostor, knave, thrive; and his/her/its attempts to defend my author's right in such a way should be qualified as a fraud. In other words, use the images from this Book, as well as from Russian version, for free, but do not forget to attribute the source.

[9] `http://mizugadro.mydns.jp/t/index.php/Kouznetsov,_permission` Open letter by Dmitrii Kouznetsov about massive removal of texts and images from Wikimedia projects with pretext of defence to the copyright of the authors. Wed, Mar 18, 2015 at 11:23

3 Physical mathematics

While writing this Book, I was asked about its genre. I searched the answer in various databases, and it seems, the closest qualification is "Physical mathematics" [78, 59, 60].

Term "physical mathematics" should not be confused with "mathematical physics". The order of words is important. Similar examples exist in the Quantum Mechanics. Let P be momentum of a particle, and Q be its coordinate; then $PQ-QP=-\mathrm{i}\hbar$.

One student could not catch the sense of non-communing operators. She tried to understand, why operator of creation does not commute with operator of annihilation. The Professor tried different explanations, then he found the strong example for female students; he asked:
Could you explain me, please, why operation **TO CLEAN-UP** *does not commute with operation* **TO DIRT-DOWN** *?*
The student was content, she said:
Thank you, Professor! This is very good example! Now I understand all the Quantum Mechanics!
Professor also got a good lesson; he realised, how to teach Quantum Mechanics, keeping women and female logic in mind [10].

Term "mathematical physics" denotes the mathematical exercises with equations that come from Physics.
Term "physical mathematics" means, that the common scientific tools are used to check mathematical conjectures; the rigorous mathematical deduction appears as a possible scientific method, but not an imperative.

4 TORI axioms

In this Book, I use the TORI Axioms. This section retells some results from publications in "Uspekhi" [67] and J. of Modern Physics [83]. These results refer to definition of term "Science". TORI Axioms appear as parts of the defination of this term.

[10] `http://pda.anekdot.ru/id/658766` Bogorad Victor. Мысли, Мысли. (2013, In Russian) `http://mizugadro.mydns.jp/t/index.php/Female_logic` D.Beklemishev. Female logic. (2013) `http://www.ams.org/notices/201309/rnoti-p1156.pdf` J.M.Deshler, E.A.Burroughs. Teaching Mathematics with Women in Mind. Notices of the AMS, V.60, No.9, p.1156-1163.

Science is kind of knowledge, activity and research, based on concepts, that have all the six properties below:

1. Applicability: Each concept has the limited range of validity, distinguishable from the empty set.

2. Verifiability: In the terms of the already accepted concepts, some specific experiment with some specific result, that confirms the concept, can be described.

3. Refutability: In the terms of the concept, some specific experiment with some specific result, that negates the concept, can be described.

4. Self-consistency: No internal contradictions of the concept are known.

5. Principle of correspondence: It the range of validity of a new concept intersects the range of validity of another already accepted concept, then, the new concept either reproduces the results of the old concept, or indicates the way to refute it. (For example, the estimate of the range of validity of the old concept may be wrong.)

6. Pluralism: Mutually-contradictive concepts may coexist; if two concepts, satisfying requirements 1-5 above have some common range of validity, then, in this range, the simplest of them has priority.

The 6 properties, requirements above are **TORI axioms.** They are specified also at `http://mizugadro.mydns.jp/t`.

The main results, that had been presented at TORI and related to this Book, are published in Scientific Journals [83, 85, 88, 91]. In order to simplify the referencing, the axioms above needed a special name. Having poor phantasy, I just keep the initial abbreviation, TORI, that means Tools for Outstanding Research and Investigation. The Russian transliteration "ТОРИ" has similar meaning ("Теоретические Основы Революционных Исследований"). Many results, reported at the conferences and in the scientific journal, look so similar, that it is difficult to find, what namely new and interesting does each of them suggest. They barely satisfy the TORI axioms, if at all. The non-trivial results appear to be "outstanding", "revolutionary", in the sense, that they stay a little bit out of the main trend of scientific research.

Articles of site `http://mizugadro.mydins.jp/t` collect tools: concepts, formulas, algorithms, pictures, that I consider to be "outstanding" and useful in the scientific research. I assume, that the "instanding" researches are already well represented in Citizendium, Wikipedia and scientific journals, and there is no need to repeat them again. In particular, I load to TORI generators of figures, used in this book. The idea is, that each code can be downloaded by the Reader, compiled, executed, and, if necessary, modified for the new (and, perhaps, unexpected for me) applications.

I load the generators also to other sites, in order not to make an impression, that the politicians may completely block the access to my results attacking another server. In this sense, *Manuscripts don't burn* [8].

Figure 1.3: Server felt down

In the Supplement, I suggest the links to other internet recourses for the case if some server is not reachable. The same, for the case, if it suddenly falls down, as it is shown in Figure 1.3.

With links, suppled to figures, the Reader can use and develop the formalism of superfunctions just from the place, until I have advanced. Just pick-up the figure you like and load its generator.

5 What for this book is

Since publication in 2009 of the article about holomoprhic tetration [54], I expected, the mathematicians can do the rest of the job by themselves. That publication refers to the special case of equation (1.1), with $T=\exp$; and the natural tetration as solution is considered, $F=\mathrm{tet}$.

Seven years passed by since [54], but superfunctions did not appear in descriptions of the algorithmic languages. Neither tetration, nor superfactorial [65], nor other superfunctions I had reported appear as built-in functions. Until now, the function Nest [97] in language "Mathematica" requires, that the number of iterate is integer. If one approximates some bell-like function, then, usually, the phantasy of colleagues does not go farther than the Gaussian exponential, hyperbolic secant or Lorentian.

One could use also, for example, the half-iterate of the exponential [10], id est, such function f, that $f(f(z))\!=\!\exp(z)$, instead of the exponent in $2/(\mathrm{e}^x\!+\!e^{-x})$, as well as other non-integer iterates of vatious functions, but one needs the formalism of superfunctions for the evaluation. This Book describes such a formalism.

Most of the Mathematical Analysis in schools and universities is built-up on the first three ackermanns: addition, multiplication, exponentiation and their inverse functions in various combinations. This arsenal could be greatly extended, including, for example, the 4th ackermann, which is tetration, or the 5th ackermann, that is pentation. However, one needs formalism of superfunctions to evaluate tetration and pentaiton [64, 89]

When physicists analyse the nonlinear response of a medium, they try to make the sample optically-thin; otherwise, the re-absorption and saturation lead to mistakes [42, 43, 44, 45, 46, 47]. For the thin sample, the precision of measurements is low. Superfunctions allow to work with thick samples, and still reconstruct behaviour of intensity inside the sample [84, 85].

Many researchers believe that namely their work is especially important. I am not an exception. From my point of view, tetration and arctetration, as well as other superfunctions and abelfunctions, will be import for science since century 22; so important, as the exponential and logarithm are important since century 19. This Book explains, how to calculate superfunctions and what can one do with them.

When the fundaments of superfunstions had been formulated, one of my coauthors, Akira Shirakava [38] had declared, that it is not possible, to recover distribution of intensity of light in the amplifier from its transfer function. [74]. Then, the algorithm for this recovery had been presented [61, 84, 85], and Shirakava said, that the formalism of superfunctions is too complicated [75]. Through all the mutual misunderstandings, I consider Shirakawa as specialist in fibre lasers, and I take into account his complains. I try to make this book popular. So, I cite not only scientific articles. Also I try to avoid complicated mathematics, at least in some figures. However, I still suppose, that the Reader can distinguish an Integral from a logarithm, and has some idea, what is square root of minus unity. If the Reader in addition, had seen some Cauchy intergal(s), this is also a good advantage.

In order to simplify the reading, I try to avoid long sentences and digressions, although not always it is possible. In order to explain this, I suggest the example:

Happy is the traveller who, after a long and heavy journey, with its cold and humidity of air-conditioners, with long waiting-lines at the passport control, curious to under ware the special control officers, delayed flights, lost luggage, going to the opposite direction taxi drivers, sees at last the familiar roof with its lights approaching to meet him. And there rise before his mind the familiar rooms, the delighted outcry of the servants running out to meet him, the noise and racing footsteps of his children, and the soothing gentle words interspersed with passionate kisses that are able to efface everything gloomy from the memory. Happy the man with a family and nook of his own, but woe to the bachelor!
Happy is the researcher who, passing by the strange results, paradoxes and their wrong, inconsistent interpretations, attaches himself to phenomena that display the loftiest virtues of scientific achievements, who, from the great whirlpool of figures flitting by him daily, has selected only the few exceptions, for which the answers are already known; who has never tuned his research to a less exalted key, has never stooped from his pinnacle to really new and unexpected phenomena. His fair portion is doubly worthy of envy; he lives in the midst of them as in the midst of his own family and, at the same time, his fame resounds far and wide. He clouds men's vision with enchanting incense; he flatters them marvellously, covering up the gloomy side of science and life, and exhibiting to them the noble man. All run after him, clapping their hands and eagerly following his triumphal chariot.
They call him a great world-famed scientist, soaring high above every other genius as the eagle soars above the other birds of the air. Young ardent hearts are thrilled at his very name; responsive tears gleam in every eye. . . . No one is his equal in power — he is a God! But quite other is the portion, and very different is the destiny of the writer who sees and reveal the phenomena strange, that are out of the commonly accepted theories and contradict to the obvious commonly-accepted common sense, calling for the revision or at least some critical analysis of the widely recognised results, that already have assigned the highly prestigious awards and huge grants and foundations.. En fin, he'll not escape from the scientific council, who keep the old paradigms and any doubt consider as a sin, as a kind of heresy.. Without compassion, such a researcher is left by the roadside like the traveller without a family. Hard is his lot and bitterly he feels his loneliness. [5]

I promise, that the above is longest digression in this Book. Trying to keep the Book thin, the philosophic exercises about the place of Science in the Human knowledge [67, 68] are published separately. It is not possible to cover Everything (id est, the Full Set). Since the beginning, I try to specify, what is in the book, and what is absent in the Book.

6 What is present in the Book

This Book suggests the general approach for construction of superfunctions and various algorithms of their evaluation. Superfunction is solution F of equation (1.1); I repeat it here: $F(z\!+\!1)=T(F(z))$. This equation is called "Transfer equation", and the given function T is called "transfer function".

In this book, I describe various algorithms for evaluation of superfunctions. The choice of the algorithm depends on the fixed points of the transfer function, which are solutions L of equation $T(L) = L$; and, also, depends on the additional requirements: what properties do we expect from the required solution F of equation (1.1). These properties, being postulated, should provide the uniqueness of the solution, but still should allow its existence and should indicate some way(s) of the evaluation.

The inverse of superfunction F is called here "abelfunction" $G=F^{-1}$. The abelfunction satisfies the Abel equation $G(T(z)) = G(z)$+1. In this book, I consider many examples of the transfer function T, superfunction F and the abelfunction G. I suggest ways of the efficient evaluation. Efficient means, that the functions (and corresponding figures) are computed in the real time and with good precision. With use of the complex double arithmetics, the superfunctions and abelfunctions are evaluated with 14 decimal digits. I load the figures together with their generators to `http://mizugadro.mydns.jp/t/index.php/Category:Book`; the readers can check them with C++ and Latex.

By default, I assume, that the parameters have real values, while the arguments of the functions may have complex values. For the illustrations, the complex maps are essential. In particular, the complex map of the natural tetration is used at the front cover of this Book.

I am more physicist, than mathematician. For physical applications, the real values of the argument are of special interest. Usually, the real values allow the straightforward interpretation and, sometimes, the application. For this reason, I supply also the graphics for the real values of the argument.

7 What is absent in the Book

In this book, there are no accurate specifications, which values are allowed for parameters and arguments of the functions considered. I specify them only in the cases, when they are essential for the consideration. For each expression, the readers are supposed to examine the range of applicability as the exercise. The reader is not a vessel to be filled, but a candle to be lighted [11].

I try to define functions in such a way, that at least some vicinity of the positive part of the real axis belongs to the range of holomorphism; although, not always this is possible. This book seems to be the first monograph with description of efficient methods of evaluation of superfunctions and abelfunctions, and this should be considered as an excuse for the heuristic style of presentation of the results.

For the same reason, in this Book, there are no rigorous proofs of convergence, existence and uniqueness of solutions. In particular, the deductions, elaborated in collaboration with Henryk Trappmann, are not included here. The Readers who like the deep drilling [12] are invited to download our original articles [61, 66, 73, 79, 86], plumb through the deduction expanded with the Saxon style of pedantic Henryk, drill the examples and ask questions. Indications of the errors, at least misprints, are of especial interest.

From my point of view, the theory of superfunctions is new branch of the Mathematical Analysis, and the serious drilling and plumbing require efforts of some institute with several research groups in order to bring it to the beautiful and rigorous shape, like the theory of the differential and Integral calculus and other mathematical analysis. Here, I deal with functions of single variable; the generalisation to multidimentional case is not presented.

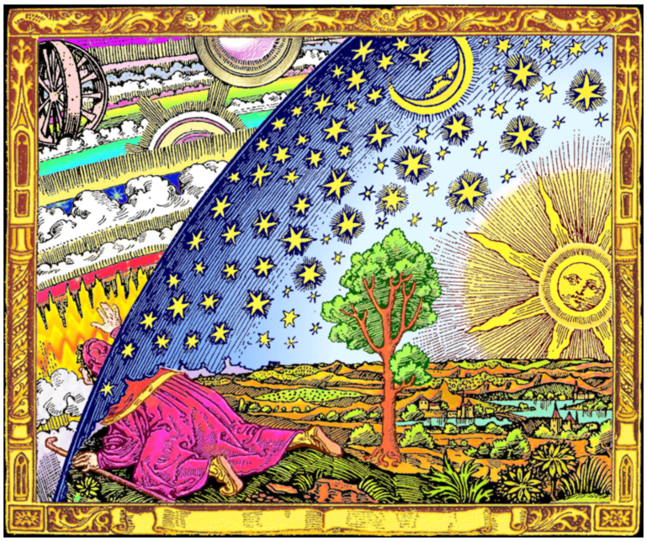
History to development of theory of superfunctions and abelfunctions begins approximately since century 19, and can be also interesting, but

[11] `http://quoteinvestigator.com/2013/03/28/mind-fire/` garson. The Mind Is Not a Vessel That Needs Filling, But Wood That Needs Igniting. March 28, 2013. .. None of the examples came with citations:
Education is the kindling of a flame, not the filling of a vessel —Socrates
Education is not the filling of a pail, but the lighting of a fire. —William Butler Yeats
Education is not the filling of a pail, but the lighting of a fire. —Plutarch
The mind is not a vessel that needs filling, but wood that needs igniting. —Plutarch ..

[12] Importance of the deep drilling can be illustrated with the limerick below:
One Chinist colleague called Lee
Was drilling his girl at the sea;
She told him: *Stop plumbing,*
Somebody is coming!
And Lee had replied: *That's me!*

it falls out of the topic of this Book. The overviews of the pre-historic publications can be found in the Internet with the following keywords: **Abel function, iterated functions, tetration, superexponential, superfunction**. Some of them can be considered as scientific, according to the definition above; I mean, the TORI axioms. This Book is dedicated to the Superfunctions, not to TORI axioms, so, these axioms are presented here only schematically; however, the more detailed description is available [67, 68, 100, 101, 102].

In this Book, there is no detailed discussion (and sometimes nor even citation) of publications of colleagues, who tried to evaluate the superfunctions during the last couple of centuries. In order to see, what had been done in the past centuries, one can look at the publications by colleagues: Neils Henryk Abel [3], Peter Fatou [6], Hellmuth Kneser [10], Jean Ecalle [17], G.Szekeres [11], Peter Walker [24]; more references are suggested in year 2014 by J.Bonet and P.Domanski [90]. In addition, some links are listed in the Supplement (page 301).



https://commons.wikimedia.org/wiki/File:Flammarion-color.png
https://en.wikipedia.org/wiki/Flat_Earth

Figure 1.4: Earth in the old model of the Universe

I see analogy with manuals on Geography and Astronomy, that have no need to begin with the historic excursus, describing the Flat Earth (Figure 1.4), nor the Geocentric model by Ptolemey. Perhaps, some Readers may consider such an analogy to be too pretentious and ambitious; but the methods reported before century 21 are, indeed, non-efficient; so inefficient, that nobody could plot a complex map of any non-trivial superfunction with methods reported in century 20. Henryk Trappmann attributes this to my routine http://mizugadro.mydns.jp/t/index.php/Conto.cin, which is now available also at various wikis. I think, the matter is not my plotter, but the beautiful and efficient representations of functions, that allow the quick and precise evaluation. In principle, these algorithms can be programmed at Mathematica, even at Maple, and still generate the same figures; although the efficiency of Maple (and, especially, Maple-10) in generating of figures allows to use that software as an illustration to the known proverb “to press a key, to have a tea”, as it is described in the Supplement, page 299. I do not consider that poem as a part of this Book, but the poem and figure 22.1 explains, why the most of figures in this Book are generated as the direct EPS graphics from the C++ programs.

I do not describe in this book algorithms suggested after year 2015, although such algorithms should appear in future. One of them is expected to be presented by William Paulsen [93].

I do not include into this Book the codes of the programs (figure 22.1 is an exception), used for evaluation of functions and plotting graphics. The algorithms are simple; any student, after a semester of programming in any language, that supports the complex arithmetics, can easy program the same, following the description from this Book. At figures, I indicate the URLs, where the code is loaded.

Some results of this book can be reproduced with the Schröder equation and the Schröder functions. In this Book, there is no detailed description of properties of the Schröder equation, nor analysis of the Schröder functions; these functions are only mentioned. I could not construct any example, that can be solved with Schröder functions, but cannot be solved with superfunctions. Some properties of the Schröder equation can be found at
`http://mathworld.wolfram.com/SchroedersEquation.html` and
`https://en.wikipedia.org/wiki/Schroeder_equation`

Preparing the results for the Book, I tried to refute each of the conjectures suggested. In principle, it is possible. It is sufficient to construct any example of a transfer function T such that the superfunction F, constructed by the methods, described in the Book, does not satisfy the transfer equation (1.1). Or to construct two different superfunctions, that satisfy the additional conditions, that are declared to provide the uniqueness. My attempts to reject my claims failed. I could not find in the literature any rejecting example, nor I could construct it by myself. Only for this reason, the rejecting, refuting examples are not included in the Book.

Many colleagues, as myself (since the childhood), prefer to watch pictures in a book, rather than to read it. Therefore, I try to illustrate each page with at least one figure. While, this goal is not yet achieved, but I hope to approach it closer in the future editions.

Chapter 2

Introduction

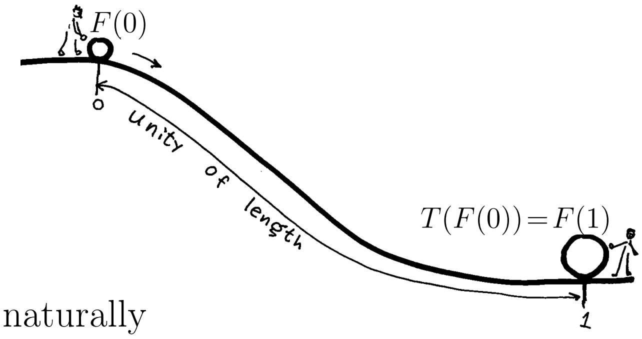


Figure 2.1: Snowball of mass $F(0)$ gets mass $T(F(0))\!=\!F(1)$ after to roll once from the hill.

Iterates and Superfunctions arise naturally at description of a sequence of similar transformations of any quantity in a homogeneous physical system. This transform may refer to accumulation of mass by a snowball, that rolls down the hill, covered with sticky snow (Fig. 2.1). This may be attenuation of a shock wave in one of the successive sections of automobile muffler. The transform may refer to change of intensity of light, passing one section of a laser amplifier or a saturable absorber. I hope, the Reader can suggest more examples of this kind.

Let the state of the system before the transformation be described with parameter x, and, after transformation - with parameter y; and let these two quantities be related with $y = T(x)$. Then, function T is called "transfer function". It describes the transformation of the signal in the system.

System, that performs the transformation, can be called "filter". Transformation of signal x in a filter with transfer function T can illustrated with expression

$$x \rightarrow \boxed{\text{filter}} \rightarrow y\!=T(x) \quad \text{[frame1]} \tag{2.1}$$

Output of one filter can be directed to the input of another identical filter. This can be expressed with

$$x \rightarrow \boxed{\text{filter}} \rightarrow T(x) \rightarrow \boxed{\text{filter}} \rightarrow T(T(x)) \quad \text{[frame2]} \tag{2.2}$$

One can write the similar expressions for combination of 3 filters and so on. In these notations, the number of filters combined is supposed to be a positive integer. The main idea of this Book is, that the number of filters has no need to be integer. In this chapter, the notations are suggested to describe this.

1 Use of superscript

From the point of view of physics, every iterate describes some combination of identical filters. Let each filter be characterised with the transfer function T. If we have signal z at the input, then, at the output, the signal is $T(z)$.

As it is mentioned in the Preamble of this chapter, the output of a filter can be directed to another similar filter, and the transfer function of the resulting combination can be expressed with

$$T(T(z)) = T^2(z) \quad \text{[IntroT2]} \tag{2.3}$$

In the right hand side of equation (2.3), the notation with superscript is used. I am not first to suggest this notation; Walter Bergweiler had used such a notation in the past century [26]. If there is superscript at name of some function, and the expression in this superscript can be interpreted as a number, then it is number of the iterate. The zeroth iterate of a function is supposed to be identical function (its value is the same as value of its argument):

$$T^0(z) = z \tag{2.4}$$

First iterate of some function T is the same function T, and the minus first iteration corresponds to the inverse function. For example, at the sinusoidal transfer function $T\!=\!\sin$, we have

$$\begin{aligned}
\sin^{-1}(z) &= \arcsin(z) && (2.5)\\
\sin^{0}(z) &= z && (2.6)\\
\sin^{1}(z) &= \sin(z) && (2.7)\\
\sin^{2}(z) &= \sin(\sin(z)) && (2.8)\\
\sin^{3}(z) &= \sin(\sin(\sin(z))) && (2.9)
\end{aligned}$$

and so on. In some textbooks, the notation is used (but not declared), when expression $\sin^a(z)$ has sense of $\sin(z)^a$. Such a notation leads to confusions: at $a=-1$, expression $\sin^a(z)$ might mean $\dfrac{1}{\sin(z)}$; but, according to another commonly used notations for the inverse function, it should mean $\sin^{-1}(z)\!=\!\arcsin(z)$.

In such a way, in this Book, the expression in the upper superscript at the name of a function is interpreted as number of iterate. However, if the superscript is just “prime”, almost vertical stick, it indicates the derivative; for example, $\sin'\!=\!\cos$; it is also usual notation.

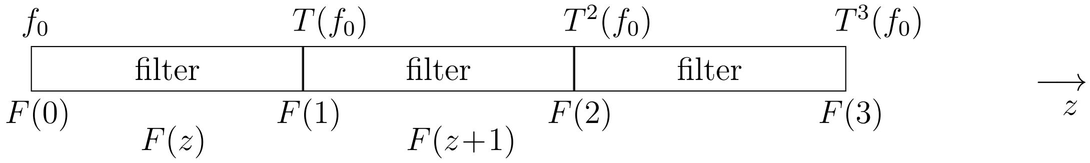


Figure 2.2: Combination of filters, each of them has transfer function T

2 Transfer equation

The use of the superscript allows to illustrate expressions (2.1) and (2.2) with figure 2.2. The figure shows the combination of identical filters with transfer function T each. I assume T to be some given, known holomorphic function. The range of holomorphism is supposed to be wide enough, to cover the needs of the following consideration.

At the left hand side of figure 2.2, in at the entry to the first filter, let the signal is characterised with some fixed value f_0. Then, after to pass the filter, it becomes $T(f_0)$; after to pass the second filter, it becomes $T^2(f_0)=T(T(f_0))$, and so on. These quantities are specified above the vertical bars, that mark the end on one filter and beginning of the next filter.

The notation can be even shortened, to count number of passes of the filter. Let f_0 be $F(0)$; let $T(f_0)$ be $F(1)$; let $T^2(f_0)$ be $F(2)$, and so on. In such a way, function F of non-negative integer argument can be defined.

Now I need the strong assumption. Let the filters, inside, are uniform, and act as some kind of continuous homogeneous nonlinear medium, that transfers the signal by some fixed (although, may be, not yet known) way; but, after to pass each section of the combined filter, the signal is transformed with known transfer function T.

We may consider coordinate z along this combined filter, and treat it as a continuous medium. It is convenient to choose the length of a single filter as a unit to measure the coordinate along the combined amplifier. Then, $F(z)$ may have sense of signal at coordinate z; for integer values of z, values of function $F(z)$ are already known. The question is, how to define, determine, evaluate function F for non-integer values of z.

If for some non-integer position z, for example, between zero and unity, the signal is $F(z)$, then, in the uniform medium, after a pass length unity along the composed filter, the signal should be transformed with the same transfer function T. This can be expressed with the transfer equation (1.1) mentioned in the Prologue. I repeat this equation:

$$F(z+1) = T(F(z)) \quad \text{[introeq]} \tag{2.10}$$

The transfer equation (2.10) specifies, how does the value of function F changes, while its argument gets the unity increment. In such a way, figure 2.2 indicates the physical meaning of the main equation in this Book; the figure suggests the physical implementation of the transfer equation (2.10).

3 Values of the argument

In this Book, I assume, that the signal, transferred through the filter, is characterised by a single number, real or complex. In principle, term "transfer function" allows generalisation to the multidimensional case; then the transfer function appears as a functional, and its argument may have sense of a vector or a function. As it is mentioned in the Preface, here I consider only the case of a single variable: it happened, that even the single-dimensional case causes a lot of confusions at the internet forums and discussions. It is methodically-incorrect, to develop the multidimensional generalisations, while the colleagues have doubts even in the single-dimensional case. According to the definition of science [67, 100] in the Prologue, first, the results bout the single-dimensional case should be presented, that still refute the commonly accepted point of view, that the recovery of the signal inside a homogeneous system from its transfer function is not possible [71, 72, 74, 75].

One of objections refers to the multitude of solutions. Consideration of complex values of argument of the transfer function allows to reduce the class of possible solutions (and sometimes even provides the uniqueness of the solution). In such a way, consideration of complex argument is more important, than analysis of some multidimensional signal. For this reason, in this Book, I assume, that the argument of the transfer function is a complex number. Also, I assume, that the transfer function is holomorphic, at least at some vicinity of some part of the real axis.

4 Nest

Figure 2.3: Nest

For iterates of functions, in algorithmic language Mathematica, there is special routine **Nest**. It requires 3 arguments. The first one indicates the name of function that should be iterated. The second argument indicates the initial value at the iterates. The third (and last) argument specifies the number of iterate. The call looks as follows:

$$\mathrm{Nest}[f, x, z] \tag{2.11}$$

In Mathematica, the arguments of a function appear in squared parenthesis; and expression (2.11) means $f^z(x)$. (However, term "Nest" may have also different meaning, as it is shown in figure 2.3.)

At least until year 2017, the implementation of Nest requires, that the last argument can be simplified to a positive integer constant. Even 0th and minus first iterates are prohibited. Perhaps, the designer assumed, that only the integer number of iterate may have sense and meaning.

Actually, the non-integer iterates do have sense, as it is mentioned in the preamble of this chapter. One more example is considered in the next section.

5 Fibre amplifier

This section describes the example, that shows the sense of the non-integer iterates. This example refers to the fibre amplifier. I type "fibre" in order to indicate, that the signal is confined in two directions, and only the power in the fibre as function of the coordinate along it is subject of consideration. A lot of physical effects are dropped out in this consideration: change of the spectral content of the signal, self-peaking, spontaneous emission, etc.. So, I consider here the simplest case, but for this model I want to get the exact solution.

Assume, some Manufacturer gives to some Physicist a piece of one meter long of the optical fibre amplifier, together with the system of pumping, and asks the Physicist to investigate, how the power of light inside grows during the amplification, but Manufacturer does not allow Physicist to cut the fibre to measure, what is inside. Physicist knows the only, that the fibre is uniformly pumped. We may assume, that some system of lateral delivery of pump is used [32, 33, 34, 41].

Physicist can measure the transfer function T of this fibre. With this transfer function, Physicist can say, what should be the transfer function of similar piece of fibre of length 2 meter. Assuming, that the source of pump is delivered together with each piece of the fibre, the transfer function of the piece of two meter is T^2; that of the 3 meter piece should be T^3 and so on; if z pieces of the fibre are combined, then, the transfer function is T^z.

Iterates T^z are pretty clear, while z is integer. But how about non-integer values? For example, what is transfer function to the piece of half meter long?

As it is mentioned in the previous section, in Mathematica, the transfer function of piece of z meter length could be expressed with $T^z(x) = \mathrm{Nest}[T, x, z]$. Unfortunately, such an expression is not yet interpreted correctly at non-integer z.

How to express the non-integer iteration of a given transfer function? These question can be analyzed with the transfer equation (2.10). The preliminary analysis is suggested in the next section.

6 Transfer equation and the Abel equation

Through this book, the transfer equation (1.1),(2.10) is repeated (and used) again and again, and, in particular, here:

$$F(z+1) = T\big(F(z)\big) \quad \text{[transfereq]} \tag{2.12}$$

I remind, that T denotes the transfer function, z may have sense of coordinate along direction of propagation of some signal, and function F expresses dependence of the signal on this coordinate. (However, coordinate z may have also any other meaning.)

In Laser Science, term “signal” denotes the wave (light), that is amplified; even if no information is transferred with this light. In this book, the length is measured in units of the length of the amplifier. The generalisation for the arbitrary units is straight-forward.

For given transfer function T, solution F of the transfer equation (2.12) is called “superfunction”. The inverse function $G = F^{-1}$ is called “Abel function” or “abelfunction” for the same transfer function T. The abelfunction G satisfies the Abel equation

$$G(T(z)) = G(z) + 1 \quad \text{[abeleq]} \tag{2.13}$$

Equations (2.12) and (2.13) can be deduced from each other other by the change of variable. Just replace z to $F(z)$ in equation (2.13) and apply function F to both side of the resulting equation. The Readers are united to make this exercise or to look for it at some Wikipedia [96]. I hope, the Soviet veterans, that remove articles about the USSR, will be stopped before they begin to vandalise the Abel equation.

Assume, that the superfunction F and the abelfunction G are somehow constructed. This gives key to the iterate T^z of the transfer function, and the number z of the iteratie has no need to be integer. This iterate can be expressed as follows:

$$T^z(x) = F(z + G(x)) \quad \text{[Tzx]} \tag{2.14}$$

In order to show an example of application of formula (2.14), return to the story about Manufacturer and Physicist from the previous section. Assume, the Physicist has found the physically-meaningful solution F of the transfer equation (2.12), and has constructed the inverse function, id est, abelfunction $G = F^{-1}$. Then, Physicist can express the transfer function F of the amplifier of arbitrary length z by formula (2.14).

For transfer function T, its superfunction F and the abelfunction G appear as two sponges of a wrench for some screw, as a tool, that allows to rotate the screw for any rational angle.

7 Multiplicity of solutions

Solution F of equation (2.12) is not unique. One can reduce multitude to solution, specifying its value at zero (or at any other point the Reader likes); id est, choose some number F_0 and add the requirement that

$$F(0) = F_0 \quad \text{[f0]} \tag{2.15}$$

In many cases, the constant F_0 does not affect the shape of iterates of the transfer function; transform

$$\tilde F(z) = F(z+x_0)\ , \quad \tilde G(z) = G(z)-x_0 \qquad \text{[zo]} \tag{2.16}$$

for some x_0 and substitution $F \to \tilde F$, $G \to \tilde G$ into equation (2.14) gives the same iterate, as the initial F, G do.

However, even after addition of requirement (2.15), the solution is not unique. If F is solution, superfunction, then another solution (another superfunction) can be constructed with

$$\tilde{f}(z) = F\big(z + \theta(z)\big) \quad \text{[tif]} \qquad (2.17)$$

where θ is periodic holomorphic function with period unity. At $\theta(0)=0$, even condition (2.15) is preserved.

The different superfunctions F, with their abelfunctions $G=F^{-1}$, give different iterates of the transfer function T. For this reason, one may think, that the non-integer iterates of a function have no meaning. In particular, in century 20, the colleagues had believed, that the half iteration of factorial, denoted with $\sqrt{!}=\text{Factorial}^{1/2}$, has no meaning.

Function $\sqrt{!}$ is especialy interesting, because since 1950, it is used as logo of the Physics Department of the Moscow State University [36]. That logo is shown in figure 2.4, borrowed from the Russian article [37]. Only in 2009, the physical and mathematical meaning of this iterate and this logo had been reported [62], when the apparatus of superfunctions had been constructed. The uniqueness of the superfunction of factorial (and, therefore, that of the half iterate of factorial) is provided by the additional requirement of holomorphism and behaviour at infinity: transformation (2.17) reduces the range of holomorphism of superfunction and that of the reconstructed non-initeger iterates.

Figure 2.4: $\sqrt{!}$ as emblema

The problem of evaluation of non-integer iterate in some physically-meaningful way arises not only at the phenomenological consideration of an idealised amplifier. Similar problem appears at the analysis of stability of jets; in the simple (single-dimensional) approximation, the appearance and disappearance of instability by the Pomeau-Manneville scenario [19, 53] can be described with some specific quadratic transfer function. Similar equation arise at the analysis of stability of attractors [20]. At least for the single-dimensional models, one can construct abelfunction and superfunction [69], and, hence, the non-integer iterates of the transfer function.

Observation of similarities in construction of superfunctions allows to formulate the problems about superfunctions and the goal of this Book. This is suggested in the next section.

8 Formulation of the problem

Suppose we have some transfer function T, holomorphic in a wide range of values of the argument, and we are interested in its iterates.

What additional conditions should be imposed on superfunctions F (or to the abelfunction $G = F^{-1}$) as solution of the transfer equation (2.12) (or that of the Abel equation (2.13)) to provide the unique solution?

How to evaluate the superfunctions and the abefunction, that satisfying the conditions chosen?

How to verify that namely this solution has the physical meaning?

The goal of this book is to answer these questions. I try to represent the answers in the most simple and explicit form. In this book, I retell the previously published articles [54, 61, 63, 69, 65, 79, 91, 88, 85]; the idea is to release the Reader from the need to drill the original publications.

The next chapter suggests examples of superfunctions and corresponding abelfunctions; other chapters suggest general algorithms for calculation superfunctions and the application to physical problems. Description of the state of a physical system with a single parameter is already a strong approximation, and at least in this case, it is desirable to represent the solution in a simple and exact form.

Perhaps, I should discuss meanings of term "simple". Applying to a function, this term may get new meaning, indicating, that values of the function belong to some finite set [1]. In the similar way, the adjective "dry", being applied to a vine or Martini, significantly change its meaning (figure 2.5). In this Book, term "simple" indicates, that the function is easy to understand, its definition is short and its implementation is fast; as for the set of values, the functions are supposed to be holomorphic and, therefore, should have values from the continuous set of complex numbers.

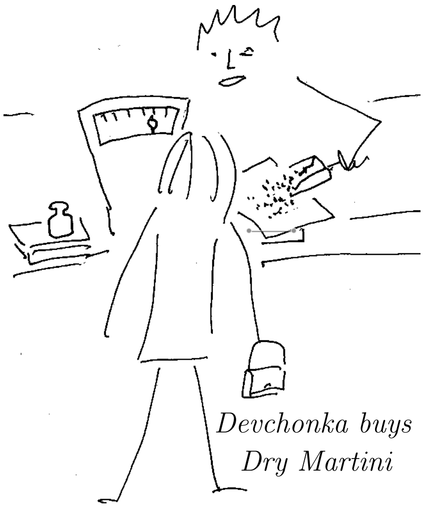


Figure 2.5: Dry martini

On this point I finish the general speculations about superfunctions and turn to the specific examples. In the following chapter, the examples of transfer function and its superfunction and abelfunction are considered.

[1] `http://en.wikipedia.org/wiki/Simple_function` In the mathematical field of real analysis, a simple function is a real-valued function over a subset of the real line, similar to a step function.

Chapter 3

Examples of superfunctions

Before to calculate the superfunctions and abelfunctions for transfer functions of general kind, it worth to see the cases, when the superfinction and the abelfunction can be easy expressed in terms of already known special functions. These examples are considered in this chapter. It can be considered as continuation of the Inroduction: the most of functions mentioned here are known since the school course of algebra.

Superfunction F for the transfer function T is solution to the transfer equation (2.12); I repeat it here:

$$F(z+1) = T(F(z))$$

In the next section, several known solutions are presented in Table 3.1. Then, in the following sections and chapters, these functions are considered with more details.

1 Table of superfunctions

There is analogy between the table of superfunctions and tables of integrals. In both cases, the direct operation is, in certain sense, more difficult, than the inverse operation. If one knows the indefinite integral of some function, and this integral is expressed in terms of elementary functions in a compact form (the writing fits the width of the column of the table), then, one can calculate the integrand with known rules of differentiation. In the similar way, if the superfunction F and the abelfunction $G = F^{-1}$ are known, the transfer function can be expressed with

$$T(z) = F(1 + G(z))$$

One of ways to built-in the table of integrals is the building of table of derivatives. One takes any short combination of basic elementary

Table 3.1: Examples of superfunctions, $T(z) = f\Big(1 + g(z)\Big)$

	$T(z)$	$f(z)$	$g(z) = f^{-1}(z)$	comment
1	c	c		
2	$z{+}1$	$b + z$	$z - b$	$b \in \mathbb{C}$
3	$b + z$	$bz + c$	$(z - c)/b$	$b \neq 0$
4	$bz + c$	$b^z + \frac{c}{1-b}$	$\log_b\!\big(z - \frac{c}{1-b}\big)$	$b{\neq}0,\ b \neq 1$, [86]
5	b^z	$\mathrm{tet}_b(z)$	$\mathrm{ate}_b(z)$	[54, 61, 79]
6	z^b	$\exp(b^z)$	$\log_b\!\big(\ln(z)\big)$	(4.19), $b{>}0,\ b{\neq}{-1}$
7	$-a^2/z$	$a\tan\left(\dfrac{2}{\pi}z\right)$	$\dfrac{2}{\pi}\arctan(z/a)$	$a > 0$
8	$\dfrac{z}{c+z}$	$\dfrac{1-c}{1-c^z}$	$\log_c\left(1 - \dfrac{1-c}{z}\right)$	$c \neq 0\ ,\ \ c \neq 1$
9	$\dfrac{z}{1+z}$	$1/z$	$1/z$	$f{=}g;\ T^n(z){=}\frac{z}{1+nz}$
10	$\ln(b{+}\mathrm{e}^z)$	$\ln(bz)$	e^z/b	$b \neq 0$
11	$(a^b{+}z^b)^{1/b}$	$az^{1/b}$	$(z/a)^b$	$a{>}0,\ b{\neq}0$
12	$2z\sqrt{1{-}z^2}$	$\sin(\pi 2^z)$	$\log_2\!\big(\arcsin(z)/\pi\big)$	
13	$2z\sqrt{1{+}z^2}$	$\sinh(2^z)$	$\log_2\!\big(\ln\big(z + \sqrt{z^2{+}1}\,\big)/\pi\big)$	
14	$2z^2 - 1$	$\cos(\pi 2^z)$	$\log_2(\arccos(z))$	
15	$2z^2 - 1$	$\cosh(\pi 2^z)$	$\log_2\!\big(\ln\big(z + \sqrt{z^2{-}1}\,\big)/\pi\big)$	
16	$2z/(1{-}z^2)$	$\tan(2^z)$	$\log_2(\arctan(z))$	
17	$2z/(1{+}z^2)$	$\tanh(2^z)$	$\log_2\left(2\ln\left(\dfrac{z{+}1}{z{-}1}\right)\right)$	
18	$\mathrm{Factorial}(z)$	$\mathrm{SuFac}(z)$	$\mathrm{AuFac}(z)$	(8.11), (8.19); [65]
19	$b\,z\,(1{-}z)$	$\mathrm{LogisticSequence}_b(z)$	$\mathrm{LogisticSequence}_b^{-1}(z)$	(7.8), (7.19); [69]
20	$\mathrm{Doya}^t(z)$	$\mathrm{Tania}(tz)$	$(z + \ln(z) - 1)/t$	(5.11), (5.3)
21	$\mathrm{Keller}^t(z)$	$\mathrm{Shoka}(tz)$	$\mathrm{ArcShoka}(z)/t$	(5.14), (5.18)
22	$\sin(z)$	$\mathrm{SuSin}(tz)$	$\mathrm{AuSin}(z)/t$	(12.8), [91]
23	$\mathrm{zex}(z){=}z\exp(z)$	$\mathrm{SuZex}(tz)$	$\mathrm{AuZex}(z)/t$	(11.1) [88]
24	$\mathrm{tra}(z){=}z{+}\mathrm{e}^z$	$\mathrm{SuTra}(tz)$	$\mathrm{AuTra}(z)/t$	(20.1) [88]
	$P(T(Q(z)))$	$P(f(z))$	$g(Q(z))$	$P(Q(z)){=}z$

functions and differentiates it. If the result can be simplified to fit the width of the column of the table, then it is declared as "function that can be analytically integrated".

In the similar way. one can deal with superfunctions. Any elementary function f, for which the inverse function $g=f^{-1}$ also can be expressed as elementary function, can be taken as an example. Then, expression

$$t(z) = f(1+g(z)) \quad \text{[tfg]} \tag{3.1}$$

should be considered. Sometimes, after simplification, this expression fits the width of the column of the table. Then, this t can be declared as "transfer function, for which the superfunction can be expressed analytically", id est, also in terms of special functions. So, this f is declared as its superfunction, and g is declared as corresponding abelfunction. The most of table 3.1 is built-up in such a way.

Not all transfer functions can be expressed as elementary functions, and not all superfunctions can be expressed through special functions. And not all abelfunctions. In this Book, I describe methods, how to deal with these cases. In general, any holomorphic function can be treated as transfer function, and the only edges of the range of holomorphism may limit the construction of the corresponding superfunction and the abelfunction. If the superfunction and the abelfunction are constructed, supplied with the specific names, described and implemented, they can be treated as a special functions. I call any function as "special function", if (and only if) the properties are revealed, described and the algorithm of the precise evaluation is supplied.

Aiming the application in physics (and, perhaps, in other sciences), I am interested, first, in those functions, that can be evaluated quickly. These functions can be used to construct new functions, and used to describe various phenomena. One can use them in the similar way, as one use other special functions (sin, bessel, erfc, etc.) These functions appear in the Table 3.1 with names; I supply also the number of formula or the cite, to indicate, where the function is described, where can one find the algorithm for the evaluation.

Table 3.1 appears as the basic toolbox for evaluation of non-integer iteration. In the following chapters, I describe, how the non-trivial superfunctions from the table can be constructed and evaluated. However, first, it worths to check properties of elementary superfunctions.

2 Construction of elementary superfunctions

As it is mentioned above, searching for elementary superfunctions, it worth to begin not with a transfer function, but with the superfunction and the abelfunction, applying formula (3.1). As an example, I show, how the 12th line of the Table 3.1 can be verified in language Mathematica:

```
f[z_] = Sin[Pi 2^z]
g[z_] = Log[ArcSin[z]/Pi]/Log[2]
f[g[z]]
T[z_]=2 z Sqrt[1 - z^2]
Simplify[T[z] - f[1 + g[z]]]
```

In Mathematica, the argument of function should appear in squared brackets. No other trick, specific for Mathematica, is used; the verification can be performed in other languages too.

Table 3.1 collects only the simplest (already described) superfunctions, They can be modified, using the last row of the Table. Any pair of mutually inverse functions P and Q determines the transform, that can be applied to any of previous rows, giving the new transfer function with corresponding superfunction and abelfunction.

The scaling transform relates the quadratic transfer function at raws 14 and 15 of the table with logistic operator (also quadratic transfer function) in raw 19, while parameter $b = 4$; in this case, the logistic sequence is expressed with elementary function and its generalisation to the non-integer values of the argument is trivial [69]. However, the superfunciton can be constructed also for other values of b; with these superfunctions, the iterates dan be calculated. This case is considered below in chapter 7.

A special case of transformation of superfunction is displacement of its argument for a constant. In some cases, it is difficult to recognise this shift. For example, superfunctions in 14th and 15th rows of the table correspond to the same transfer function $T(z)=2z^2-1$; these superfunction are related with translation of the argument for constant $\mathrm{i}\pi \ln(2)/2$. Similar relations take place also for other superfunctions. For superfunctions of $\exp_b$ for $1<b<\exp(1/\mathrm{e})$ discussed in [61], this case is mentioned in Chapter 9.

I invite the Reader to add some new raws to the table, following the trick above: choose some special function f, for which $g=f^{-1}$ is also implemented, and try to simplify t, determined by equation (3.1).

This chapter only declares the superfunctions. Some elementary superfunctions are considered with more details in the next chapter.

Chapter 4

Elementary superfunctions

In order to use superfunctions, described in this chapter, the Reader has no need to know, that they are superfunctions. In the similar way, during 40 years, Joudrain has no need to know, that he speak in prose [2]. However, many properties of elementary superfunctions are the same, as properties of other, nontrivial superfunctions, that cannot be easily expressed through elementary functions. In order to show these properties, in this chapter I consider elementary superfunctions.

I would not like the colleagues to say, that *the formalism of superfunctions is too complicated* [75]. So I begin with very simple example, with linear function.

Figure 4.1: M. Joudain: *These forty years now, I've been speaking in prose without knowing it!* [2]

1 Iteration of linear function

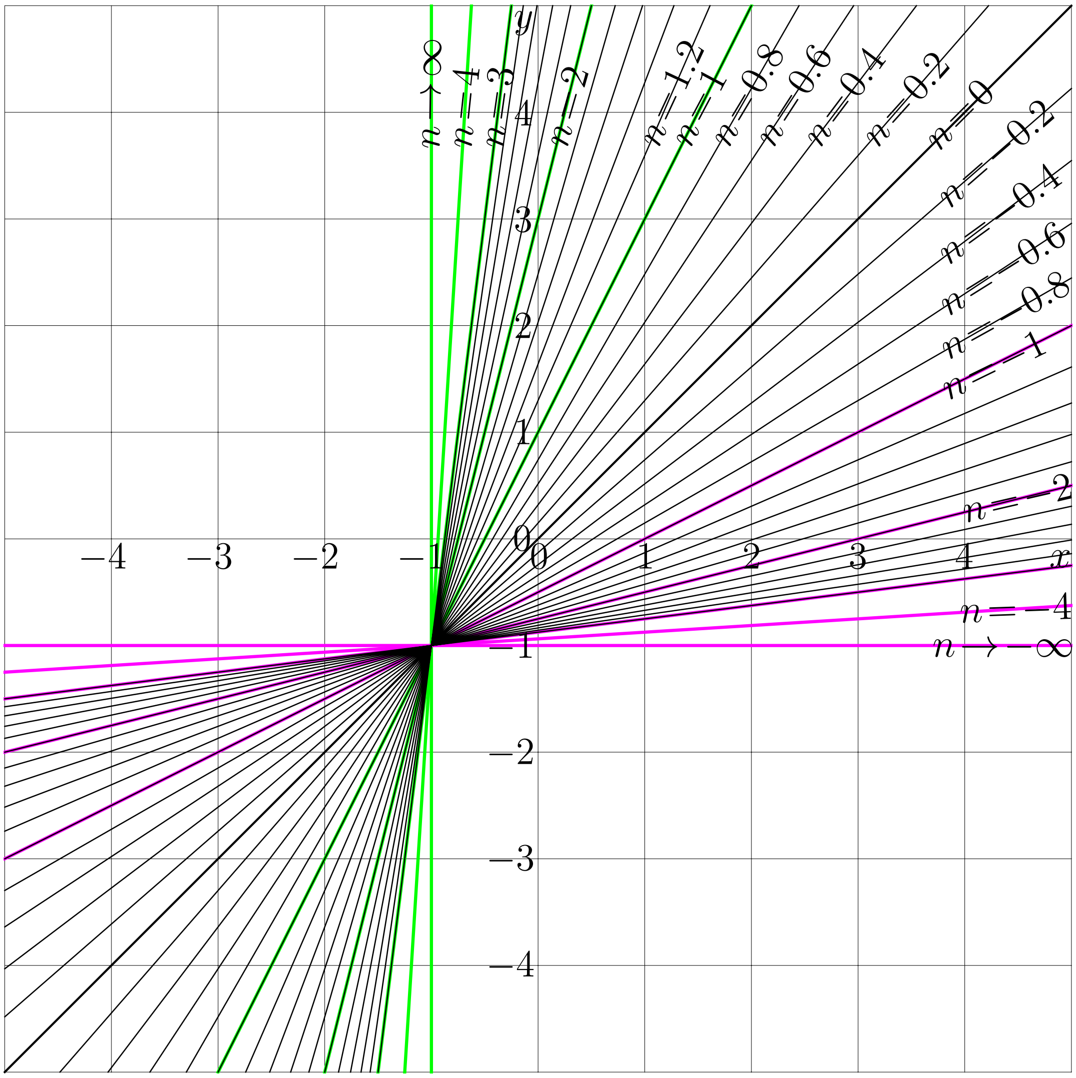
Consider the linear transfer function

$$T(z) = A + Bz \quad \text{[TABz]} \tag{4.1}$$

where A and B are constants. For $A\!=\!1$, $B\!=\!2$, iterates of this function are shown in figure 4.2. Graphic $y\!=\!T^n(x)$ is plotted versus x for various values of the number n of the iterate.

The nth iterate of function T can be written as follows:

$$T^n(z) = A\frac{B^n - 1}{B - 1} + B^n z \quad \text{[TABzn]} \tag{4.2}$$



http://mizugadro.mydns.jp/t/index.php/File:Itelin125T.jpg

Figure 4.2: Iterates of linear function (4.1) at $A\!=\!1,\ B\!=\!2$; $y\!=\!T^n(x)$ versus x for various n [itelin125]

This representation is used to plot figure 4.2. The graphics are straight lines. They cross at the point (L,L) of the coordinate plane. The fixed point L is determined by the equation $A+BL=L$, that gives

$$L = A/(1-B) \quad \text{[Llin]} \tag{4.3}$$

At $B\!\to\!1$, the fixed point runs to infinity, and the graphics in the analogy of figure 4.2 become parallel. For $A\!=\!1,\ B\!=\!2$, this is case of figure 4.2, we get value $L\!=\!-1$; so, lines in figure 4.2 cross in point $(-1,-1)$.

Representation (4.2) can be obtained with the general formula (2.14) with superfunction

$$F(z) = A\frac{1-B^z}{1-B} \quad \text{[FTABz]} \tag{4.4}$$

and abelfunction

$$G(z) = \log_b\left(1 + \frac{B-1}{A}z\right) \quad \text{[GTABz]} \tag{4.5}$$

In the special case $B=1$, the transfer function has no fixed points and representations (4.2), (4.4), (4.5) cannot be used. For this case, id est, for $T(z) = A+z$, superfunctions and abelfunctions can be written as follows:

$$F(z) = Az~, \quad G(z) = z/A \tag{4.6}$$

Their combination gives

$$T^n(z) = F(1+G(z)) = A~(n+z/A) = An+z \quad \text{[Anz]} \tag{4.7}$$

Those, who still think, that the formalism is too complicated [75], are invited to check the deduction above and verify, that this case corresponds to the level of a junior high school.

To iterate the linear function (4.1), the use of superfunction (4.4) and abelfunction (4.5) can be qualified with term "use sledgehammer to crac a nut", as it is shown in figure 4.3. However, this example is important: it is simple and it shows, how the superfunction and abelfunction can be used together to express iterates with formula (4.7).

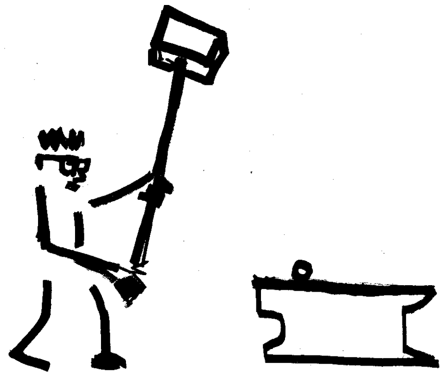


Figure 4.3: Mathematician uses sledgehammer to crack a nut

2 Rational function

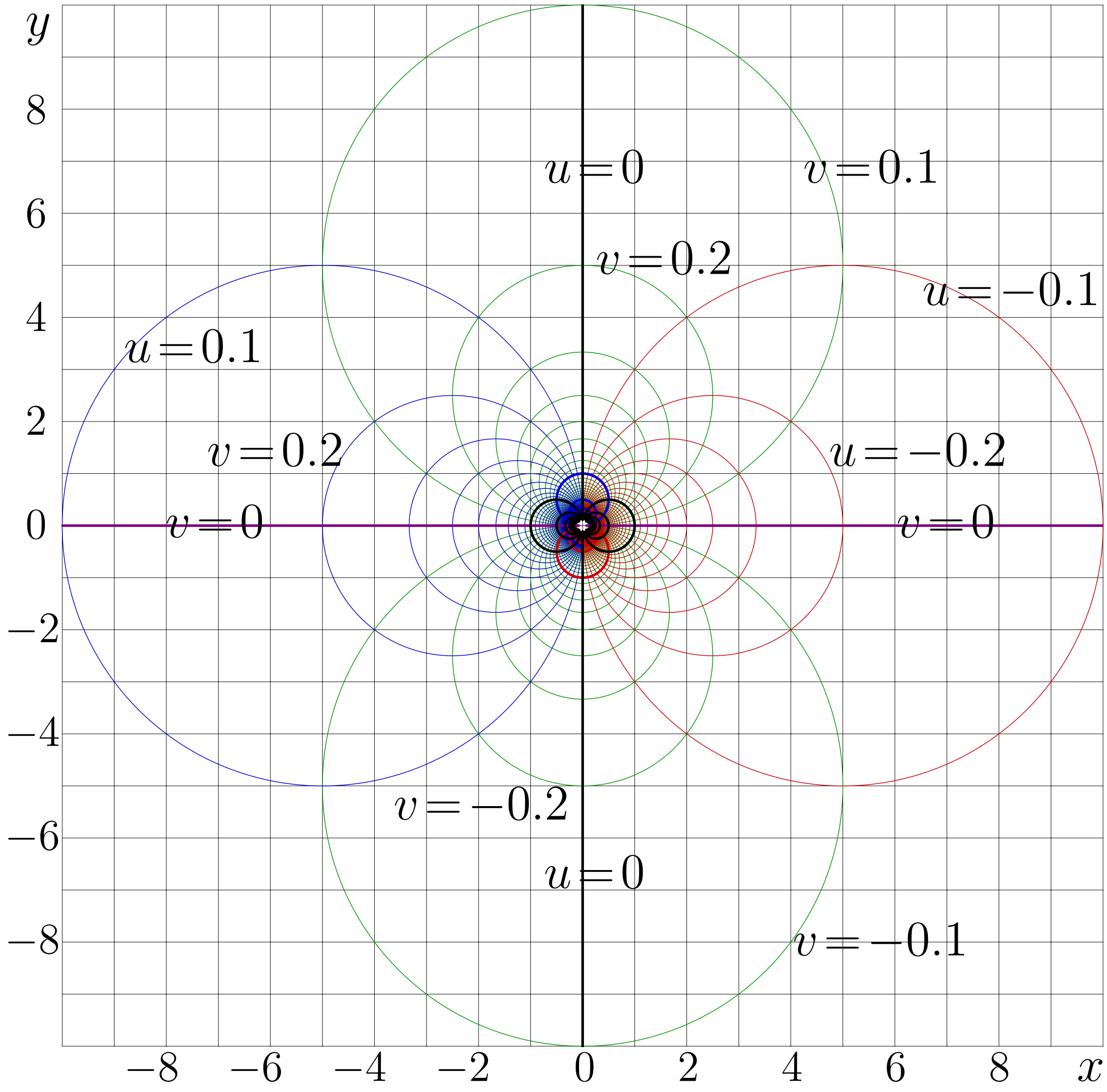
The linear fraction, or "rational function" can be considered as generalisation of the linear function. Let

$$T(z) = \frac{U+Vz}{W+z} \quad \text{[Tuvwz]} \tag{4.8}$$

where U, V and W are constant parameters. First, as an example, consider function

$$T(z) = -1/z \quad \text{[Tzm1z]} \tag{4.9}$$

This example corresponds to $U=-1$, $V=0$, $W=0$ in formula (4.8). Negative value of U is chosen in order to have positive derivative at the



`http://mizugadro.mydns.jp/t/index.php/File:F1xmapT.jpg`

Figure 4.4: $u+\mathrm{i}v=T(x+\mathrm{i}y) = 1/(x+\mathrm{i}y)$ [f1xmap]

positive values of the argument; iterates of a growing function are easy to interpret in terms of modelling of physical process. Iterates of decreasing function, contrary, imply dealing with complex values. I am interested mainly in the applications for physics; so, I consider mainly the growing transfer functions and growing real-holomorphic superfunctions.

Compex map of function T by equation (4.9) is shown in figure 4.4. For this function, the levels of constant real part and levels of constant imaginary part are circles, and all these circles pass through the origin of coordinates.

For real values of the argument, iterates of function T by (4.9) are shown in figure 4.5. Lines $y=T^n(x)$ are plotted versus x for various values of n. Below I describe, how these iterates are calculated.

For transfer function T by (4.9), the superfunction F and abelfunction

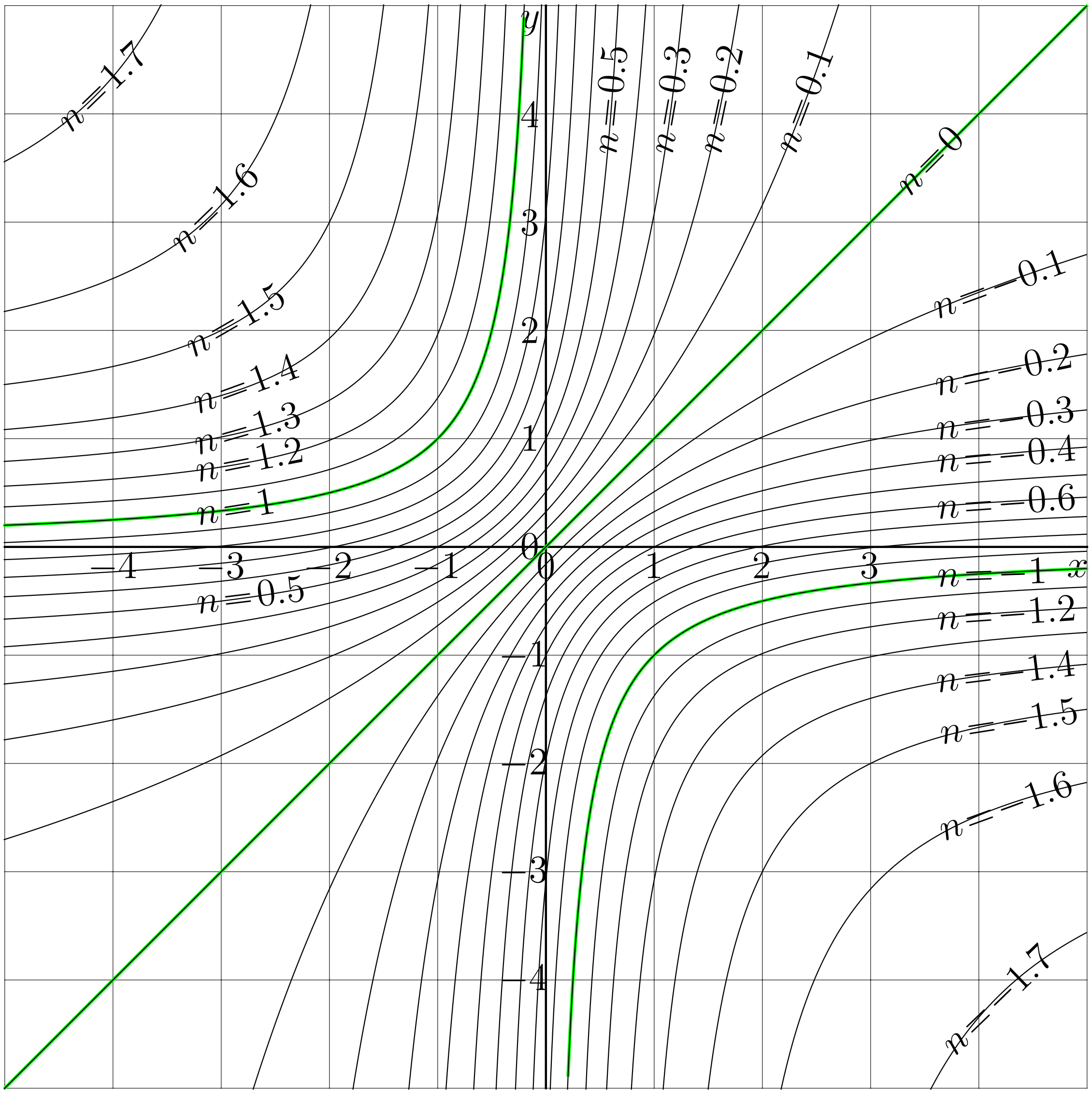


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Figure 4.5: $y\!=\!T^n(x)$ for $T(z)\!=\!-1/z$ by formula (4.12) at various n

G can be written as follows:

$$F(z) \;=\; \tan\left(\frac{\pi}{2}z\right) \qquad (4.10)$$

$$G(z) \;=\; F^{-1}(z) = \frac{2}{\pi}\arctan(z) \qquad (4.11)$$

Then, the nth iterate of the transfer function $T^n(z) = F(n+G(z))$ appears to be

$$T^n(z) = \frac{-1-\cot\left(\frac{\pi}{2}n\right)\,z}{-\cot\left(\frac{\pi}{2}n\right)\,+\,z} \quad \text{[linfrac1ite]} \qquad (4.12)$$

This case is represented in row 7 of table 3.1. Expiression (4.12) is used to plot figure 4.5.

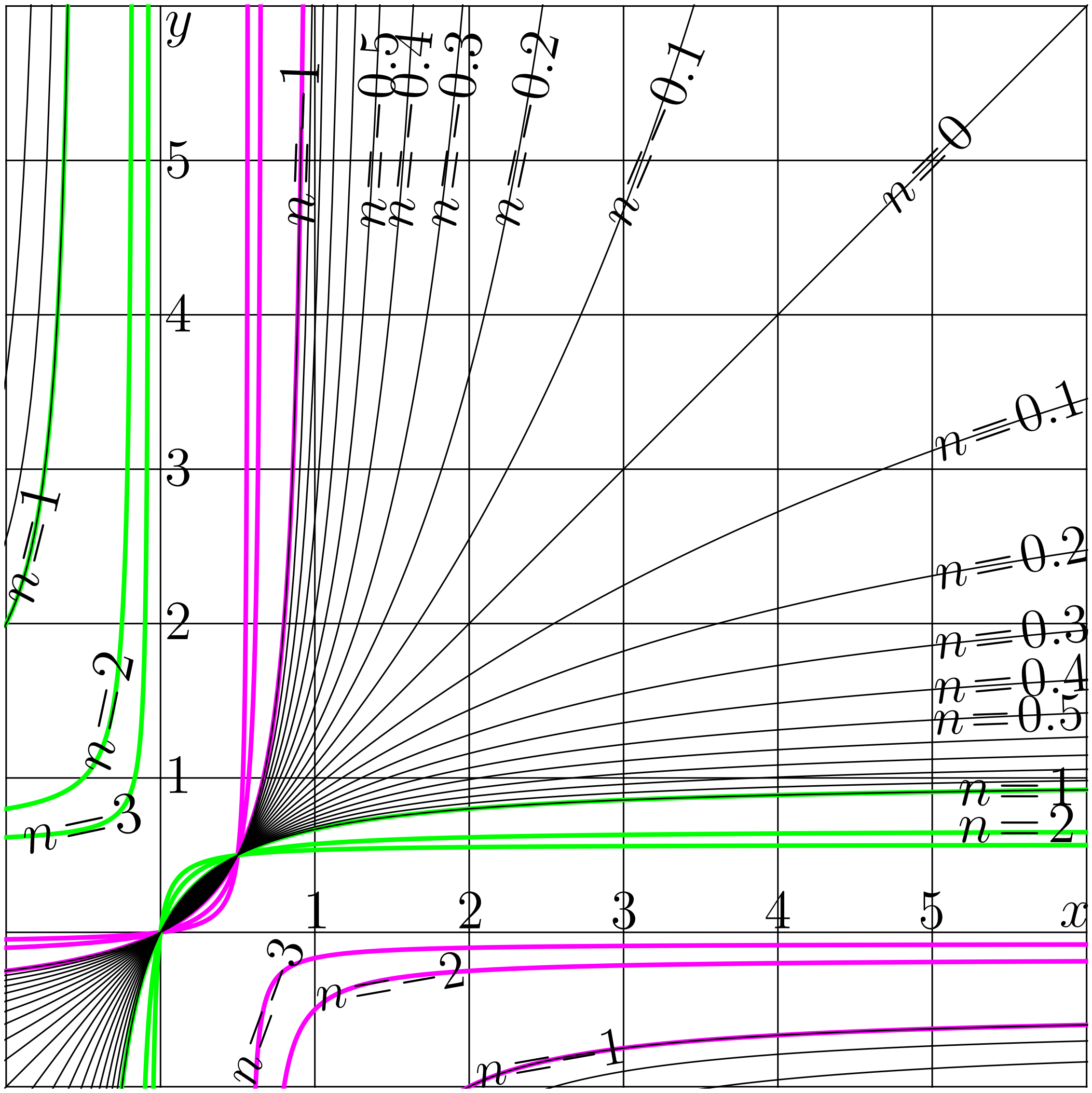

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Figure 4.6: $y=t^n(x)$ by (4.13) at $c=0.5$ [c05]

Consider one more special case of formula (4.8). Let

$$t(z) = \frac{z}{c+z} \quad \text{[tzcz]} \tag{4.13}$$

where c is constant. Then

$$t^n(z) = \frac{z}{c^n + \frac{1-c^n}{1-c} z} \quad \text{[tnzc]} \tag{4.14}$$

Iterates of function t for $c=0.5$ are shown in figure 4.6. The same for $c=1$ are shown in figure 4.7, and the same for $c=2$ are shown in figure 4.8. Below I show, how these iterates are evaluated.

For the transfer function t by (4.13), superfunction f can be written as

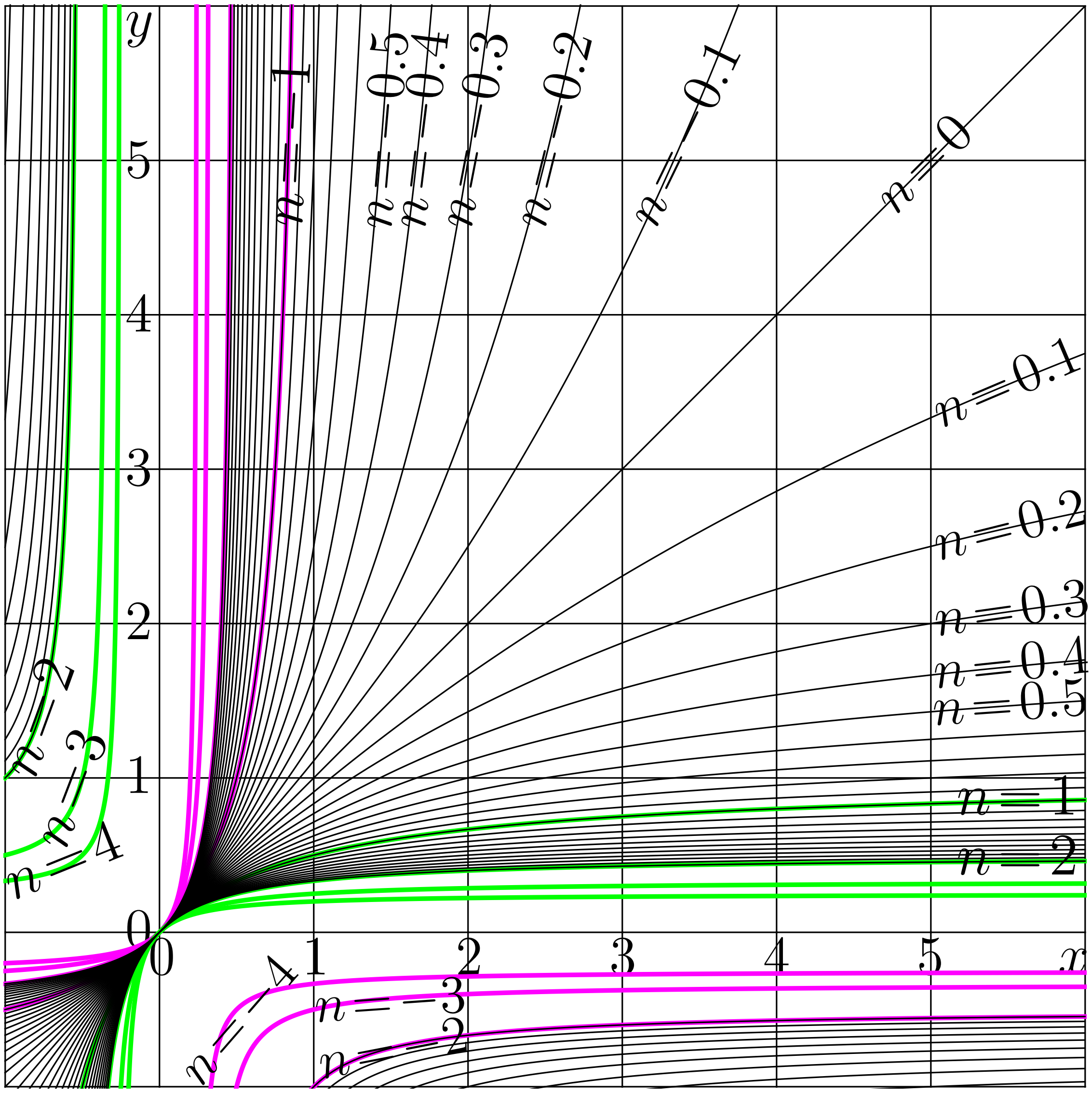

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Figure 4.7: $y=t^n(x)$ by (4.13) at $c=1$ [c10]

follows:

$$f(z) = \frac{c-1}{c^z - c} \quad \text{[fzfrac]} \tag{4.15}$$

Corresponding Abel function $g = f^{-1}$ is

$$g(z) = \log_c\left(1 + \frac{c-1}{z}\right) \quad \text{[gzfrac]} \tag{4.16}$$

Expressions (4.15) and (4.16) are used to evaluate $t^n(z) = f(n+G(z))$ and plot figures, in particular, figures 4.6, 4.7 and 4.8. The Reader is invited to plot iterates of function t by formula (4.15) at other values of c too.

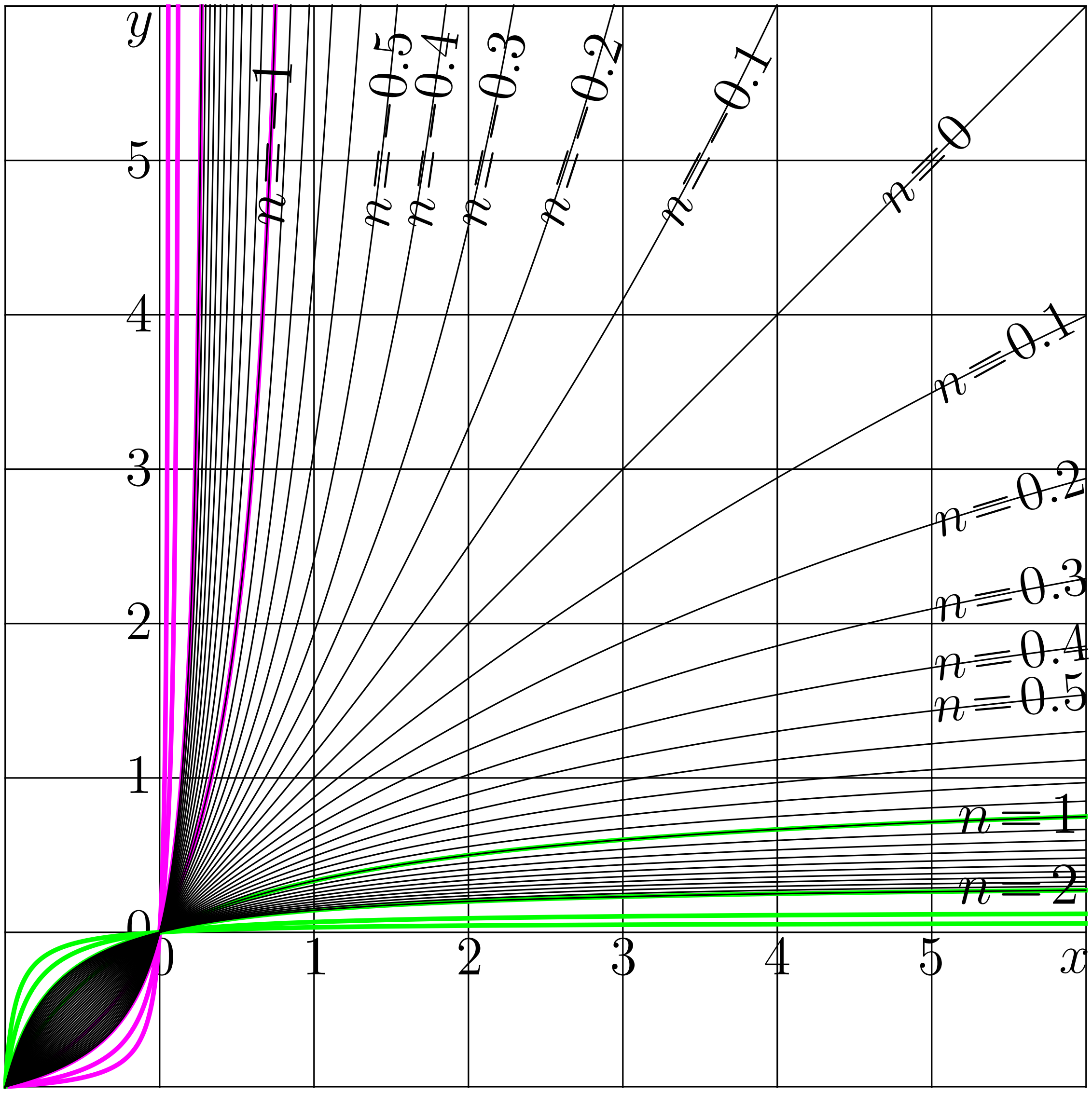

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Figure 4.8: $y\!=\!t^n(x)$ by (4.13) at $c\!=\!2$ [c20]

The iterates of the rational function of real argument show the smooth transition from function t to the identity function and then to inverse function t^{-1}. All the curves in figures 4.6, 4.7 and 4.8 pass through the fixed point. This value is mapped to itself at the iterates of the transfer function. This property is not specific feature of the rational function. Other functions, considered on the following chapters, have the same property.

For construction and uniqueness of superfunctions in the following sections, it is important to consider them in the complex plane. The complex maps help to understand properties of these functions. Following

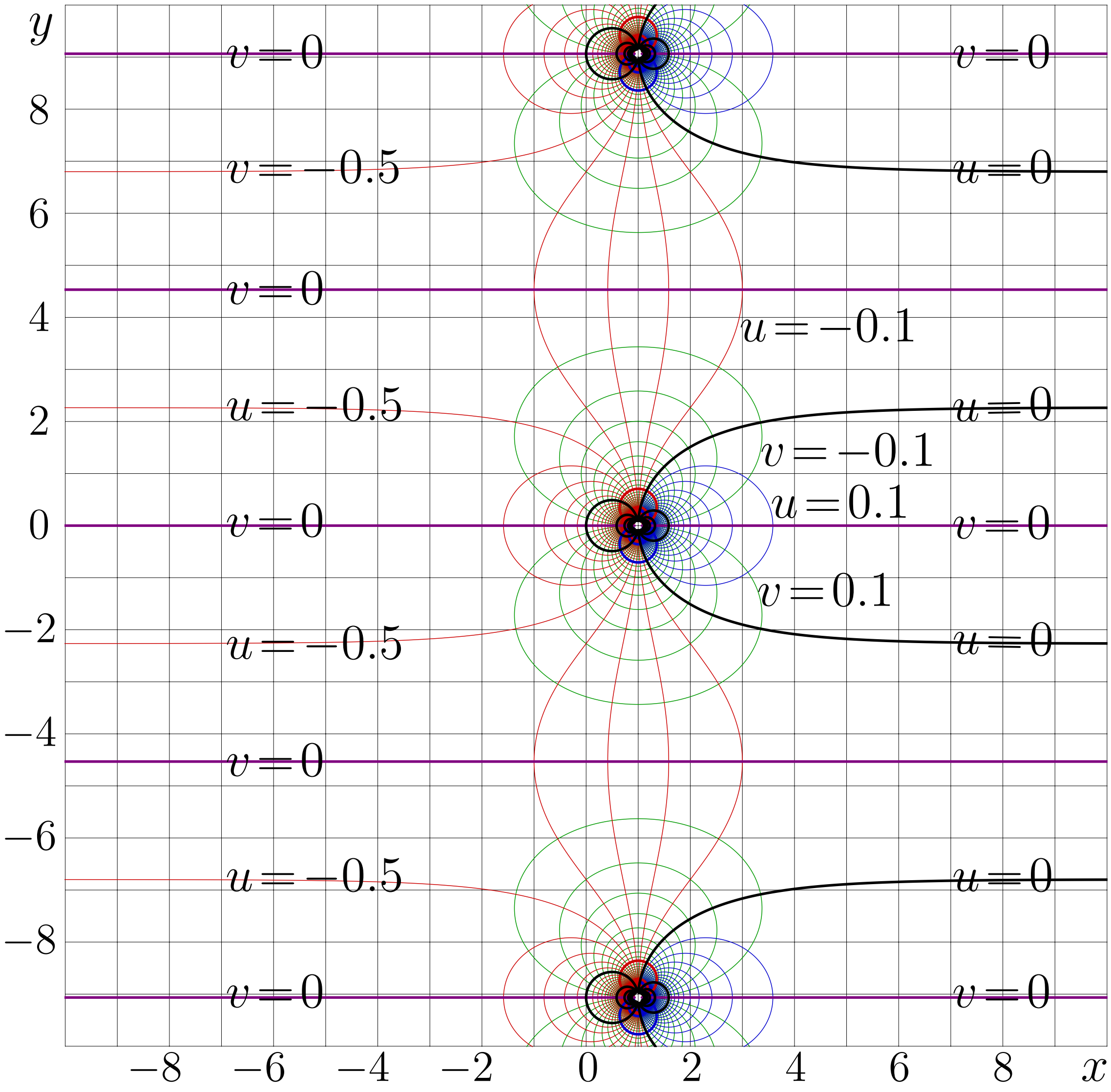

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Figure 4.9: $u+\mathrm{i}v = f(x+\mathrm{i}y)$ by (4.15) at $c=2$ [fzfracmap]

the idea to begin with simple examples, I suggest the complex maps of superfunction f and abelfunction g. For $c=2$, the complex map of superfunction f by formula (4.15) is shown in figure 4.9. Similar map of the Abel function g is shown in figure 4.10.

Function f by formula (4.15) is periodic; its period

$$P = 2\pi\mathrm{i}/\ln(c) \tag{4.17}$$

At $c=2$, this period $P=2\pi\mathrm{i}/\ln(2)\approx 9.06472\,\mathrm{i}$. Vertical size of figure 4.9 covers a little bit more that two periods of this function. For real c, the period is pure imaginary; the map reproduces itself at the translations for integer factor of $|P|$ along the imaginary axis.

The inverse function $g=f^{-1}$ is expressed by formula (4.16). It is Abel

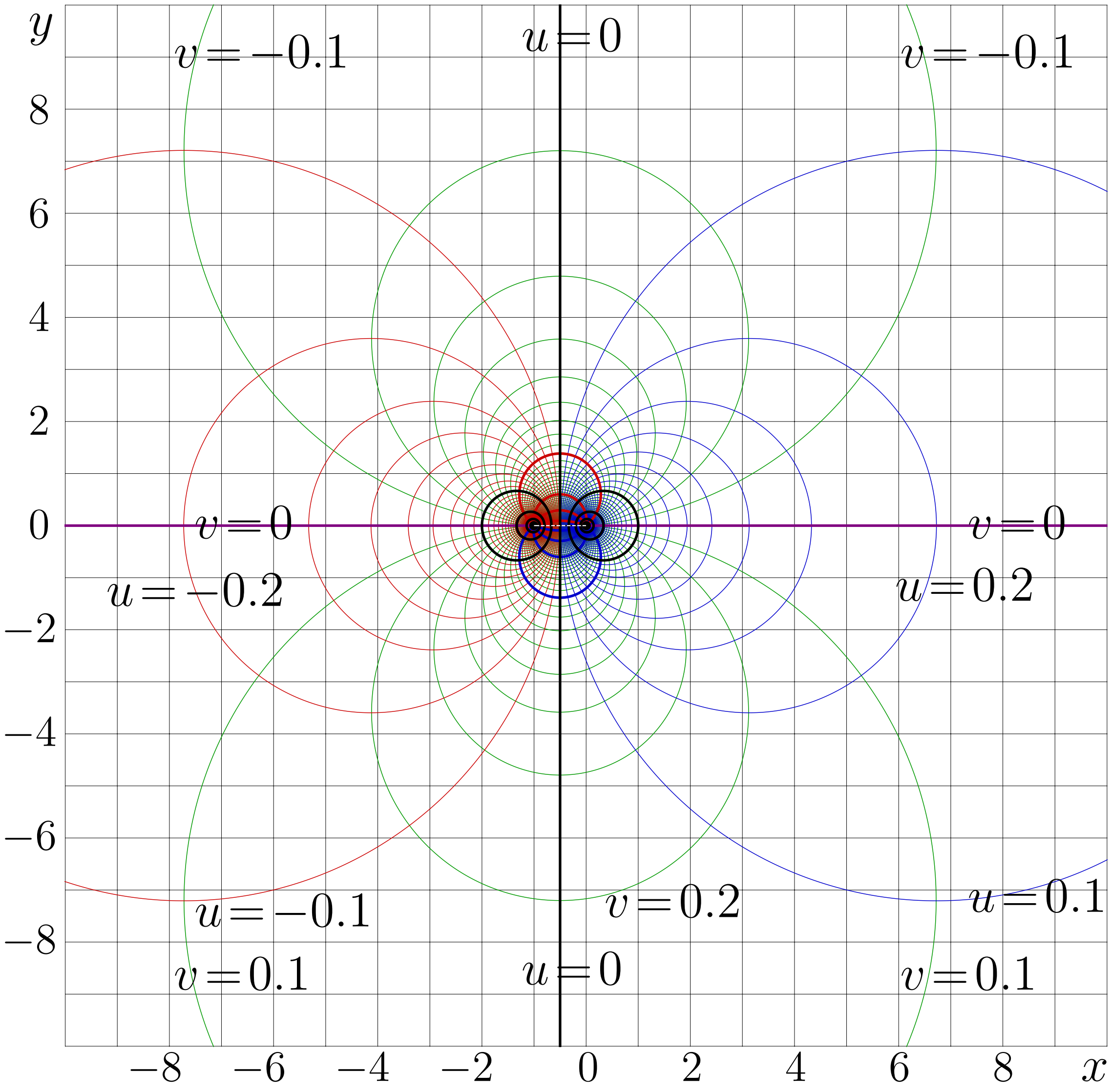

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Figure 4.10: $u\!+\!\mathrm{i}v = g(x\!+\!\mathrm{i}y)$by (4.16) at $c\!=\!2$ [gzfracmap]

function for the transfer function t. Abelfunction g is shown with its complex map in figure 4.10 for the same value of parameter, $c=2$, as function f in figure 4.9

Function g has two branch points, $c-1$ and zero. For real c, the cut line between these two points belongs to the real axis. Equilnes are symmetric with respect to reflections from line $\Re(x) = (c-1)/2$.

With superfunction f and abelfunction g by formulas (4.15) and (4.16), iterates of the transfer function t can be written as usually,

$$t^n(z) = f(n + g(z)) \quad \text{[tnzfg]} \tag{4.18}$$

The readers are invited to check, that this representation agree with expression (4.14). It is better to do this exercise for the simple. Then it will be easier to reproduce the similar exercises for other superfunctions in the following sections.

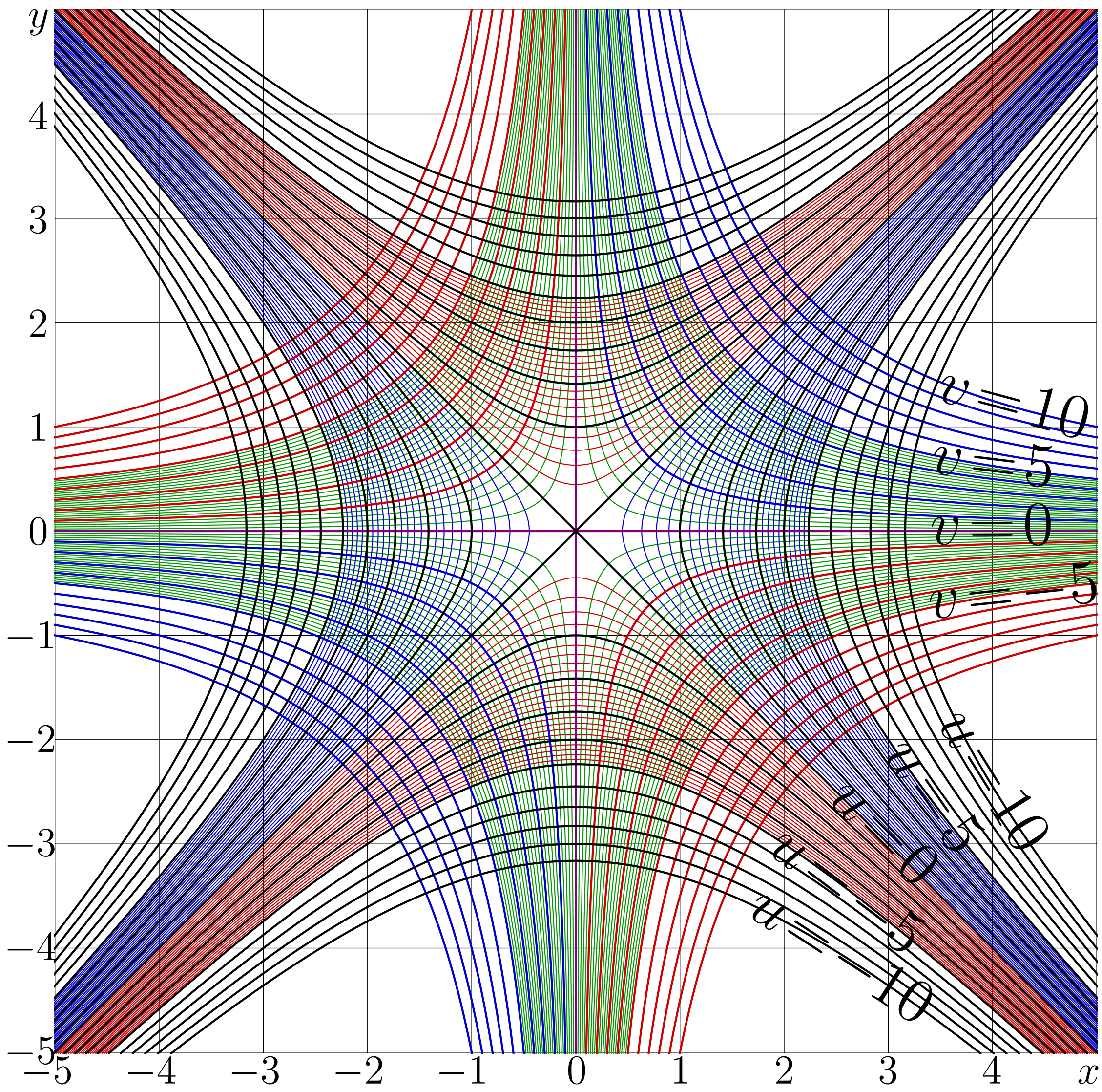

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Figure 4.11: $u+\mathrm{i}v = T(x+\mathrm{i}y) = (x+\mathrm{i}y)^2$, by (4.19) for $a=2$ [z2itmap]

3 Power function and its iterates

Consider the power function

$$T(z) = \mathrm{Pow}_a(z) = z^a \quad \text{[Pow]} \quad (4.19)$$

For $a=2$, the map of function T is shown in figure 4.11. Those, who are looking for some superpower, should be especially interested in this section: Here, I suggest the superfunction of the power function, id est, the superpower function, as it is shown in figure 4.12.

Figure 4.12: Superpower

Consider iterates of the power function T by (4.19), they can be written

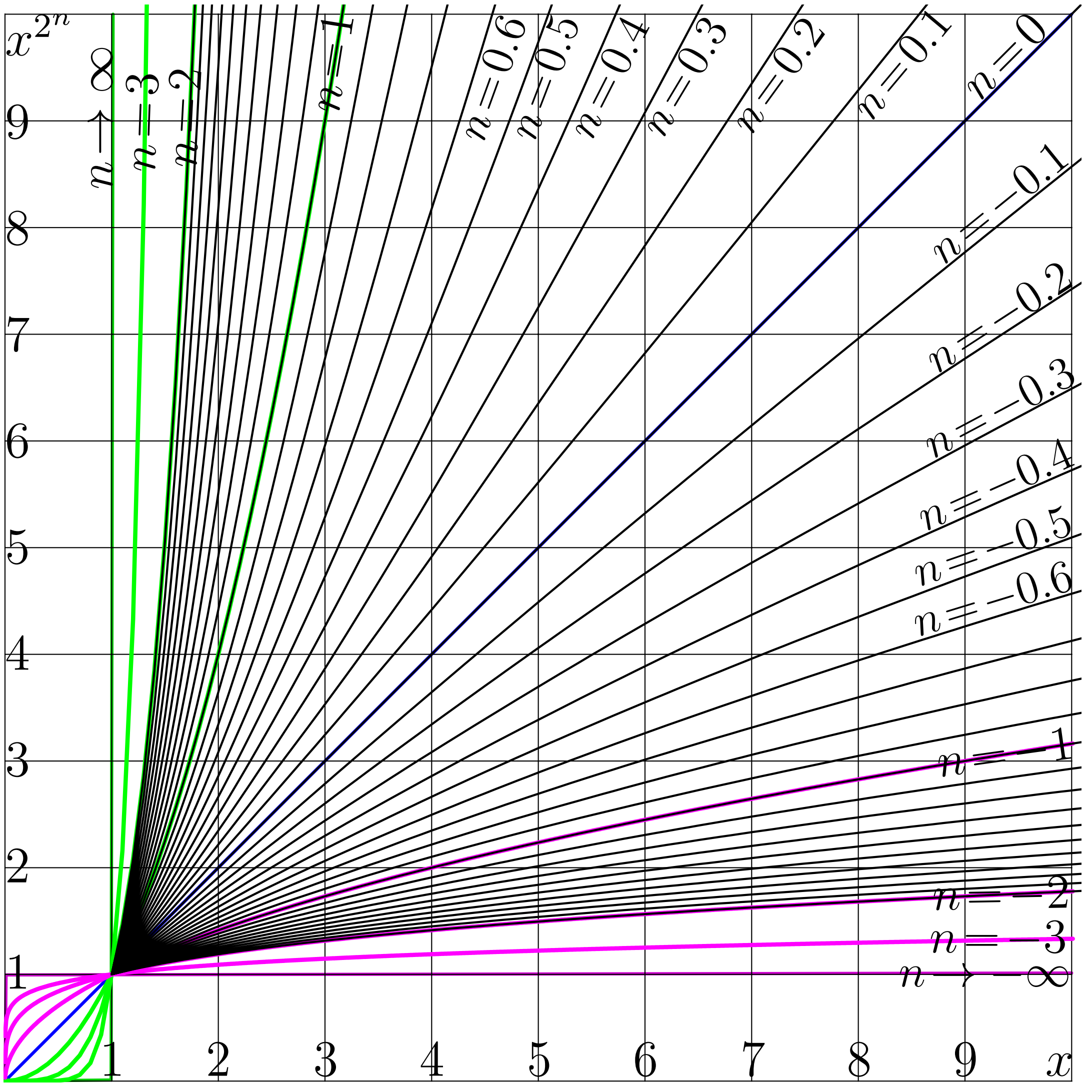

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Figure 4.13: $y=T^n(x)=\mathrm{Pow}_2{}^n(x)$ для различных n. [IterPowPlot]

as follows:

$$T^n(z) = \mathrm{Pow}_a{}^n(z) = z^{a^n} = \underbrace{\mathrm{Pow}_a\Big(\mathrm{Pow}_a\big(...\mathrm{Pow}_a(z)..\big)\Big)}_{n \text{ evaluations of function } \mathrm{Pow}_a} \qquad (4.20)$$

Figure 4.13 shows T^n by formula (4.20) for $a=2$ at various n. Iterates of the power function can be expressed also with the general formula,

$$T^n(x) = F\Big(n + G(x)\Big) \quad \text{[againTc]} \qquad (4.21)$$

where F is superfunction of the transfer function T and $G=F^{-1}$ is the abelfunction.

For the power function, the superfunction (id est, the superpower function) can be written as follows:

$$F(z) = \exp\Big(\exp(\ln(a)\, z)\Big) = \exp^2\big(\ln(a)\, z\big) \quad \text{[powF]} \tag{4.22}$$

Inverting this representation, one can get the abelfunction; id est, the "abelpower" function,

$$G(z) = \ln\Big(\ln(z)\Big)/\ln(a) = \ln^2(z)/\ln(a) \quad \text{[powG]} \tag{4.23}$$

These formulas correspond to the 6th row of table 3.1.

For the transfer function T by (4.19), iteration (10.16) can be simplified,

$$T^n(z) = \exp^2\Big(\ln(a)\big(n+\ln^2(z)/\ln(a)\big)\Big) = z^{a^n} \quad \text{[zbc]} \tag{4.24}$$

leading to the 3d expression in equation (4.20).
In such a way, iterates of the transfer function T by (4.19) appear to be function of the similar kind (also power function). I invite the Reader to plot the complex maps of superpower function F by (4.22), abelpower function G by (4.23) and iterates of power function by (4.24).

For $T(z)\!=\!z^a$, the simple relations take place:

$$T(z^b) = T(z)^b \quad \text{[fnz1]} \tag{4.25}$$

At $b\!=\!a\!=\!2$, in addition, the following relation hakes place:

$$T^b(z) = T(z)^b \quad \text{[fnz2]} \tag{4.26}$$

There is common confusion (the wrong public opinion), that some simple equivalents of relations (4.25), (4.26) should take place for other values of b and for other functions too. In particular, one of critics of the publication about half iteration of factorial [62] had insisted, that $\mathrm{Factorial}^{1/2}(z) = \mathrm{Factorial}(z)^{1/2}$, and I was not successful explaining, that it is not the case, even if notation $\sqrt{\mathrm{Factorial}} = \mathrm{Factorial}^{1/2}$ is used. The factorial is considered in this Book in the Chapter 8. However, before to deal with factorial, I would like to consider simple example. The power function is one of the simple examples. I invite the Reader to check, that even for $a\!=\!3$, relation (4.26) is not valid (except of some specific values of z).

Having the explicit representation, one can calculate all the iterates necessary, including the non-integer iterates. But not so many elementary functions are inverse of some other elementary functions, Perhaps, it is better to refer to special functions. The simple example of the non-elementary function is considerd in the next chapter.

Chapter 5

Tania and Shoka

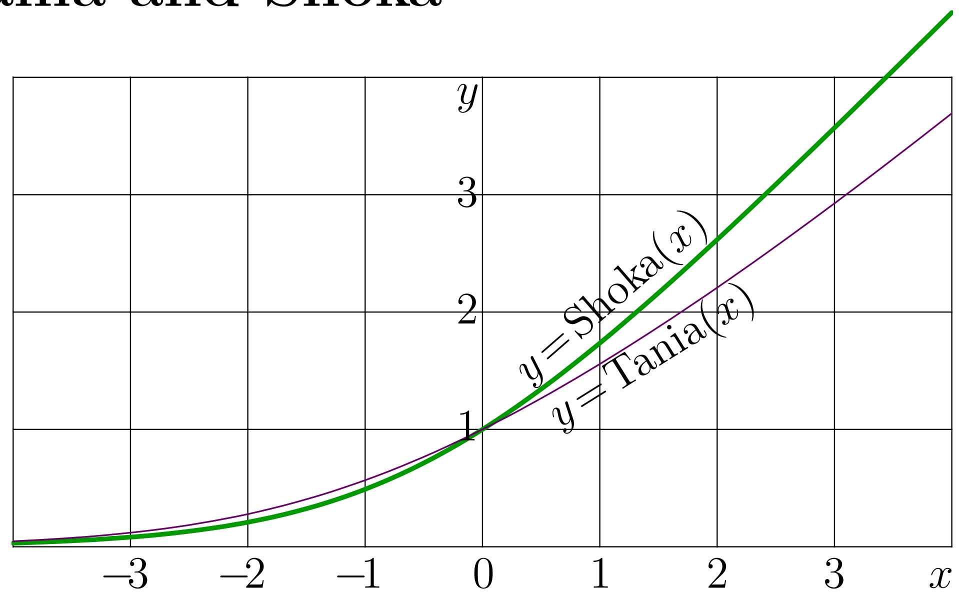

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Figure 5.1: Functions Tania and Shoka by (5.1) and (5.2) [shokataniaplot]

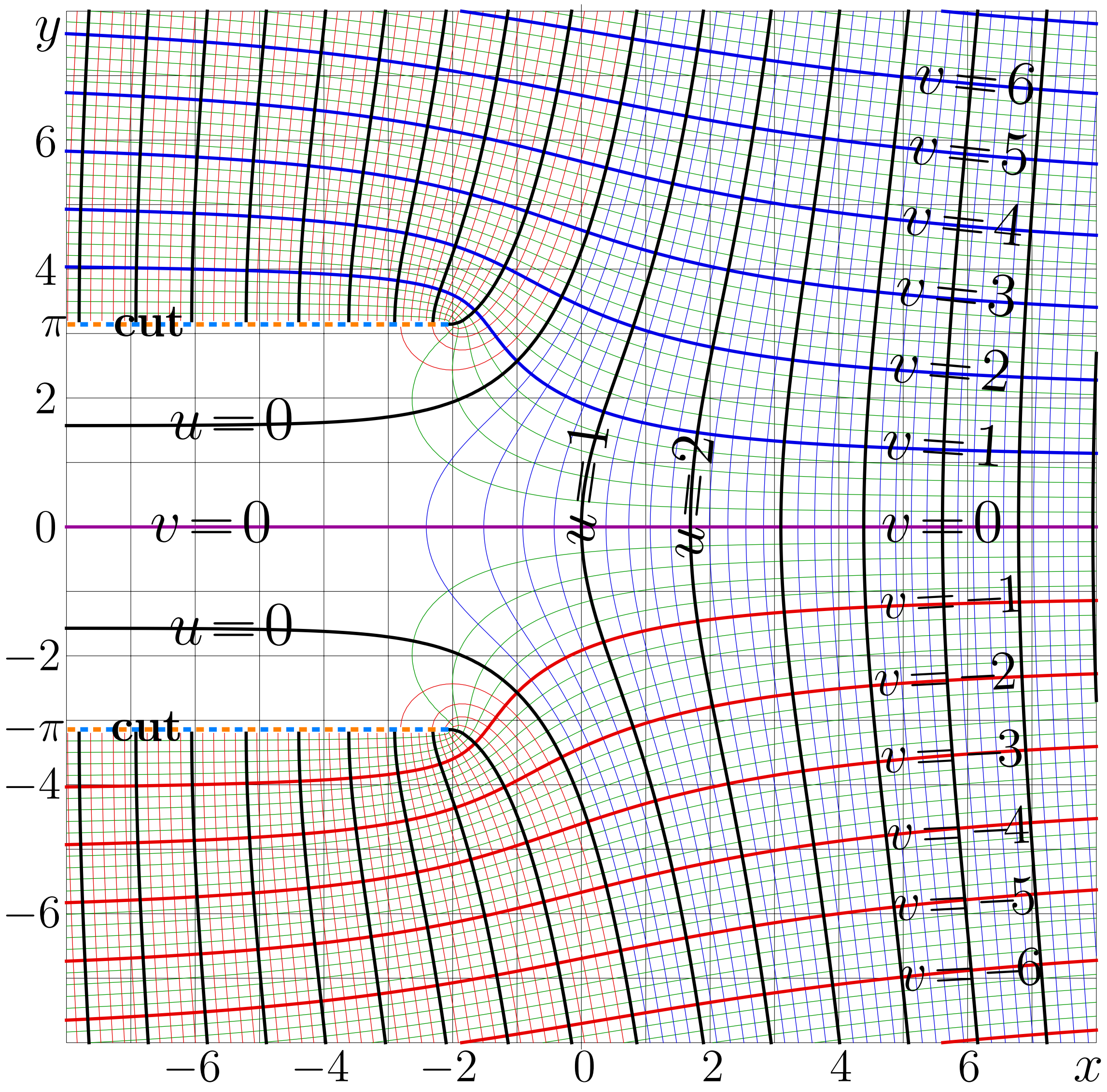
This chapter considers functions, that have applications in laser science. I call them Tania and Shoka. For some vicinity of the real axis, these functions can be defined as follows:

$$\mathrm{Shoka}(z) = z + \ln(\mathrm{e}^{-z} + \mathrm{e} - 1) \quad \text{[shoka0]} \tag{5.1}$$

$$\mathrm{Tania}(z) = \mathrm{WrightOmega}(z+1) \quad \text{[Tania0]} \tag{5.2}$$

At $|\Im(z)| \geq \pi$, function WrightOmega(z) behaves in a way I dislike. To avoid confusions, for the function I like, I use name Tania. Below I define functions Shoka and Tania for the complex argument. However, the representations (5.1) and (5.2) are sufficient to plot Shoka and Tania in figure 5.1.

This chapter retells the contents of articles [84, 85]. This chapter should be especially important for the narrow specialists, who works in the laser science.



`http://mizugadro.mydns.jp/t/index.php/File:TaniaContourPlot100.png`

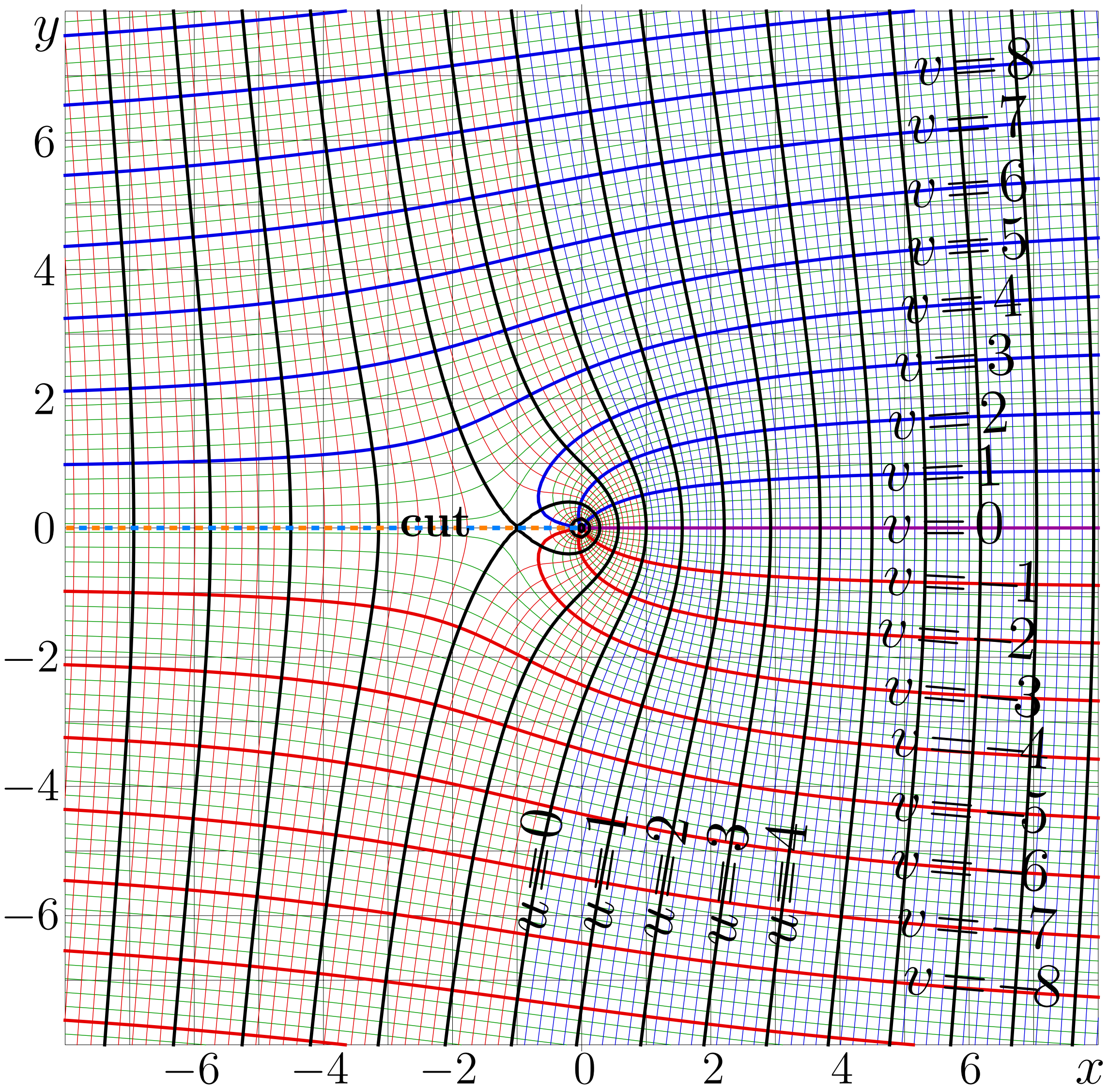
Figure 5.2: Complex map of function Tania, $u+\mathrm{i}v = \mathrm{Tania}(x+\mathrm{i}y)$ [TaniaMap]

1 Tania and Arctania

Let function Tania be solution f of the differential equation

$$f'(z) = \frac{f(z)}{1+f(z)} \quad \text{[taniaprim]} \tag{5.3}$$

with additional condition $f(0)=1$, where contour of integration of (5.3) for $f(z)$ goes first from zero to imaginary part of z along the imaginary axis, and then, along the line, parallel to the real axis, goes to point z. Figure 5.1 shows function Tania of real argument. For moderate values of imaginary part of the argument, solution $f=\mathrm{Tania}$ of equation (5.3) is expressed through the special function WrightOmega with equation (5.2). Complex map of function Tania is shown in figure 5.2.



http://mizugadro.mydns.jp/t/index.php/File:ArcTaniaMap.png

Figure 5.3: $u+\mathrm{i}v=\mathrm{ArcTania}(x+\mathrm{i}y)$ [ArcTaniaMap]

For real values of argument, Tania is positive and grows monotonously. Toward the negative values of the argument, Tania decays exponentially. At zero, Tania grows with tangent $1/2$. At large positive values of the argument, Tania grows in a way, similar to the linear function with tangent unity.

Function Tania, shown in figure 5.2, has two branch points, $-2 \pm \mathrm{i}\pi$. The cut lines are directed parallel to the real axis toward its negative direction; they are determined by specification of path of integration of equation (5.3).

Complex map of the inverse function, id est, $\mathrm{ArcTania} = \mathrm{Tania}^{-1}$ is shown in figure 5.3. It can be expressed as elementary function,

$$\mathrm{ArcTania}(z) = z + \ln(z) - 1 \qquad [\mathrm{ArcTaniaz}] \tag{5.4}$$

There are no complex constants in the representations of functions Tania and Arctania; these functions are real-holomorphic:

$$\mathrm{Tania}(z^*) = \mathrm{Tania}(z)^* \ , \quad \mathrm{ArcTania}(z^*) = \mathrm{ArcTania}(z)^* \tag{5.5}$$

Functions Tania and ArcTania look similar to the linear function at large values of the argument; the lines of levels of constant real part and those of constant imaginary part form almost rectangular grid. Functions ArcTania also have almost linear asymptotic.

The almost linear asymptotic behaviour of Tania(z) at large $|z| \gg 1$ holds for the most of the complex plane, except the strip $\Re(z) < 0$, $|\Im(z)| \le \pi$. In the strip $|\Im(z)| < \pi$, at $\Re(z) \to -\infty$, function Tania(z) decays exponentially, this agrees with graphic of this function for real argument at figure 5.1.

For moderate values of the imaginary part of the argument, function Tania can be expressed through the known special function WrightOmega [12, 115] with equation (5.2). In particular, one has no need to make any difference between Tania(z) and WrightOmega(z+1) for real z.

For the efficient evaluation of Tania, its asymptotic expansions can be used. The whole complex plane can be covered with these expansions.

At large values of the argument, Tania can be expanded as follows:

$$\mathrm{Tania}(z) = z + 1 - \ln(z) + \frac{\ln(z) - 1}{z} + \frac{\ln(z)^2 - 4\ln(z) + 3}{2z^2} + .. \tag{5.6}$$

The effective small parameter of the expansion (5.6) is $\ln(z)/z$. For negative values of $\Re(z)$, this expansion is valid while $|\Im(z)| > \pi$ However, this representation does not work between the cut lines at figure 5.2. For the representation of Tania in this half-strip, I define the new variable $\varepsilon = \exp(1+z)$; then Tania can be expanded as follows:

$$\mathrm{Tania}(z) = \varepsilon - \varepsilon^2 + \frac{3}{2}\varepsilon^3 - \frac{8}{3}\varepsilon^4 + \frac{125}{24}\varepsilon^5 + O(\varepsilon^6) \tag{5.7}$$

The expansions above are not good for the moderate values of the argument, and especially poor is the approximation in vicinity of the branch points. For the branch point, the following expansion takes place:

$$\mathrm{Tania}(z) = -1 + 3t - 3t^2 + \frac{3}{4}t^3 + \frac{3}{10}t^4 + \frac{9}{160}t^5 + .. \tag{5.8}$$

where $\quad t = \mathrm{i}\sqrt{\frac{2}{9}\big(z\!+\!2\!-\!\pi\mathrm{i}\big)}$.

In addition, one may use the Taylor expansion at zero:

$$\mathrm{Tania}(z) = 1 + \frac{z}{2} + \frac{z^2}{16} - \frac{z^3}{192} - \frac{z^4}{3072} + \frac{13z^5}{61440} - \frac{47z^6}{1474560} + .. \qquad (5.9)$$

With the representations above, for every point of the complex plane, for Tania(z), one can get the "zeroth" approximation, let it be called s_0, with few correct significant figures. Then, in order to get the maximal precision for the "complex double" variable, it is sufficient to make three or four iterations by the Newton method

$$s_{n+1} = s_n + \frac{z - \mathrm{ArcTania}(s_n)}{\mathrm{ArcTania}'(s_n)} \qquad \text{[ssTania]} \qquad (5.10)$$

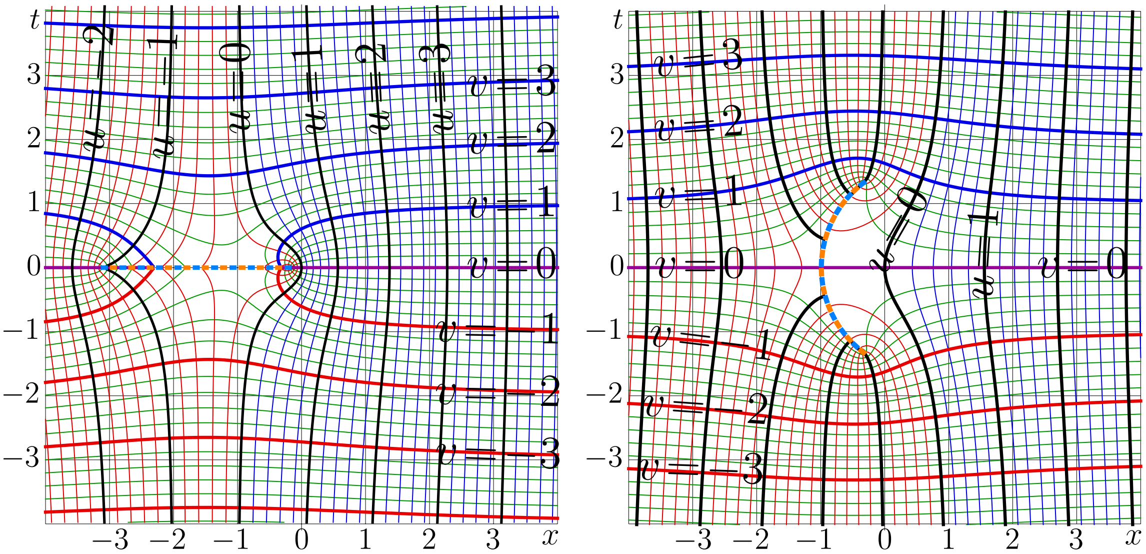
where $\mathrm{ArcTania}'(z)\!=\!1\!+\!1/z$ is derivative of ArcTania. I remind that, ArcTania by equation (5.4) is ementary function. In such a way, Tania can be evaluated quickly and precisely. The C++ implementation of function Tania with this algorithm for "complex double" variables is loaded as `http://mizugadro.mydns.jp/t/index.php/Tania.cin`

Function Tania is simpler than function WrightOmega. If necessary, Tania can be used to evaluate WrightOmega.

Function Tania has simple physical meaning. It represents dependence of intensity of light on the length of its propagation in laser with simple model of the active medium. The argument has sense of coordinate measured in units of the inverse increment of the low signal. The returned value has sense of intensity, measured in the units of saturation.

In this Book, function Tania is used many times. In this chapter, Tania appears as superfunction of the special function Doya, considered in the nest section. I use the trick, mentioned in the chapter 3: First, I choose superfunction, id est, Tania, and then, I construct the transfer function for it; I call this transfer function "Doya". I am grateful to Valérie Doya (Figure 5.4), she kindly allowed to use her name for the function, that appeared during our collaboration at Nice in 2010.

Figure 5.4: V.Doya [116]



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http://mizugadro.mydns.jp/t/index.php/File:Doyam10map4bT100.png

Figure 5.5: $u+\mathrm{i}v=\mathrm{Doya}(x+\mathrm{i}y)$, left, and $u+\mathrm{i}v=\mathrm{Doya}^{-1}(x+\mathrm{i}y)$, at right

2 Transfer function Doya

Functions Tania and ArcTania from the previous section allow to build-up the "solvable" transfer function. I call it Doya,

$$T(z) = \mathrm{Doya}(z) = \mathrm{Tania}\big(1 + \mathrm{ArcTania}(z)\big) \quad \text{[Doya]} \tag{5.11}$$

Complex map of function Doya, and also map of its inverse function ArcDoya $=$ Doya^{-1} are shown in figure 5.5. At large values of the argument, each of these functions looks similar to identity function, but they have the branch points and cuts in the central part of the maps.

While functions Tania and ArcTania are already implemented, the evaluation of function Doya is straightforward. In addition, in vicinity of the real axis, Doya can be expressed through the known special function LambertW [111]:

$$\mathrm{Doya}(z) = \mathrm{LambertW}\big(z\,\mathrm{e}^{z+1}\big) \quad \text{[DoyaLambertW]} \tag{5.12}$$

According to definition (5.11), Tania is superfunction of Doya, and ArcTania is its abelfunction. The nth iterate of Doya can be written as follows:

$$\mathrm{Doya}^n(z) = \mathrm{Tania}\Big(n + \mathrm{ArcTania}(z)\Big) \quad \text{[DT]} \tag{5.13}$$

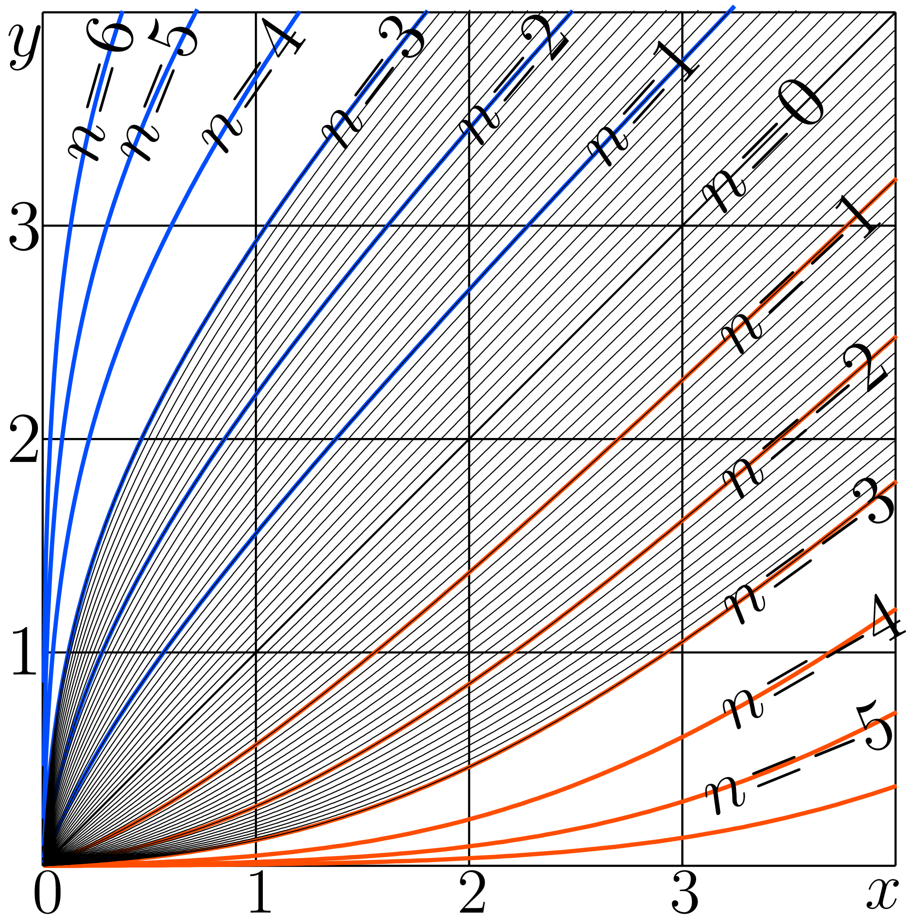

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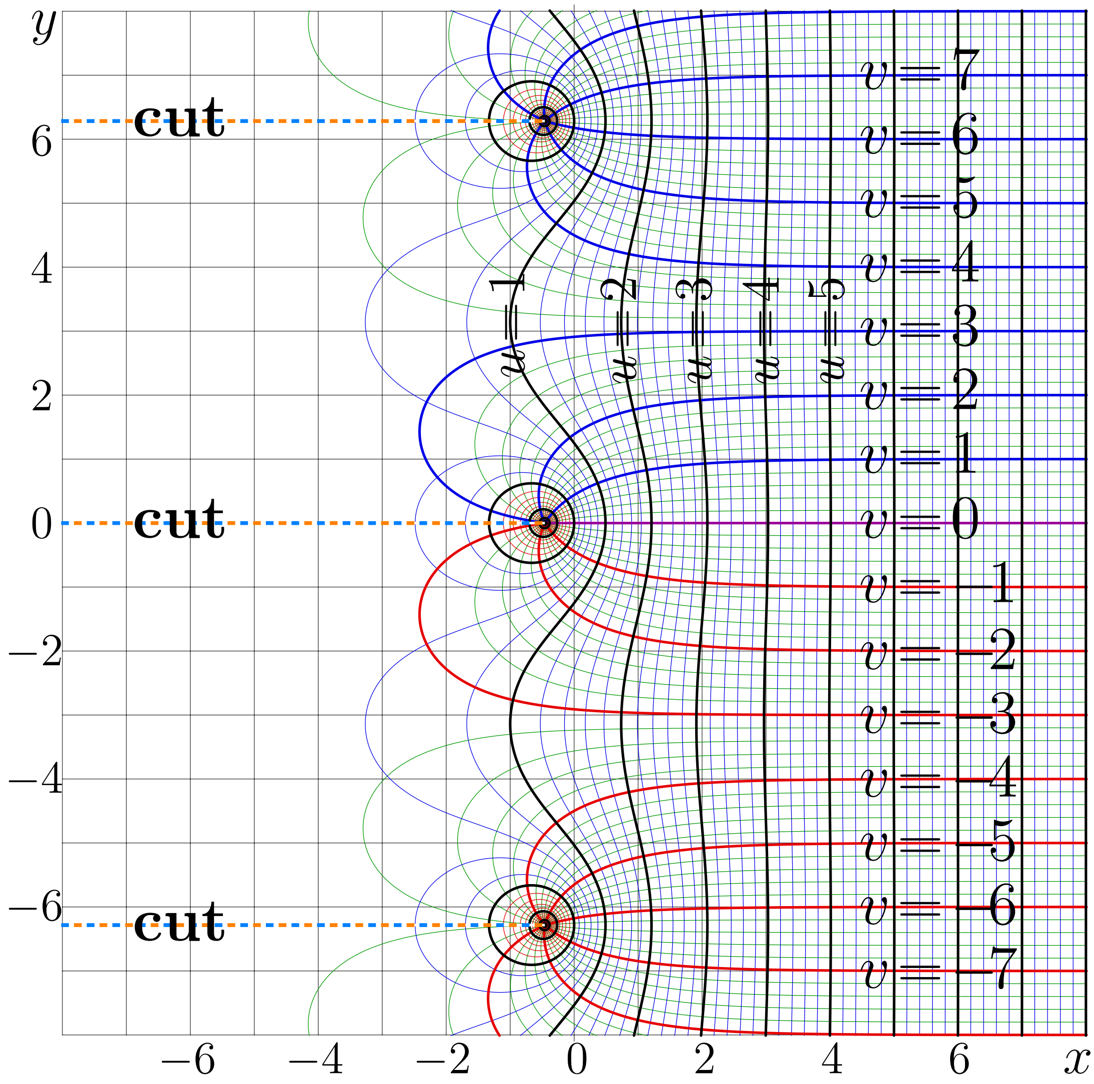
Figure 5.6: $y = \mathrm{Doya}^n(x)$ по формуле (5.13) [doyaplo]

These iterates are shown in figure 5.6. The graphics represent $y = \mathrm{Doya}^n(x)$ versus x for various values of the number n of iterate.

As it is mentioned above, function Tania has simple physical sense; it describes evolution of a signal in a simple model of a uniform saturable amplifier (or absorber) at the appropriate choice of units of length and units of intensity. Similar sense can be given to function Doya, it appears as dependence to the output intensity of this amplifier on its input intensity [84, 85].

Functions Doya, Tania and ArcTania give an example, when the transfer function, the superfunction and the abelfunction can be expressed in terms of special functions, already described in the literature of 20th century. In the following section, one more example of this kind is considered.

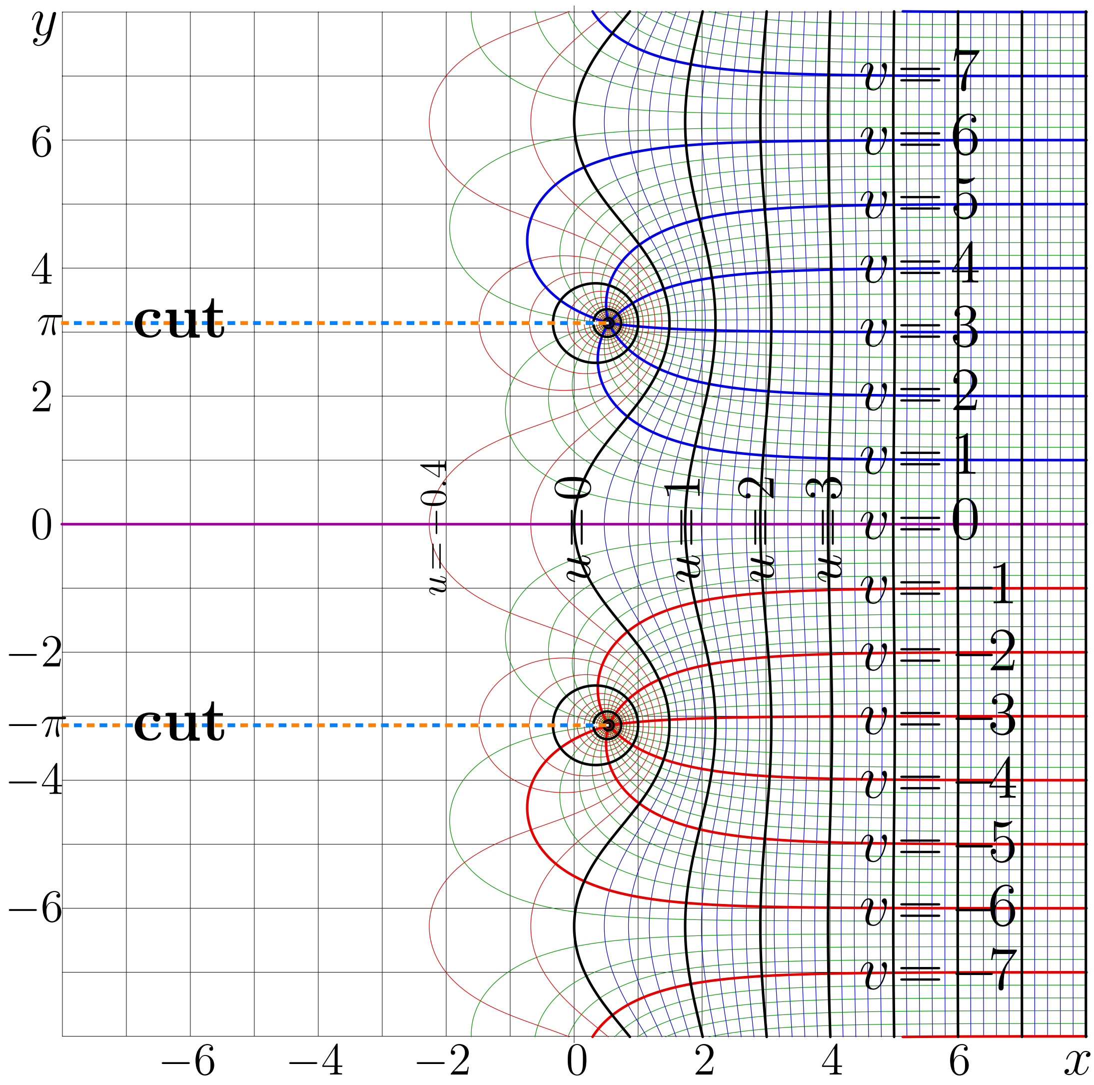


http://mizugadro.mydns.jp/t/index.php/File:KellerMapT.png

Figure 5.7: $u+v = \mathrm{Keller}(x+\mathrm{i}y)$ by (5.14). [KellerMap]

3 Keller, Shoka and ArcShoka

Wave packet of light (or any other quasi–classical bosons) in a uniform amplifier can be characterised with its energy or its fluence; this quantity can be denoted as "signal". Roughly, fluence is energy of pulse per area of its transversal cross-section. In analogy with the continuous-wave amplifier, the signal at the output of the amplifier can be considered as function of the input; and this dependence can be interpreted as the transfer function of the amplifier. For the simple model of the active medium, this function can be expressed as elementary function. Complex map of this function is down in figure 5.7; I call it "Keller function" and define it as follows:



http://mizugadro.mydns.jp/t/index.php/File:KellerMapT.png

Figure 5.8: $u+v = \mathrm{ArcKeller}(x+\mathrm{i}y)$ by (5.16). [ArcKellerMap]

$$\mathrm{Keller}(z) = z + \ln\Big(\mathrm{e} - \mathrm{e}^{-z}(\mathrm{e}-1)\Big) \qquad \text{[KellerDef]} \qquad (5.14)$$

Complex map of this function is shown in figure 5.7.

The inverse function $\mathrm{ArcKeller} = \mathrm{Keller}^{-1}$ can be written as follows:

$$\mathrm{ArcKeller}(z) = z + \ln\left(\frac{1}{\mathrm{e}} + \frac{\mathrm{e}-1}{\mathrm{e}}\,\mathrm{e}^{-z}\right) \qquad \text{[ArcKellerDef]} \quad (5.15)$$

Complex map of function ArcKeller is shown in Figure 5.8. Maps of functions Keller and ArcKeller look similar; the following relation takes place:

$$\mathrm{ArcKeller}(z) = \mathrm{Keller}(z - \mathrm{i}\pi - 1) - 1 + \mathrm{i}\pi \qquad (5.16)$$

In publications [30, 35] by Ursula Keller (fig.5.9), another representation is used,

$$\mathrm{Keller}(z) = \ln\Big(1 + \mathrm{e}\,(\mathrm{e}^z - 1)\Big) \qquad \text{[KellerLit]} \quad (5.17)$$

Figure 5.9: U. Keller

This representation is equivalent of expression (5.14), while $|\Im(z)|<\pi$. In particular, representation (5.17) can be used instead of (5.14) for real positive values of z.

Function Keller of real argument is compared to function Doya in figure 5.10. These functions look similar.

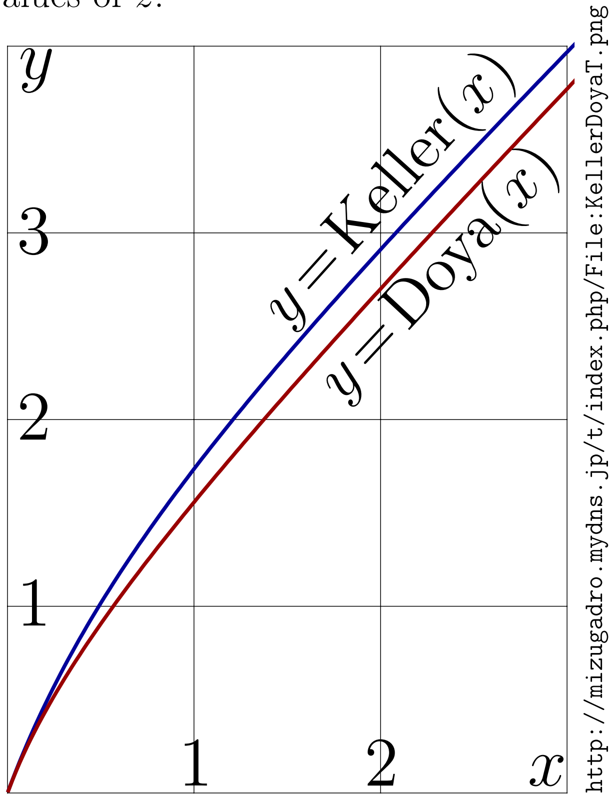


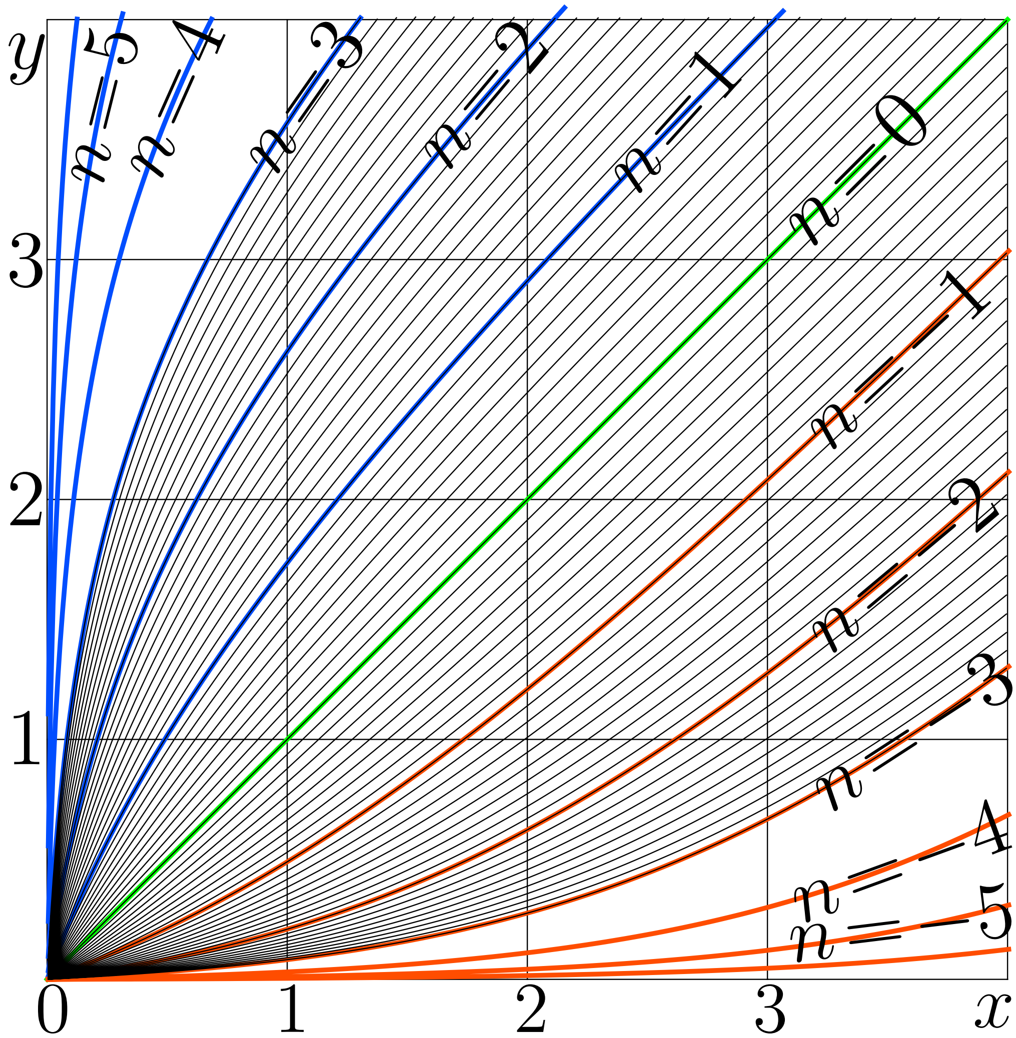
Figure 5.10: Comparison of functions Keller и Doya by (5.14) and (5.11)

Description of pulses usually is more difficult, than consideration of the stationary regime; for pulses, there is additional parameter, time. However, for the simple model above, the pulsed regime is simpler, than the steady-state case, because, for the pulses, the transfer function, and the superfunction can be expressed as elementary functions. In such a way, function Shoka is simpler than function Tania, and function Keller is simpler than function Doya.

Iterates of function Keller are shown in figure 5.11. They look similar to iterates of function Doya shown in figure 5.6. By general formula 2.14, the iterates of the Keller function can be expressed through its superfunction $F=$Shoka and abelfunction $G=$ArcShoka; these functions can be expressed as follows:

$$\mathrm{Shoka}(z) = z + \ln\Big(\mathrm{e}^{-z} + \mathrm{e} - 1\Big) \qquad \text{[Shoka]} \qquad (5.18)$$

$$\mathrm{ArcShoka}(z) = z + \ln\left(\frac{1-\mathrm{e}^{-z}}{\mathrm{e}-1}\right) \qquad \text{[ArcShoka]} \qquad (5.19)$$



http://mizugadro.mydns.jp/t/index.php/File:KellerPlotT.png

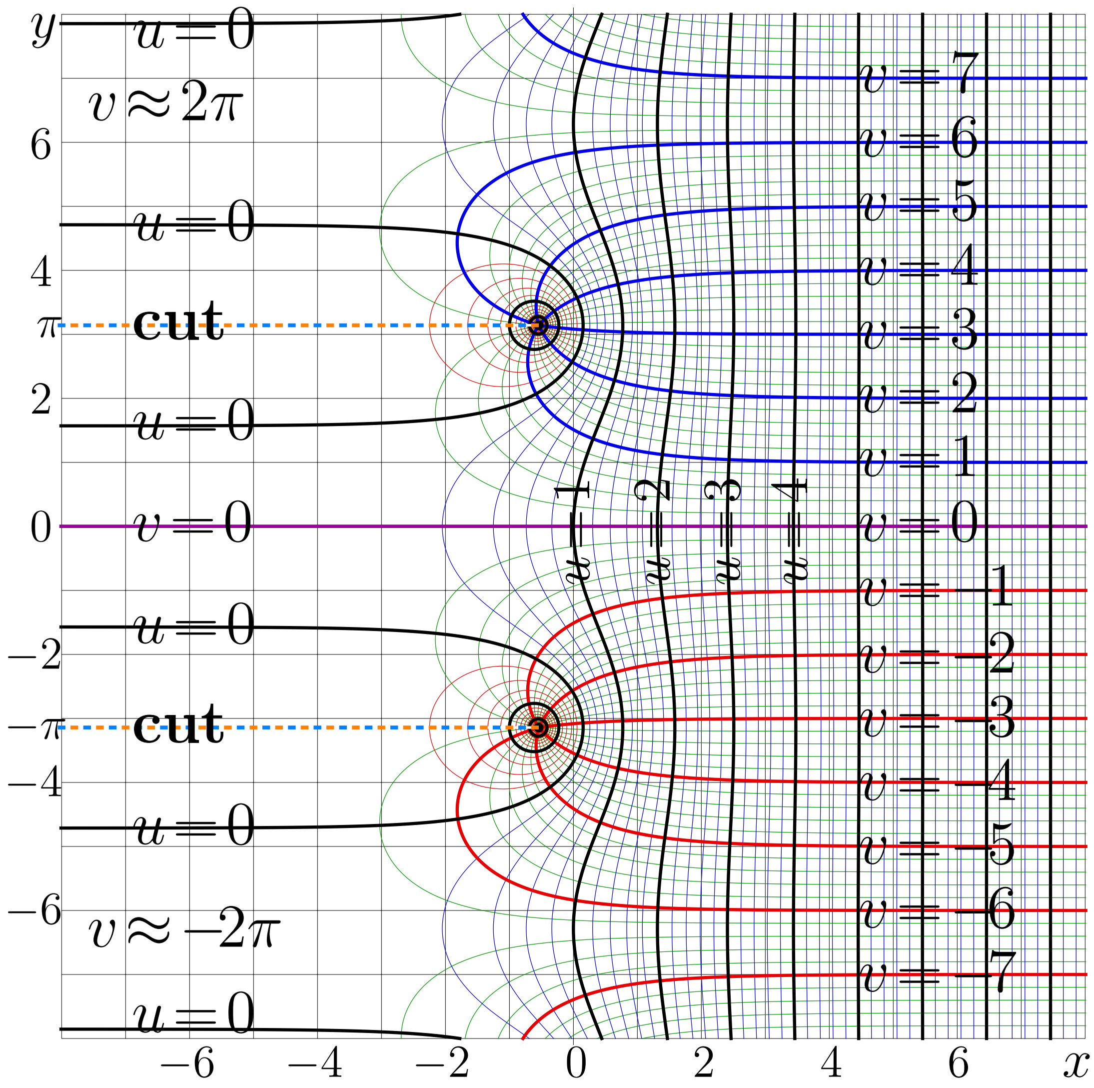
Figure 5.11: $y = \mathrm{Keller}^n(x)$. [kellerite]

Complex maps of functions Shoka and ArcShoka are shown in figures 5.12 and 5.13. These maps look similar to those for functions Tania and ArcTania shown in figures 5.2 and 5.3.

For real values of the argument, functions Tania and Shoka are compared in figure 5.1 mentioned in the preamble of this chapter. Both functions in the left hand side of the graphic have the exponential growth with increment unity; both pass thorough point (0,1) and both grow almost linearly in the right hand side of the graphics.

Complex maps of functions Shoka and ArcShoka in figures 5.12 and 5.13 look similar. One of them can be obtained from another with constant displacement of the argument and addition of some constant to its value. This can be expressed with relation

$$\mathrm{ArcShoka}(z) = \mathrm{Shoka}\big(z - \mathrm{i}\pi - \ln(\mathrm{e}-1)\big) - \ln(\mathrm{e}-1) + \mathrm{i}\pi \qquad (5.20)$$



http://mizugadro.mydns.jp/t/index.php/File:ShokaMapT.png

Figure 5.12: $u+\mathrm{i}v=\mathrm{Shoka}(x+\mathrm{i}y)$ [ShokaMap]

In vicinity of the positive part of the real axis, the map of function Shoka in figure 5.12 looks similar to the map of function Tania at Figure 5.2. However, I cannot suggest any simple expression of function Tania through function Arctania. I know no analogy of formula (5.20) for functions Tania and ArcTania.

There are also qualitative differences between Tania and Shoka. Tania had only two branch points, and correspondently, two cut lines. Shoka has countable set of branch points and cuts.

All the six functions Tania, ArcTania, Doya, Shoka, ArcShoka and Keller look similar to the linear function at the large values of the real part of the argument. The right hand side of the complex maps in figures 5.2, 5.3, 5.5, 5.12 5.13, 5.7, the structure of levels of constant real part and

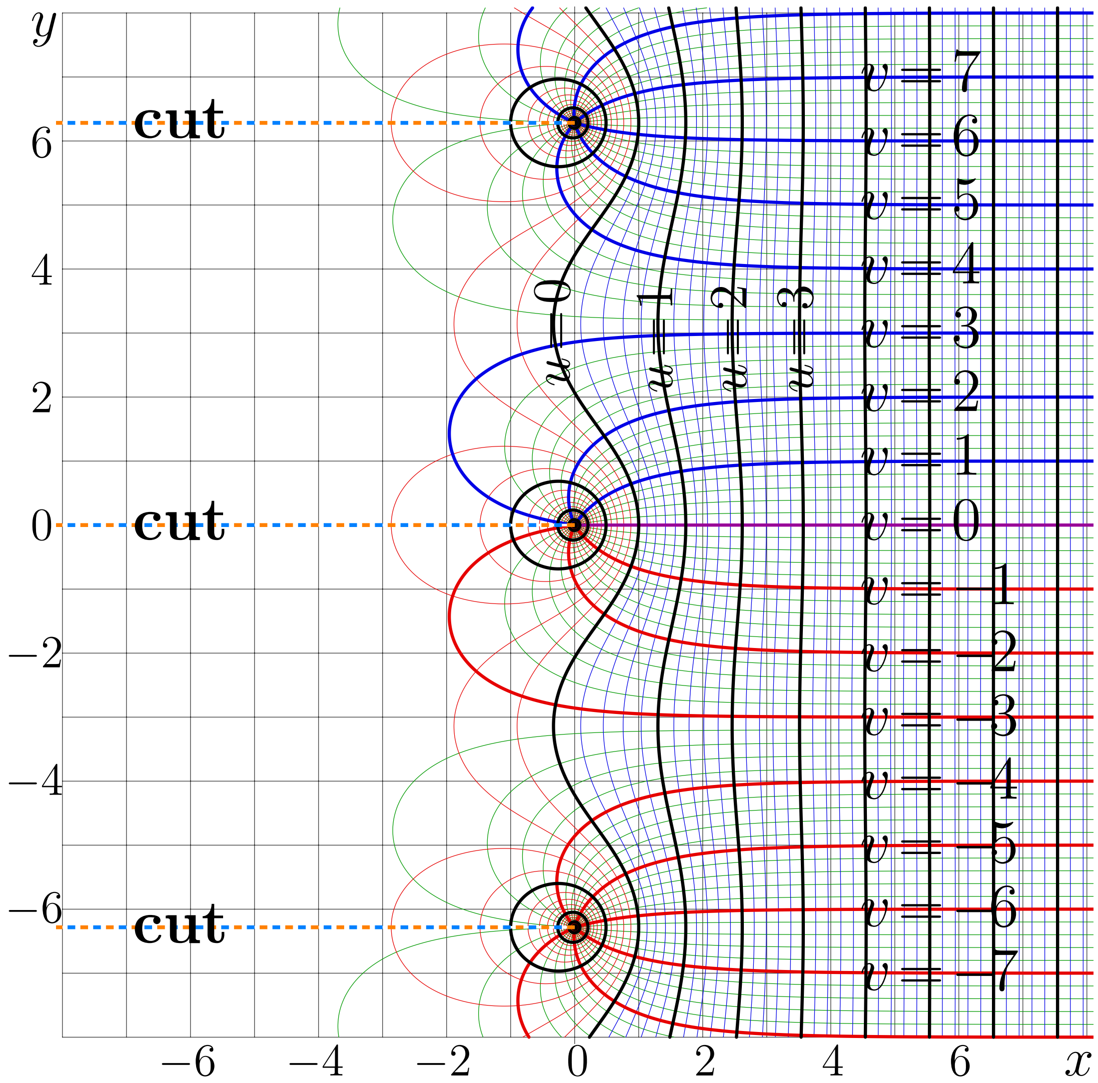


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Figure 5.13: $u+\mathrm{i}v=\mathrm{ArcShoka}(x+\mathrm{i}y)$. [ArcShokaMap]

levels of constant imaginary part form grid of lines, almost parallel to the real or imaginary axes.

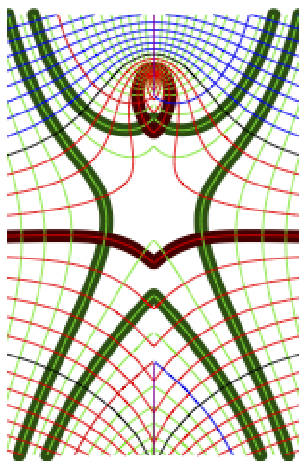

Figure 5.14: Eureka!

In order to catch (or to remember) the properties of function Doya, the left hand side part of figure 5.5 is repeated in figure 5.14. The map is shown with $u+\mathrm{i}v=\mathrm{Doya}(x+\mathrm{i}y)$, but the picture is rotated for 90^{o} in the positive direction (counterclock wise). The thick lines show levels $u=-0.4$, $v=\pm 1.2$, $v=\pm 1.4$ The map looks as a draw of a Researcher, who has found something extraordinary in a big Book. Perhaps, that Book is about superfunctions. Generator of this image is loaded as `http://mizugadro.mydns.jp/t/index.php/File:Doya500.png` . The Reader is invited to check that values of the 3 parameters, indicated above, provide the structure, shown in figure 5.14. The range of the almost linear transfer appears above the "head" of the contour of the "human" in the figure.

Similarity of the transfer functions Doya and Keller and, correspondently, similarity of their superfunctions Tania and Shoka indicate, that for analysis of the nonlinear media, these functions should be measured with several significant figures; over-vice, the measurement will not allow to make choice between divergent models. At the convenient measurement of the nonlinear response of the medium, variation of the intensity in a sample should be small: over-vice, it is difficult to guess, namely which intensity does the gain or absorption correspond to. At small variation of the intensity, the precision of its measurement is poor. This difficulty can be avoided with superfunctions. The transfer function of the optically-thick sample should be measured, and then, the superfunction, id est, the evolution of intensity inside the sample can be reconstructed. In such a way, the number of parameters of the model, can be reduced: one of parameters, namely, the length of the amplifier, can be excluded from the model.

My particular interest in superfunctions is related to the ability of the precise measurement of the gain (or absorption) in the nonlinear medium versus intensity. The application may refer to investigation of limits of validity of the commonly-used model of the Yb-doped crystals, glasses and ceramics. The new effects are expected to appear at the edge of the limit of applicability. One of such effects is described in the article about switching of emissivity of Yb-doped samples, or the "Bisson effect" [48], but namely in that case, the main mechanism of the phenomenon seems to have thermal origin: the sample warms, and this warming enhances the absorption (and heating), leading to the avalanche behaviour, that is difficult to interpret in terms of the transfer function of a single variable, considered in this Book [1] .

[1] The name "bisson effect" or 'bison effect" refers to the analogy of the avalanche of the tribe of running bisons, who awake, mock and force to run the new and new bisons. However the analogy is limited, because the electrons, that were expected to be popped up into the conduction band, are neither bisons, nor even bosons, but fermions.

4 Overview

The six functions, (Tania, Shoka, ArcTania, ArcShoka, Doya and Keller) are defined in this chapter. These functions show the realistic examples, the physically-meaningful transfer functions and the corresponding meaningful superfunctions are expressed in terms of elementary functions. These example will be used in this book later, to illustrate more general methods of construction and evaluation of superfunctions.

My main claim is that I can construct the superfunction, abelfunction and, therefore, the non-integer iterates of any growing real-holomorphic function [2]. Even more, everybody, after to read this Book, also can do the same.

In the following chapters, I consider methods, that can be used for various transfer functions, even if the superfunction cannot be expressed through the special functions known since century 20. One of these cases is considered in the next chapter.

[2]With the same methods, the non-real, but still holomorphic functions also can be treated; such an example is considered in chapter 18.

Chapter 6

Regular iteration

Regular iteration is way to construct iterates of the transfer function, that are regular in vicinity of its fixed point. The transfer function is supposed to be holomorphic. In the most of examples, considered in this Book, the transfer function is assumed to be also real-holomorphic. In addition, in this chapter, I assume, that the fixed point is real. [1]

This Book deals with solutions F of the transfer equation (2.12); I repeat it here

$$F(z+1) = T\big(F(z)\big)$$

Assume, that the transfer function T is real-holomorphic, and its fixed point L is real, id est,

$$T(L) = L \quad , \quad L = L^* \tag{6.1}$$

In addition, I assume, that $T'(L)\!>\!0$, this case is easier to interpret [2] . For this case, In this chapter, the specific iterates of the transfer function are constructed, that can be expanded into convergent Taylor series at least in vicinity of the fixed point L. In this sense, they are regular. On the other hand, the iterates are constructed with definite iteration procedure, described below, in a regular straightforward way. Hence, there are at least two independent reasons to use name "regular iteration" for the procedure below.

[1] The same regular iteration can be applied also for the case of non-real fixed point, but the resulting iterates of the transfer function are not real-holomorphic. The real-holomorphic functions often are easier to interpret (and to apply in Physics), than the more general complex case. This Book is planned as applied, so, the most of examples here refer to the real-holomorphic functions.

[2] Assumption $T'(L)\!>\!0$ is natural. Case $T'(L)\!<\!0$ is difficult to interpret in terms of real valued non-integer iterates, because, ${T^n}'(L)$ should approach unity at $n\!\to\!0$, but, for realistic cases, should avoid zero. Practically, this means that the non-integer iterates are complex, not real, as in the case $T'(L)\!>\!0$

1 General formula

Assume that L is fixed point of the transfer function T, id est, $T(L) = L$. For the transfer equation (2.12), I search for the asymptotic solution F in the following form:

$$\tilde{F}(z) = L + \varepsilon + a_2\varepsilon^2 + a_2\varepsilon^3 + .. \quad \text{[Lea2general]} \qquad (6.2)$$

where

$$\varepsilon = \exp(kz) \quad \text{[Lea2]} \qquad (6.3)$$

Here, k is constant, that bas sense of the increment, and a refer to some set of real constants, that do not depend on z. As it is indicated in the Preamble of this chapter, I keep in mind the case when $T(z^*) = T(z)^*$ и $F(z^*) = F(z)^*$; however, in principle, the expansion (6.3) can be used for more complicated case too (and in this case the coefficients a should be complex).

The consideration could be generalised, adding new terms in to the right hand side equation (6.2), for example, terms, that are not polynomial with respect to ε. Following my wishes about the popularity [3], below I consider the simplest (but still non-trivial) case.

Substituting $F \to \tilde{F}$ into the transfer equation (2.12), which is

$$F(z+1) = T(F(z))$$

in the left hand side I get the following

$$\tilde{F}(z+1) = L + \mathrm{e}^k\varepsilon + a_2\mathrm{e}^{2k}\varepsilon^2 + a_3\mathrm{e}^{3k}\varepsilon^3 + .. \quad \text{[iteraz1]} \quad (6.4)$$

and the right hand side gives

$$\begin{aligned} T(\tilde{F}(z)) &= L + T'\cdot\varepsilon + T'\cdot a_2\varepsilon^2 + T'\cdot a_3\varepsilon^3 + .. \\ &\quad + \frac{T''}{2}(\varepsilon + a_2\varepsilon^2 + ..)^2 + \frac{T'''}{6}(\varepsilon + ..)^3 + .. \quad \text{[i2]} \qquad (6.5) \end{aligned}$$

where $T' = T'(L)$, $T'' = T''(L)$, $T''' = T'''(L)$, .. are derivatives of the transfer function T at the fixed point L.

coefficients at the same power of ε in expression (6.4) and in expression (6.5) should be equal. This gives the equations for increment k and coefficients a:

$$\begin{aligned} \mathrm{e}^k &= T' && (6.6) \\ \mathrm{e}^{2k}a_2 &= T'a_2 + T''/2 && (6.7) \\ \mathrm{e}^{3k}a_3 &= T'a_3 + T''a_2 + T'''/6 && (6.8) \\ .. & \end{aligned}$$

[3]Here, the term "pipuliqrity" may have each of the two its meanings

The chain of these equations determines

$$
\begin{aligned}
k &= \ln(T') && \text{[k38]} && (6.9)\\
a_2 &= \frac{T''/2}{(T'-1)T'} && \text{[a2.39]} && (6.10)\\
a_3 &= \frac{T''a_2 + T'''/6}{((T')^2-1)T'} && \text{[a2.40]} && (6.11)\\
&..
\end{aligned}
$$

Typically, for a simple special function T, the Mathematica, the Maple or any similar software allow to evaluate tens of coefficients a in real time.[4] The truncated series in representation (6.2) provides good approcimation for F at $\varepsilon \ll 1$. For positive values k, this corresponds to the large negative values of $\Re(z)$. For other values, the approximation can be improved with

$$F(z) \approx T^n\Big(\tilde{F}(z-n)\Big) \quad \text{[regi]} \qquad (6.12)$$

for the large enough positive values of n at positive k, and for the large enough negative values of n at negative k. Case $k=0$ is qualified as exotic and is considered below in the special chapter about exotic iterates.

In may cases, representation (6.12) allows to evaluate the superfunction with the required precision. In particular, this allows to evaluate the super exponentials to base $b < \exp(1/\mathrm{e})$ [61], superfactorial [65], and holomorphic extension of the logistic sequence [69], and even the holomorphic extension of the Collatz subsequence [110]. Some of these examples are considered in the following chapters.

I call this case "regular iteration", as the iteration by (2.14), id est, $T^n(z) = F\big(n+F^{-1}(z)\big)$ is holomorphic ("regular") function in vicinity of the stationary point $z=L$. For real fixed point L, I expect, namely regular iteration corresponds to the physically-meaningful solution F of the transfer equation (2.12). This statement can be qualified as conjecture. Following the TORI axioms, I tried to negate, refute this conjecture, but I got confirmations instead. These confirmations, examples form the significant part of this Book.

[4] My experience indicates, that, for applications, the evaluation of all quantities in real time is so important, as the real holomorphism of the functions. In such a way, I try to be realistic, dealing with real quantities in the real time. However, even for the real-holomorphic superfunctions, their behaviour at complex values of argument is important and should be considered; but even in this case I try to keep algorithms short and fast, to make evaluations in the real time.

2 Exact solution

The approximate equality in expression (6.12) should not make an impression, that the regular iteration gives only approximation of the superfunction.[5] For the case of $\Re(k)>0$ (or, in particular, for $k>0$), the exact superfunction can be expressed with limit

$$F(z) = \lim_{n\to\infty} \tilde{T}^n\big(F_m(z-n)\big) \quad \text{[regitexa]} \tag{6.13}$$

where

$$F_m = L + \varepsilon + \sum_{\ell=2}^{m} a_n \varepsilon^n \tag{6.14}$$

and k, ε and coefficients a are defined in the previous section; while $m \geq 2$ is some integer number. Due to the asymptotic properties of the solution $\tilde{F}$, the limit in expression (6.13) does not depend on m. However, the convergence is faster for larger m. Similar limit can be written for the case $\Re(k)<0$.

The series (6.3) can be inverted, giving the asymptotic approximation $\tilde{G}$ of the abelfunction $G = F^{-1}$. Then, the exact abelfunction can be expressed with

$$G(z) = \lim_{n\to\infty} \ln\left(\frac{1}{k}\tilde{G}_m(T^{-n}(z))\right) + n \tag{6.15}$$

where $\tilde{G}_m$ is some truncation of the asymptotic expansion for the abelfunction. The iterate of the transfer function, $T^n(z) = F(n + G(z))$ happen to be regular in vicinity of $z = L$. In the following chapters, this statement is verified for many specific realisations of the transfer function T.

Before to apply the regular iteration to the new transfer functions, for which the superfunctions cannot be easy expressed through the superfunctions, known in century 20, it worth to check the method for some easy function, for which the answer is known. First, I use the method of regular iteration to construct the superfunction for the transfer function Doya, considered in the previous chapter. I do it, as if I would not know that its superfunction is Tania. This regular iteration is considered in the next section, and compared to the exact solution Tania.

[5] One of my coauthors, until now, believes, that even π is approximate number. This can be considered as a kind of a mental illness, as well as some kind of religion.. However, if for any given $\alpha > 0$, the error of evaluation of some quantity can be done smaller, than α, then the quantity is considered as exact. Then, we may consider the spu time and number of flops required to get some given precision of the approximate evaluation of superfunction F in (6.12).

3 Example with known solution: again Doya

In this section, I show, how does the regular iteration work. I use the example with transfer function Doya, described in the previous chapter. For real value of argument, graphic of this function is shown in figure 6.1 with thick curve.

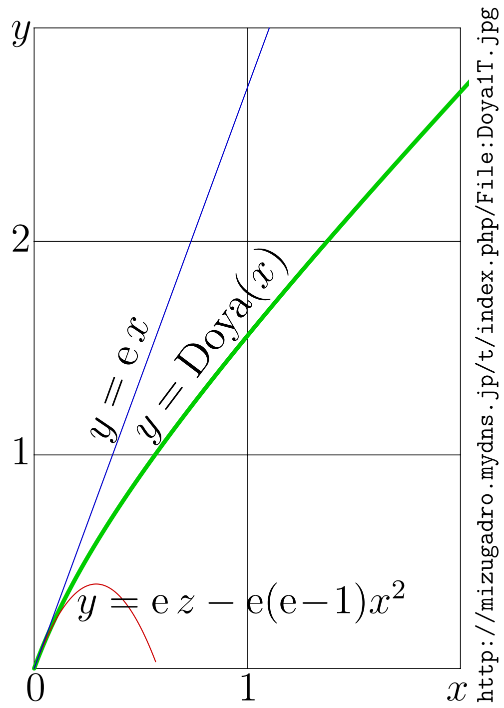

Figure 6.1: Function Doya and its polynomial approximations

For transfer function T = Doya, the superfunction is known, it is function $F=$Tania, and the corresponding abelfunction is $G=$ArcTania. Properties of thiese function are described in publications [85, 106, 107] and in the previous chapter. But, in order to check the regular iteration, assume for a minute, that we do not know the analytic expression for the superfunction, and want to construct it with the regular iterations described above. After to construct it, we may compare the result with the special function Tania.

Let the transfer function, at least in some vicinity of the real values of the argument, can be expressed with formula (5.12), I repeat it here:

$$T(z) = \mathrm{Doya}(z) = \mathrm{LambertW}(1+z\,\mathrm{e}^z) \quad \text{[doya1]} \quad (6.16)$$

Properties of function LambertW are known [111, 112, 113, 114]. This function can be used even without to refer to function Tania; this emulates situation, when the superfunction F is not known.

Graphic of function Doya is shown in figure 6.1 with thick curve. This transfer function describes some idealised amplifier with saturation, neglecting the spontaneous emission (and many other physical effects). In this case, the fixed point $L=0$ should be considered; $T(0)=0$. At small intensity at the input, the amplification coefficient is e, but it saturates at the intensity of order of unity. For small argument, the transfer function can be expanded as follows:

$$T(z) = \mathrm{Doya}(z) = \mathrm{e}\, z - \mathrm{e}\,(\mathrm{e}-1) z^2 + O(z^2) \qquad \text{[DoyaExpa]} (6.17)$$

In this case, $T'(0)=\mathrm{e}$ and $T''(0)=-2\mathrm{e}(\mathrm{e}-1)$. The corresponding linear and quadratic approximations are shown in figure 6.1 with thin lines.

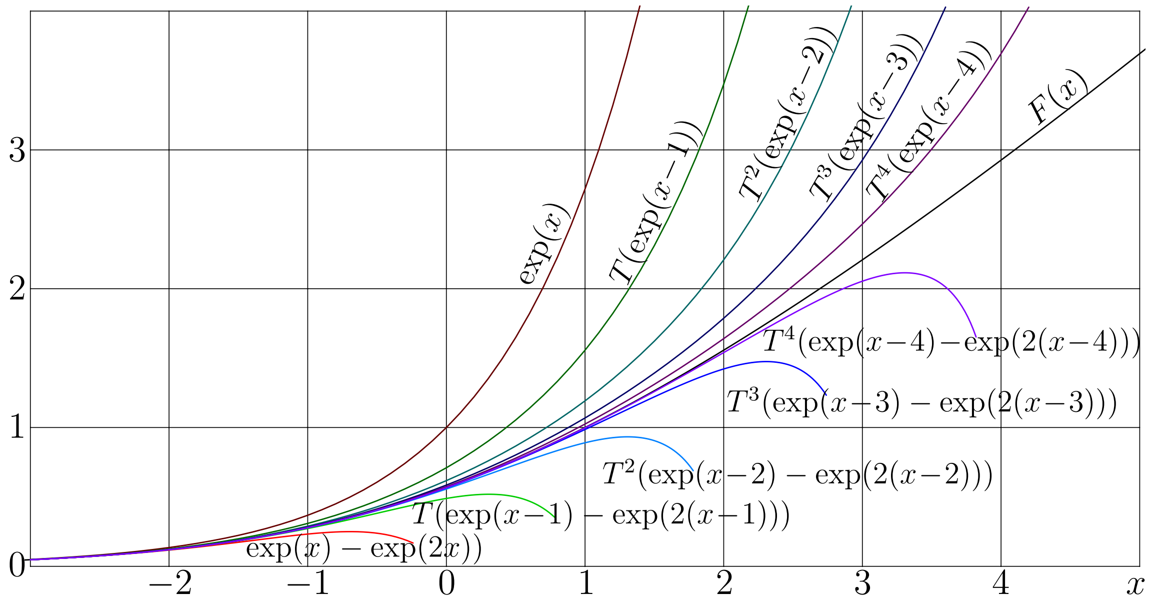

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Figure 6.2: Superfunction $F=$Tania and its approximations $\tilde{F}$ by (6.12) for $n=0..4$

For regular iteration of transfer function Doya, using formulas (6.9) and (6.10), I found $k=1$ and $a_2=-1$. In such a way, the primary approximation with singe term is

$$\tilde{F}(z) = \exp(z) \quad [\text{refitF1}] \tag{6.18}$$

and the primary approximation with two terms is

$$\tilde{F}(z) = \exp(z) - \exp(2z) \quad [\text{refitF2}] \tag{6.19}$$

These two primary approximations are shown in figure 6.2 with uppest and lowest curves. At the same figure, the four iterations of these primary approximations by (6.12) are shown for $n=0..4$. These approximations approach the exact solution

$$F(z) = \text{Tania}(z-1) = \text{WrightOmega}(z) \tag{6.20}$$

This function F is shown in figure 6.2 with thick line. It remains between approximations obtained with iterate from the primary approximation $\tilde{F}$ with single term, id est, (6.18) and those with tho terms by (6.19).

Example with the transfer function Doya shows the efficiency of the regular iteration. In the following chapters, the regular iteration is used also for other transfer functions; I mean, for the cases, when the superfunction cannot be easy expressed in terms of the special functions, known in century 20.

4 Schröder equation

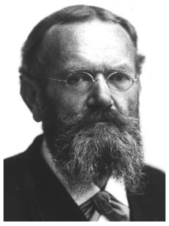


Figure 6.3: E.Schröder

For the regular iteration, instead of superfunctions (presented in this Book), another formalism can be used, namely, the formalism of the Schroeder (Schröder) functions; they are called after Ernest Schröder, shown in figure 6.3, and described many years ago [11, 27].

Let transfer function T be real-holomorphic, and let $L=0$ be its fixed point. The generalization to the case of other values of L is straightforward, so, here I consider only case $L = 0$. Let $S = T'(0)$. The schröderfunction is solution g of the Schröder equation

$$g(T(z)) = S\, g(z) \quad \text{[schroederEq]} \tag{6.21}$$

where s is some constant. Usually, it is assumed that $T(0) = 0$, id est, zero is fixed point of the transfer function T. One can search for solutions in the asymptotic form (6.21)

$$g(z) = \sum_{p=1}^{\infty} a_p\, z^p \quad \text{[schroederexpa]} \tag{6.22}$$

where coefficients a are constants, id est, do not depend on z. Usually, the series in the right hand side of (6.22) diverges, but this representation can be used for approximation of g at small values of $|z|$. Then, any required precision can be achieved, applying the Schroeder equation (6.21) or its inverse

$$g(z) = S\, g\Big(T^{-1}(z)\Big) \quad \text{[schroederEr]} \tag{6.23}$$

looking, what is smaller, $|T(z)|$ or $|T^{-1}(z)|$. This method is especially explicit for a real-holomorphic transfer function T, that monotonously grows along the real axis.

In certain parti f the complex plane, the abelfunction G can be expiressed through the Schroeder function g:

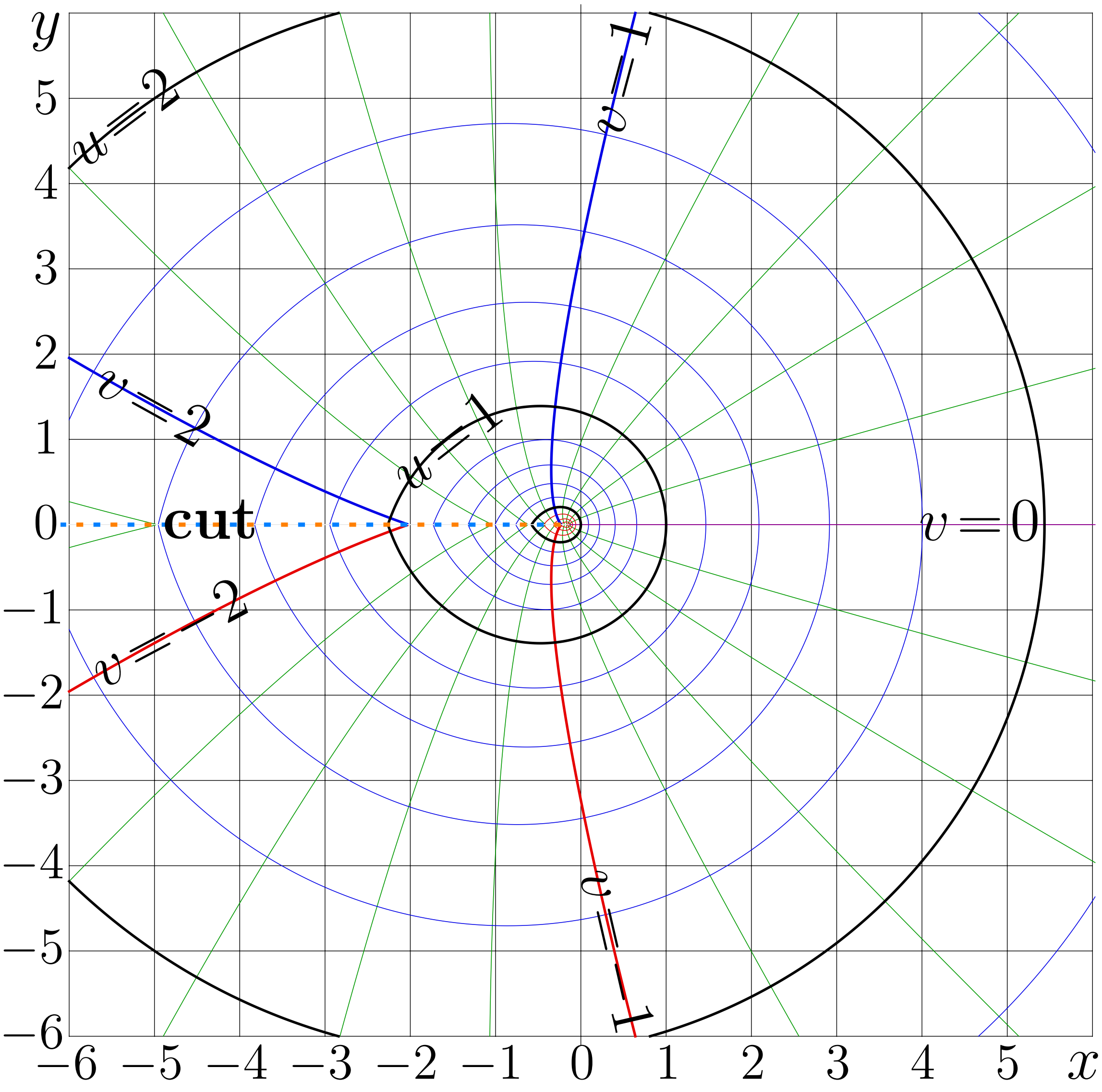
$$G(z) = \log_S(g(z)) \quad \text{[shroederabel]} \tag{6.24}$$

Taking logarithm to base S of the both sides of equation (6.21), I get

$$G(T(z)) = 1 + G(z) \quad \text{[analo]} \tag{6.25}$$

this equation is just Abel equation for the same transfer function T.

In the similar way one can express the analogy of superfunction. It can be called "scaling function" or even "scalingfunction". As examples of the



http://mizugadro.mydns.jp/t/index.php/File:Olga6map.jpg

Figure 6.4: $u+\mathrm{i}v = \mathrm{Olga}(x+\mathrm{i}y)$ by (6.26)

Schroeder functions and the scaling functions, figure 6.4 and 6.5 show complex maps of functions Olga and Anka. They are Schroeder function and the scaling function for the transfer function $T = \mathrm{Doya}$, considered in chapter 5 and in the previous section.

The Schroeder function Olga can be expressed through function Tania:

$$\mathrm{Olga}(z) = \mathrm{Tania}\Big(\ln(z)\Big) \quad [\mathrm{olga}] \tag{6.26}$$

The Scaling function Anka can be expressed through the ArcTania:

$$\mathrm{Anka}(z) = \exp\Big(\mathrm{ArcTania}(z)\Big) \quad [\mathrm{anka}] \tag{6.27}$$

For the transfer function Doya, the scaling factor $S=\mathrm{e}$; so, the natural logarithm and the natural exponent appear in equations (6.26) and (6.27). Definitions (6.26) and (6.27) imply, that $\mathrm{olga}=\mathrm{anka}^{-1}$.

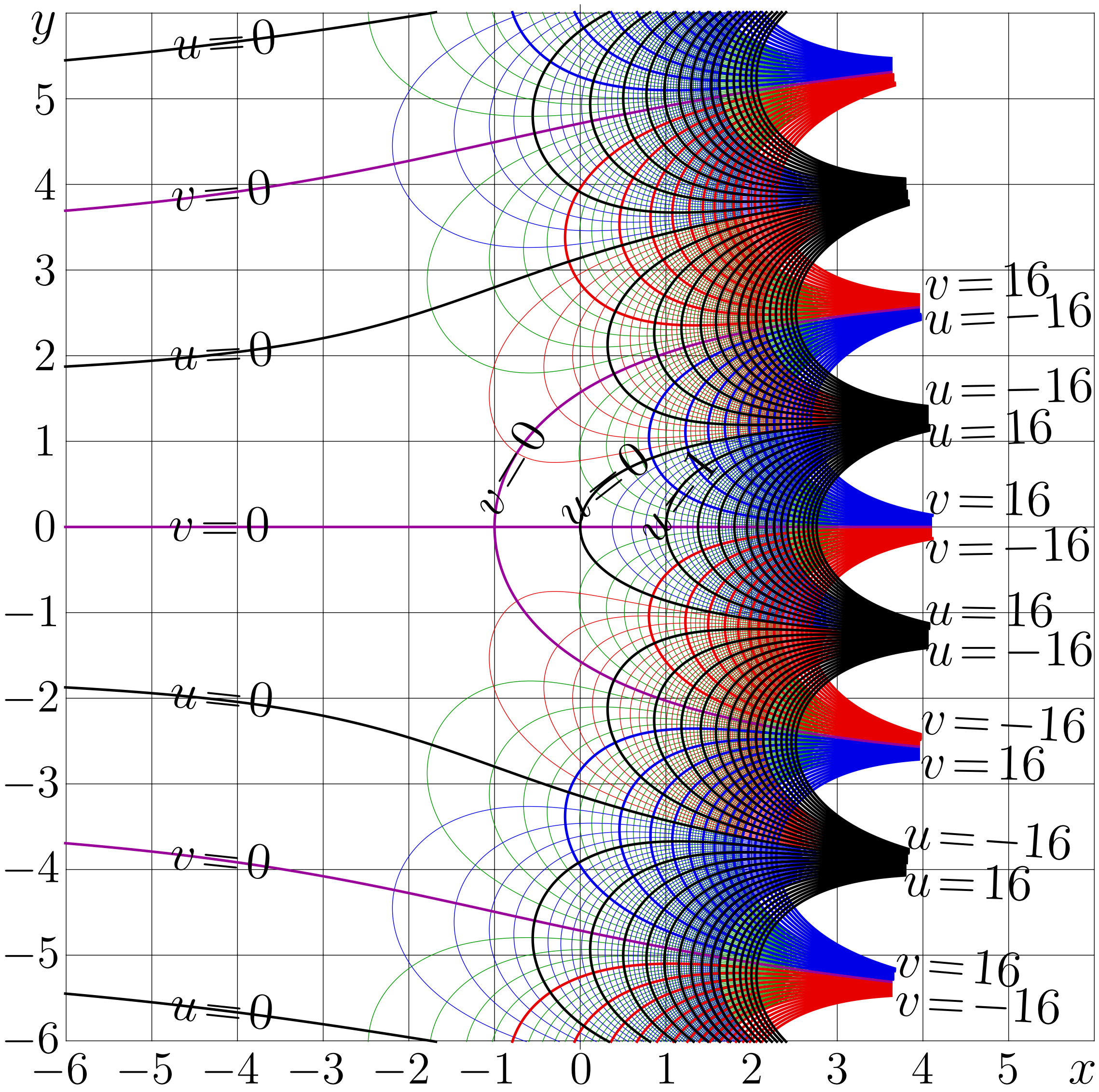


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Figure 6.5: $u+\mathrm{i}v = \mathrm{Anka}(x+\mathrm{i}y)$ by (6.27)

In this section, I use the example, when some dependences can be expressed in terms of elementary functions. In particular, this refers to function ArcTania by equation (5.4); in the most of the complex plane (except the negative part of the real axis), $\mathrm{ArcTania}(z) = z + \ln(z) - 1$. Then, function Anka can be expressed as follows

$$\mathrm{Anka}(z) = \frac{z}{\mathrm{e}} \exp(z) \qquad \text{[ankae]} \tag{6.28}$$

Functions Anka and Olga satisfy the scaling equation

$$\mathrm{Doya}(\mathrm{Anka}(z)) = \mathrm{Anka}(\mathrm{e}\, z) \tag{6.29}$$

and the Schroeder equation

$$\mathrm{Olga}(\mathrm{Doya}(z)) = \mathrm{e}\, \mathrm{Olga}(z) \tag{6.30}$$

The Reader is invited to investigate the ranges of validity of equations

$$\mathrm{Olga}(\mathrm{Anka}(z)) = z \tag{6.31}$$

$$\mathrm{Anka}(\mathrm{Olga}(z)) = z \tag{6.32}$$

In wide range, that includes the positive part of the real axis, iterates of function $T = \mathrm{Doya}$ can be expressed through its scaling function $f\!=\!\mathrm{Anka}$ and Schröder function $g = \mathrm{Olga} = \mathrm{Anka}^{-1}\!=\!\mathrm{ArcAnka}$:

$$T^n = f(S^n\, g(z)) \quad [\mathrm{Tnfg}] \tag{6.33}$$

for some appropriate constant S. Readera are invited to check, that formula (6.33) gives the same iterates, as the expression through the superfunction $F\!=\!\mathrm{Tania}$ and Abel function $G\!=\!\mathrm{ArcTania}$

$$T^n = F(n + G(z)) \quad [\mathrm{TnFG}] \tag{6.34}$$

While the real-holomorphic transfer function T has real fixed point L with the scaling factor S, and this fixed point is used to construct the superfunction F and the scalingfunction f, formulas (6.33) and (6.34) are equivalent. Several examples from this book, considered with superfunctions and abelfunctions, can be treated also with scalingfunctions and schroöderfunctions.

Relation (6.34) is more general than (6.33). Here I announce few cases, when the Schroeder functions fail.

If $T'(L) = 1$, then the scaling factor S becomes unity, and expression (6.24) fails. In particular, this is case of $T(z)\!=\!\exp(z/\mathrm{e})$, case of $T(z)\!=\!\mathrm{zex}(z)\!=\!z\, \exp(z)$, and that for $T(z)\!=\!\sin(z)$, see Table 3.1.

The transfer function T may have no real fixed points, as it takes place for $T\!=\!\exp$.

In addition, it may happen, that the transfer function T has no fixed points at all, as it takes place for $T(z)\!=\!\mathrm{tra}(z)\!=\!z\!+\!\exp(z)$.

Iterates of these functions T are straightforward with the superfunctions and abelfunctions. These examples are mentioned in the Table 3.1 and considered in the following chapters of this Book. However, these cases are difficult to treat with the scalingfunctions and the schroederfunctions, if at al. The Schroeder functions, if they can be applied, do not give any new in compare to use of the superfunctions. For this reason, this Book is dedicated to superfunctions and abelfunctions, and not to schroederfunctions.

I hope, the example above is sufficient to feel relation between Abelfunctions and Schroederfunctions. On this point I stop speculations about the Schroeder functions, Schroeder equations, Scaling equations, and scaling functions, and return to superfunctions. Superfunctions for the specific quadratic transfer function are considered in the next chapter.

Chapter 7

Logistic map

Figure 7.1: P.V.Elutin

Term “Logistic map” [1] may refer to the quadratic transfer function

$$T(z)=\mathrm{Elu}_s(z)=s\,z\,(1-z) \quad \text{[logisticop]} \qquad (7.1)$$

Usually, parameter s is assumed to be a real number. Term “map” is used also in the context of the “complex map”, and this may cause confusions. For this reason I give this function name Elu_s. The thee characters of the name are taken from the last name of Pawel Elutin (see Figure 7.1), my teacher of Quantum Mechanics. He asked me to construct the analytic extension of iterates of function T by (7.1) in the private communication [58]. In such a way, this chapter could be called also “Elutin function”. Here I present some results of publication [69] that appeared as the answer on the request by Elutin.

Historically, iterates of another transfer function had been considered before the iterates of the Elutin function. I mean, iterates of factorial [65]. The request to do the same of the logistic map appeared as result of the publication about the half iterate of factorial. In this Book, I do not follow the history; first, I describe the superfunction, abelfunction and iterates of the logistic map, or iterates of the Elutin function by (7.1); these iterates seem to be simpler than those for the factorial. For various values of parameter s, iterates of Elu_s are shown in figure 7.2. In order to plot this figure, I use the specific superfuction and the corresponding abelfunction. I describe them in this chapter.

The logistic sequence is solution F of the logistic equation

$$F(z+1)=\mathrm{Elu}_s(F(z)) \quad \text{[LOGEQ]} \qquad (7.2)$$

[1] http://en.wikipedia.org/wiki/Logistic_map

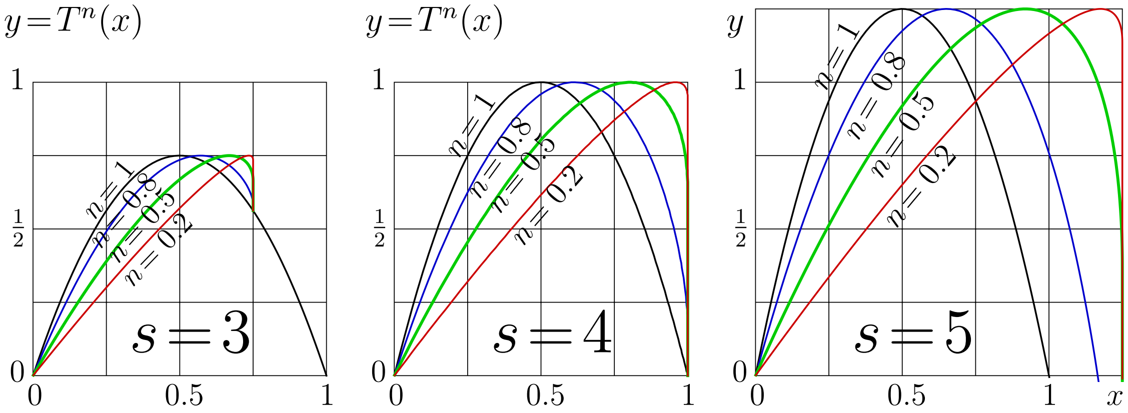

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Figure 7.2: Iterates of the Elutin function (7.1): $y = \mathrm{Elu}_s^n(x)$ for $s=3$, left; for $s=4$, center; for $s=5$, right; curves for $n=1$, $n=0.8$, $n=0.5$, $n=0.2$ are drawn.

Equation (7.2) is, actually, the transfer equation (2.12), with the Elutin function (logistic map) Elu_s by (7.1) as the transfer function T. In order to define the sequence, the initial value $F(0)$ should be specified.

In publications about the logistic equation (7.2) with transfer function (7.1), the argument of function F is assumed to be an integer number [25, 29, 31, 56]. For integer argument of the solution F of equation (7.2), the solution catches some properties of transition of the physical systems to chaos [55, 22, 21]. Iterates of the transfer function T by (7.1) appear as a rough description of the stochastic physical systems, as a simple, heuristic approach for the problems of hydro- and aero- dynamics, and also the transition regime of the stochastic lasers, in vicinity of the single-mode regime of generation. Here I consider the holomorphic extension of the logistic sequence for not only integer, but complex values of the argument.

1 Logistic sequence

Iterates of the Elutin function (logistic map) $T = \mathrm{Elu}_s$ by (7.1), as functions of real argument, are shown in figure 7.2 for $s=3$ (left picture) $s=4$ (central picture) and $s=5$ (right picture); $y = \mathrm{Elu}_s^n(x)$ is plotted versus x for $n=0.2$, $n=0.5$, $n=0.8$ and $n=1$. All the graphics in figure 7.2 are plotted with the same formula

$$\mathrm{Elu}_s^n(z) = F(n + G(z)) \quad \text{[ite]} \tag{7.3}$$

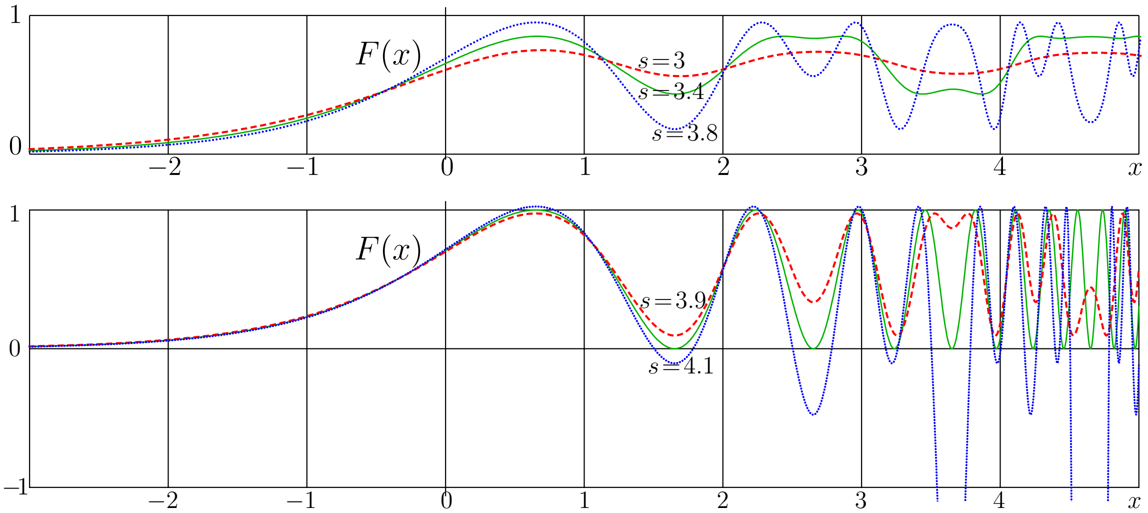

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Figure 7.3: $\mathrm{Pav}_s(x)$ by formulas (7.4),(6.12), (7.8) for $s = 3$, $s = 3.4$, $s\!=\!3.8$, at the top picture, and for $s\!=\!3.9$, $s\!=\!4$, $s\!=\!4.1$, bottom. [logi56]

where F is specific superfunction, solution of equation (7.2), and $G = F^{-1}$ is the inverse function (id est, abelfunction of Elu_s). Superfunction F appears as holomorphic extension of the logistic sequence [69]. Superfunction F can be constructed with the regular iteration, described in the previous chapter. This construction is described below.

At the construction of a superfunction, the key question is about the fixed points of the transfer function. For transfer function $T\!=\!\mathrm{Elu}$ by (7.1), equation $\mathrm{Elu}_s(z) = z$ has two solutions, $z = 0$ and $z = 1-1/s$. First of these solutions does not depend on s. This solution is used for construction and evaluation of the "holomorphic extension of the logistic sequence", id est, the specific solution $F = \mathrm{Pav}_s$ of equation 7.2, shown in figure 7.3 below. Then I can construct the inverse function $G = F^{-1}$, which is the corresponding abelfunction. These F and G allow to plot figure 7.2 by formula (7.3). Construction of these functions is described in the next section.

2 Fixed point $L\!=\!0$

For the Elutin transfer function by (7.1), with the regular iteration at the fixed point $L\!=\!0$, I construct the superfunction $F = \mathrm{Pav}_s$. Graphics $y\!=\!\mathrm{Pav}_s(x)$ are shown in figures 7.3 for different s. The complex maps are shown in figures 7.4, 7.5, 7.6. Below I describe the construction and evaluation of this superfunction.

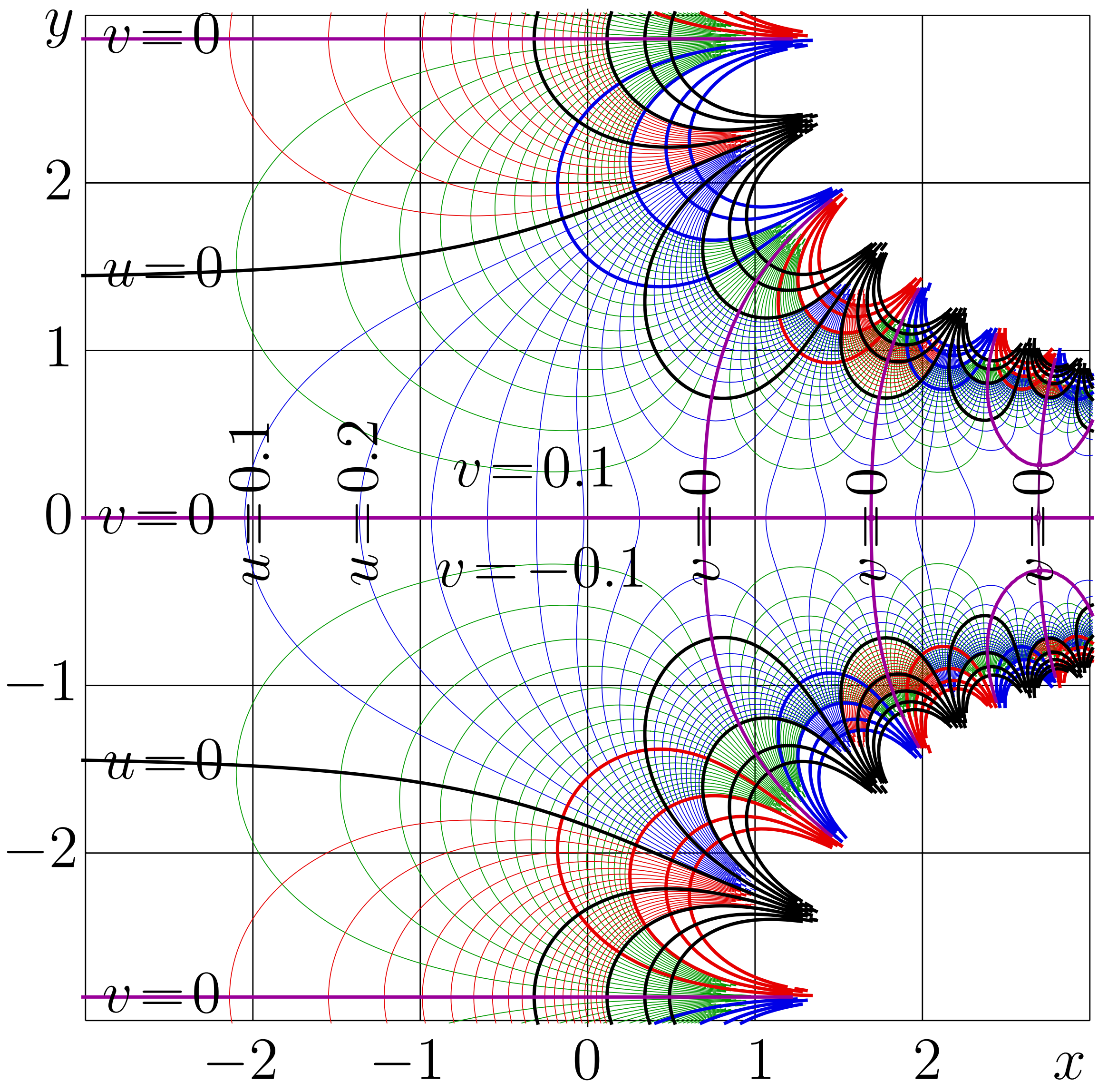

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Figure 7.4: $u+\mathrm{i}v=\mathrm{Pav}_3(x+\mathrm{i}y)$ by (7.8) [logi2c3]

I need some name to denote superfunction for the transfer function Elu_z; For this superfunction, I suggest name "Pav". In this name, I use first 2.5 characters of the first name Pawel of my teacher (who had asked me to construct this function; half of letter w appears as v). First three characters of his last name Elutin are already used below to denote the quadratic function (7.1), known also as "logistic map". My excuse for defining of the new name is the following: the "logistic map", as it is called in the literature, is not actually map in the common sense of this word; it is holomorphic function; and the similar note refers to the so-called "logistic sequence". So, I use term "Elutin function" insteat of "logistic map", in the context of this book, it is holomorhic function rather than a map.

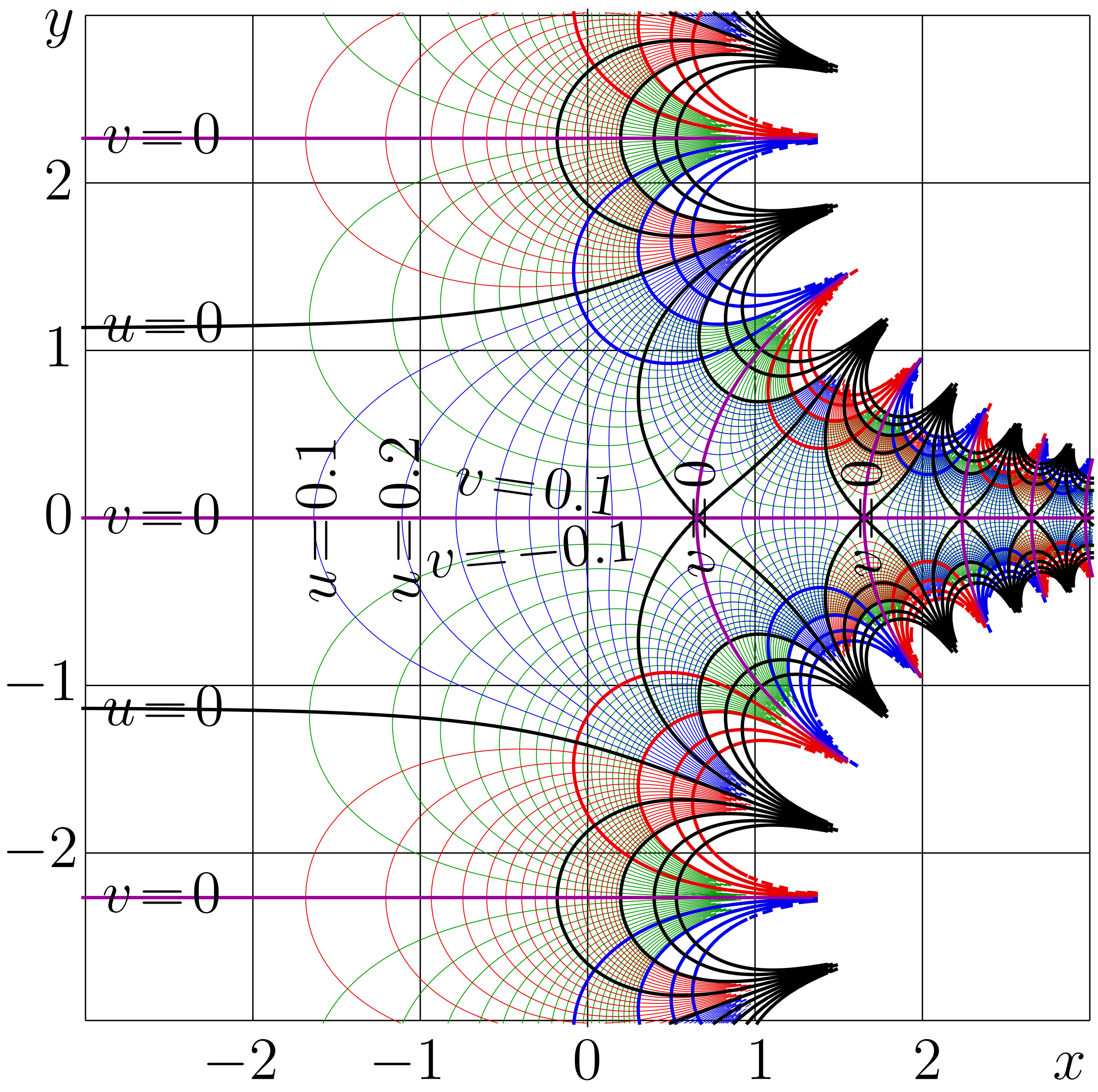

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Figure 7.5: $u+\mathrm{i}v=\mathrm{Pav}_4(x+\mathrm{i}y)$ by (7.8) [logi2c4]

In order to define function Pav_s, first, I construct its asymptotic. For the fixed point $L=0$, in the expansion (6.2), increment $k=\log s$, and the expansion parameter $\varepsilon=s^z$; This gives the primary expansion of the superfunction

$$\tilde{F}(z) = \sum_{n=1}^{N-1} a_n s^{nz} + \mathcal{O}(s^{Nz}) \quad \text{[as]} \qquad (7.4)$$

For the representation (6.2), I set also $a_1 = 1$. Variation of this parameter causes only the displacement of the argument of the resulting superfunction; this does not affect the iterates of the transfer function

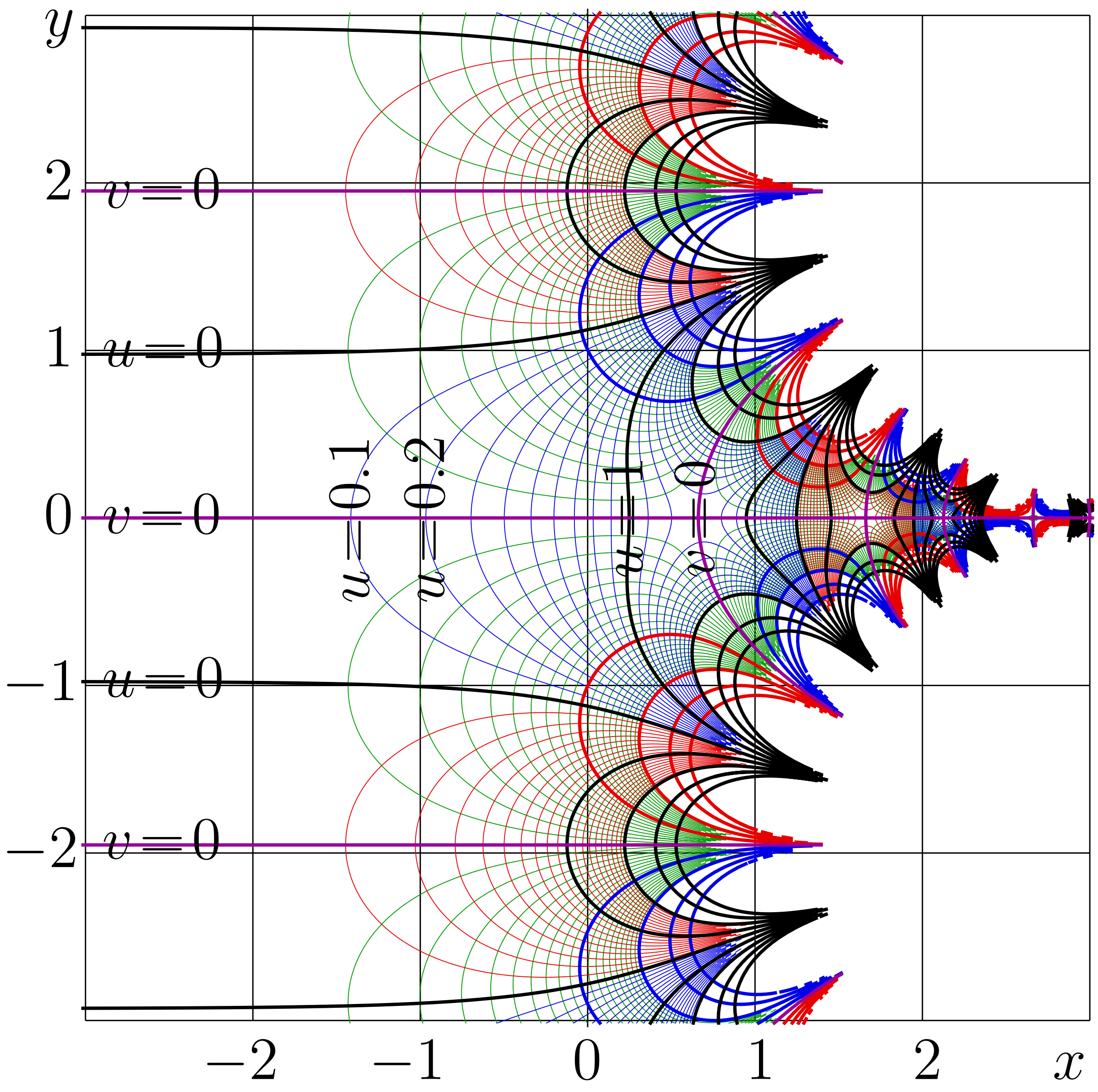

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Figure 7.6: $u\!+\!\mathrm{i}v\!=\!\mathrm{Pav}_5(x\!+\!\mathrm{i}y)$ by (7.8) [logi2c5]

Elu_s. Expressions (6.10) determine the coefficients a. In particular,

$$a_2 = \frac{1}{1\!-\!s} \qquad \text{[logia2]} \qquad (7.5)$$

$$a_3 = \frac{2}{(1\!-\!s)(1\!-\!s^2)} \qquad \text{[logia3]} \qquad (7.6)$$

$$a_4 = \frac{5+s}{(1\!-\!s)(1\!-\!s^2)(1-s^3)} \qquad \text{[logia4]} \qquad (7.7)$$

The primary representation $\tilde F(z)$ by (7.4) allows the accurate (precise) evaluation of $F(z)$ at large negative values of $\Re(z)$. Then, the continual extension of the logistic sequence, shown in figure 7.3, appears as limit

$$F(z) = \mathrm{Pav}_s(z) = \lim_{n\rightarrow\infty} \mathrm{Elu}_s{}^n\Big(\tilde F(z\!-\!n)\Big) \qquad \text{[logilim]} \qquad (7.8)$$

For real argument, graphics of superfunction $F=\mathrm{Pav}_s$ are shown in figure 7.3 for $s=3$, $s=3.4$, $s=3.8$ at the top picture and for $s=3.9$, $s=4$, $s=4.1$ at the bottom picture. For $s=3$, $s=4$ and $s=5$, the complex maps of function $F=\mathrm{Pav}_s$ are shown in figures 7.4, 7.5, 7.6 announced above.
While $s<3.5$ (id est, does not exceed the "Pomequ-Mannevill constants" [69, 19, 53]), the logistic sequence F (as for integer values of the argument, as for the real values) shows pretty boring and regular oscillations. At larger values of s, at the argument grows, the oscillations become dense. Observation of $F(n)=\mathrm{Pav}_s(n)$ at the integer values of n make an impression of quasi-random sequence. While $s\le 4$, at real x, function $F(x)$ oscillates within segment [0,1], and only at $s=4$ it touches the borders of this segment. At $s>4$, the function has "gaps"; they become deeper and deeper at the increase of the argument x or parameter s.

At $s=4$, the holomorphic extensiion $F_s=\mathrm{Pav}_s$ of the logistic sequence can be represented as the elementary function,

$$\mathrm{Pav}_4(z) = (1-\cos(2^z))/2 \quad \text{[logicos]} \tag{7.9}$$

The curve, corresponding to $s=4$ at figure 7.3, could be plotted even without the regular iteration; the same applies to map at figure 7.5.

At consideration of the holomorphic extension of the logistic sequence in the complex plane, its behaviour is regular. However, at large s, the oscillations become more and more dense with the increase of the argument.

Superfunction F_s, built with regular iteration at the fixed point $L=0$, is entire and periodic. The period is pure imaginary (for real s):

$$P = 2\pi \mathrm{i}/\ln(\mathrm{s}) \quad \text{[logiperiod]} \tag{7.10}$$

This periodicity is seen in figures 7.4, 7.5, 7.6: The isolines are reproduced with the corresponding translations along the imaginary axis. A little bit more than one period fit the height of the map in figures 7.4, 7.5, 7.6.

With holomorphic extension of the logistic sequence, id est, superfunction Pav_s, one can construct the non-integer iterates of the logistic map, id est, iterate the Elutin function Elu_s by (7.1); in particular, the half iterate of this transfer function can be constructed. However, for this, the inverse function is also required, I mean, the abelfunction $G_s=\mathrm{ArcPav}_s=\mathrm{Pav}_s^{-1}$.

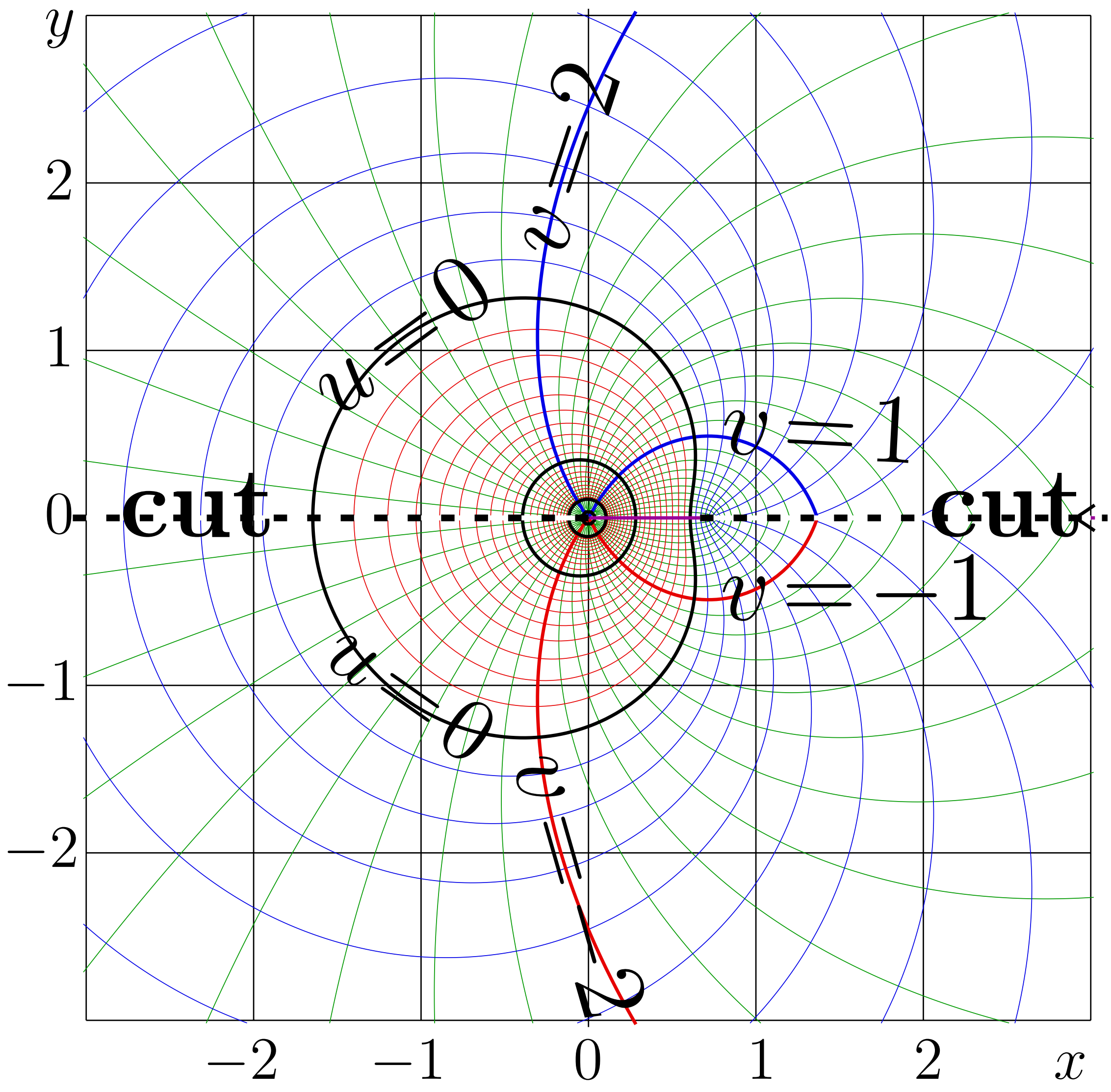

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Figure 7.7: $u+\mathrm{i}v = \mathrm{ArcPav}_3(x+\mathrm{i}y)$ [logi2d3]

The complex maps of the abalfunction ArcPav_s are shown in figures 7.7, 7.8, 7.9 for $s=3$, $s=4$ and $s=3$. This abelfunction is described in the next section.

3 Abelfunction for the Elutin function

This section describes the Abel function of the Elutin function Elu_s by (7.1). I call this function ArcPav_s. It is inverse function of function Pav_s by (7.8), described in the previous section.

Function $G=G_s=\mathrm{ArvPav}_s=\mathrm{Pav}_s^{-1}$ satisfies the Abel equation

$$G_s(\mathrm{Elu}_s(z)) = G_s(z) + 1 \quad \text{[GT]} \tag{7.11}$$

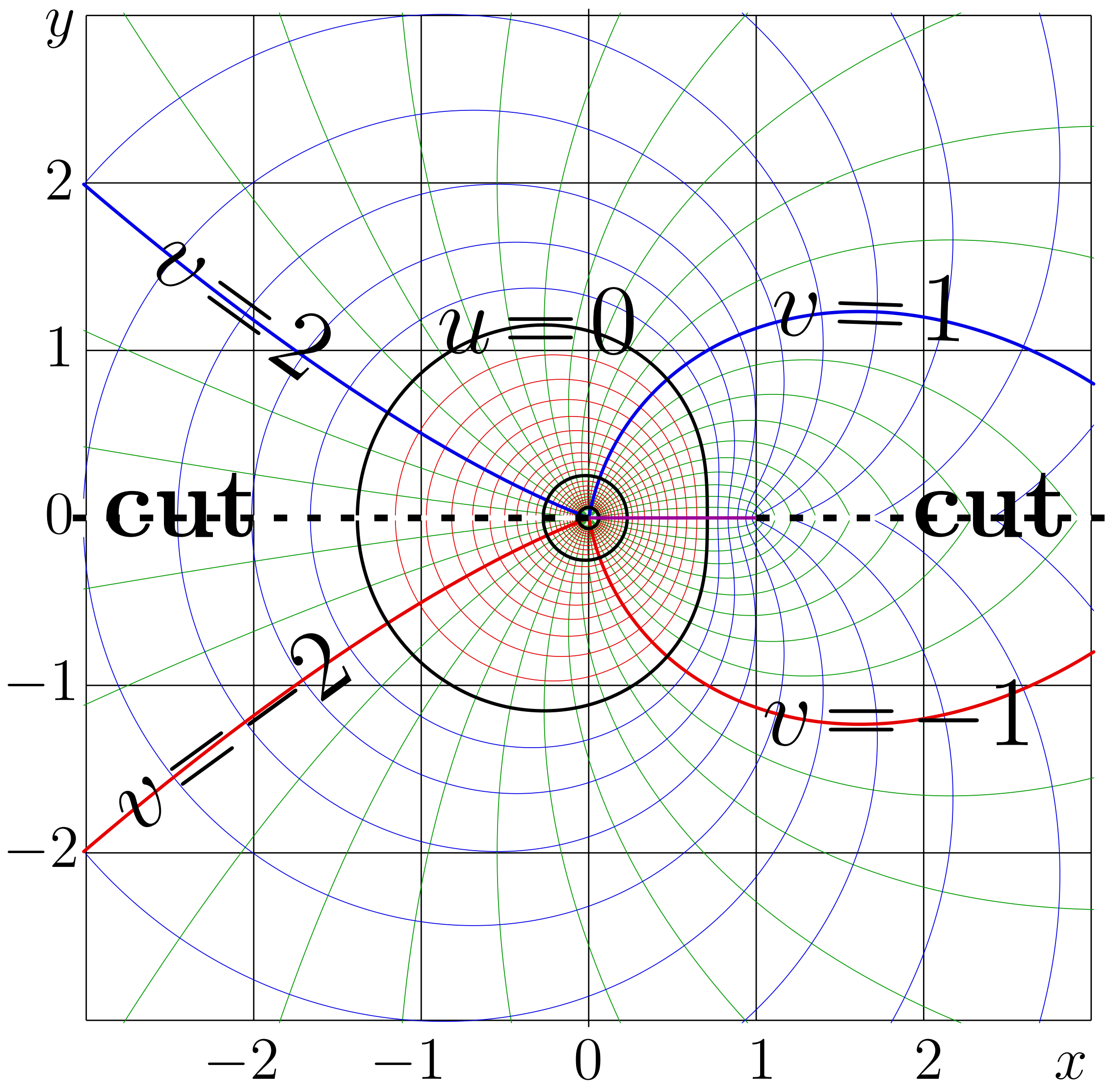

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Figure 7.8: $u+\mathrm{i}v = \mathrm{ArcPav}_4(x+\mathrm{i}y)$. [logi2d4]

This equation is the same as (2.13), the only additional subscript $_s$ is gaffed to indicate the parameter in the transfer function $T = \mathrm{Pav}_s$. Complex maps of function G_s are shown in figures 7.7, 7.8, 7.9 for various values of s. This section describes evaluation of this function.

The asymptotic expansion $\tilde G$ for abelfunction G can be found with inversion of expansion (7.4) for superfunction F; it has the following form:

$$\tilde G(z) \;=\; \log_s \left(\sum_{n=1}^{N-1} C_n z^n + O(z^N) \right) \qquad \text{[GC]} \qquad (7.12)$$

In the software Mathematica, there is special procedure **InverseSeries** for such inversions.

Coefficients C in equation (7.12) depend also on parameter s, but this

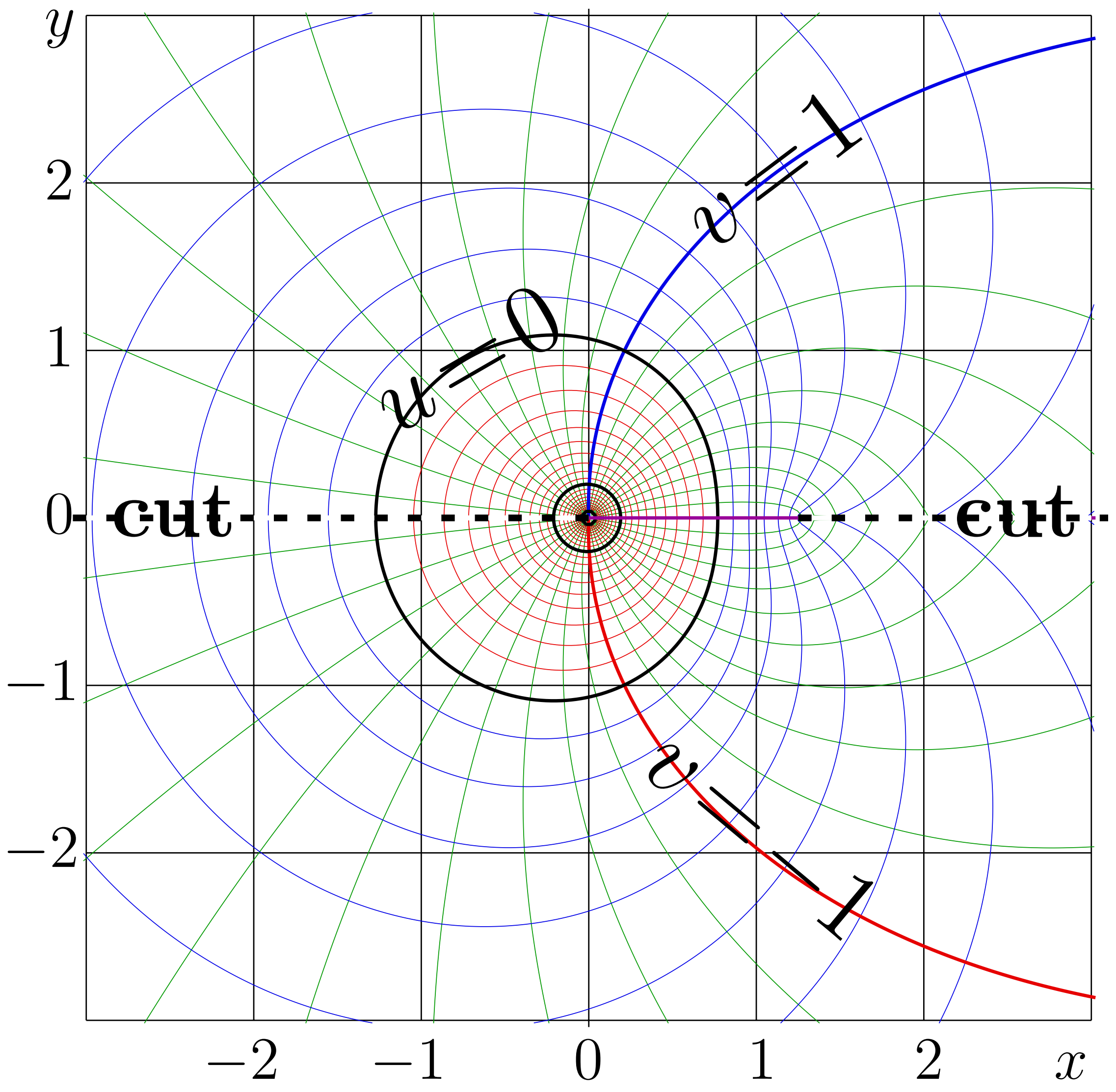

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Figure 7.9: $u+\mathrm{i}v = \mathrm{ArcPav}_5(x+\mathrm{i}y)$. [logi2d5]

dependence is not indicated explicitly, in order to keep the expression compact. The same coefficents can be found also substituting the expansion (7.12) into the Abel equation (7.11) with transfer function Elu_s and equalising the coefficients at equal powers of z in the left and in the right hand sides. In particular,

$$C_1 = 1 \tag{7.13}$$

$$C_2 = \frac{1}{s-1} \tag{7.14}$$

$$C_3 = \frac{3s}{(s-1)(s^2-1)} \tag{7.15}$$

$$C_3 = \frac{(s^2-5)s}{(s-1)(s^2-1)(s^3-1)} \tag{7.16}$$

The truncated series of expansion (7.12) gives way to evaluate abefunction G at small values of the argument. At large values, the representation can be extended with

$$G(z) \approx \tilde{G}(\mathrm{Elu}_s^{-n}(z)) + n \quad \text{[Git]} \tag{7.17}$$

for some large enough integer n. The negative integer iterates of the transfer function can be evaluated, using representation of the inverse of the transfer function:

$$\mathrm{ArcElu}_s^{-1}(z) = \mathrm{ArcElu}_s(z) = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{z}{s}} \tag{7.18}$$

Symbol "≈" in expression (7.17) should not make an impression, that only the approximation of the Abel function is constructed. The exact Abel function can be expressed through the limit

$$\mathrm{ArcPav}_s(z) = G(z) = \lim_{n\to\infty} \Big(\tilde{G}(\mathrm{Elu}_s^{-n}(z)) + n \Big) \tag{7.19}$$

where symbol $\tilde{G}_s$ denotes the truncated series in the expansion (7.12). This representation is used in generators of figures 7.7, 7.8, 7.9 . The same representation is used to plot iterates of the transfer function in figure 7.2. at $s\!=\!3$, $s\!=\!4$ and $s\!=\!5$.

Function $\mathrm{ArcElu}_s(z)$ by (7.18) has the branch point $z = s/4$. This determines also the branch of the abelfunction ArcPav_s and cuts in the right hand sides of maps in figures 7.7, 7.8, 7.9 . The non-integer iterates of the transfer fiunction, shown in figure 7.2, have the similar branching. The half iterate is special case of the non-integer iterate; this case shown in figures 7.10, 7.11, 7.12 and described in the next section.

4 Halfiteration of logistic operator

With superfunction F_s and abelfunction $G_s = F_s^{-1}$, iterates of the transfer function T_s are expressed with formula (7.3). This representation is used to generate figure 7.2. In particular, $T_s^{0.5}$ by (7.3) is solution of the problem, formulated by P.Elutin: The function h_s is built as

$$h_s \!=\! \mathrm{Elu}_s^{0.5} \qquad \text{[logih]} \tag{7.20}$$

such that its second iterate gives the logistic operator (7.1); id est, for some range of values of z, the relation below holds:

$$h^2(z) = h(h(z)) = T(z) = \mathrm{Elu}_s(z) = s\, z\, (1\!-\!z) \tag{7.21}$$

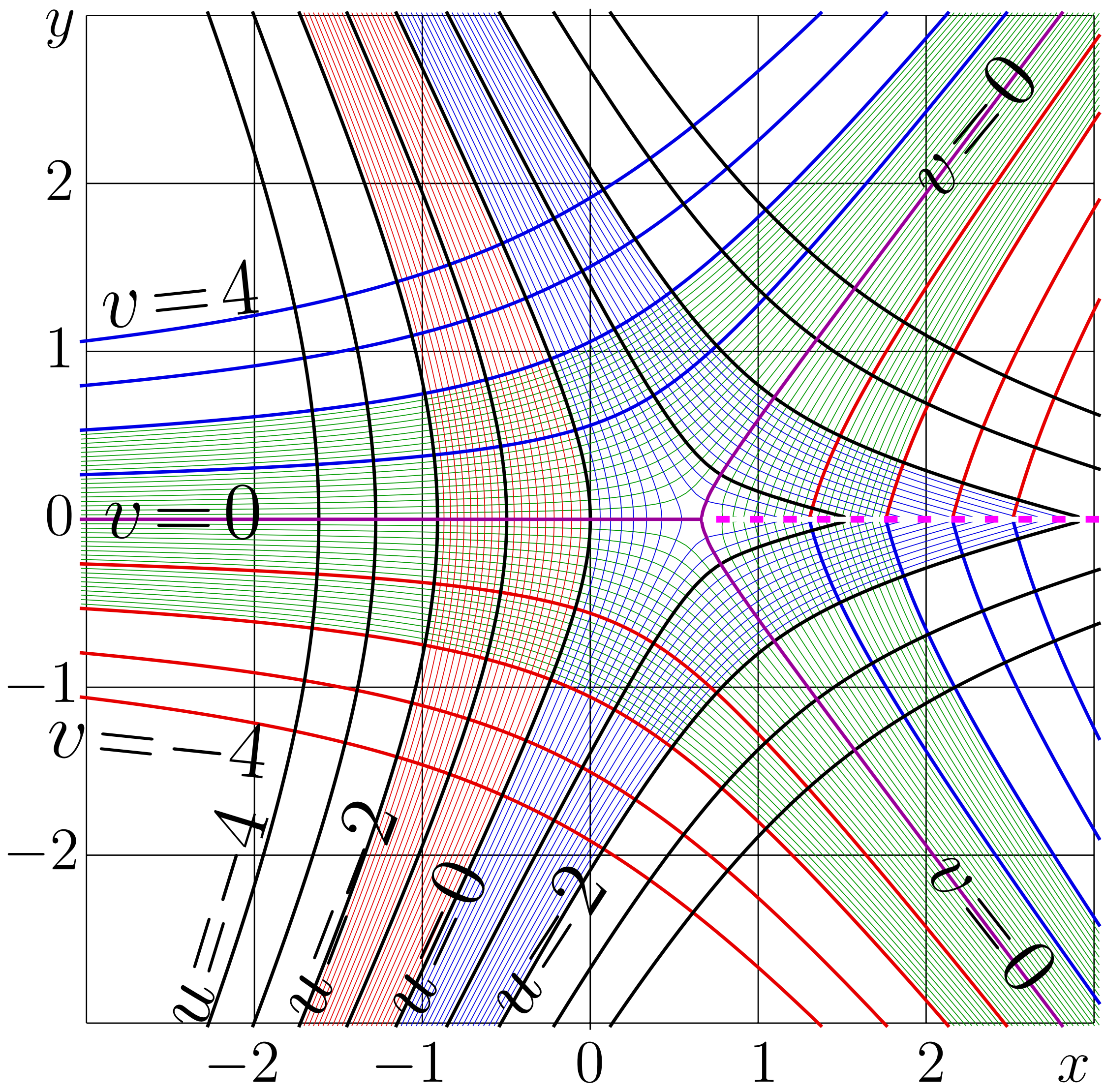

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Figure 7.10: $u+\mathrm{i}v = \mathrm{Elu}_3^{1/2}(x+\mathrm{i}y)$. [logi2b3]

Complex maps of function h_s are shown in figures 7.10, 7.11 and 7.12 for various values of parameter s. These maps look similar, because, at large values of the argument z, the approximate relation takes place:

$$\mathrm{Elu}_s(z) \approx \tau(z) = -s(-z)^2 \quad \text{[lopbig]} \tag{7.22}$$

Iterates of function τ by (7.22) can be calculated in analogy with iterates of the transfer function by (4.19).

The readers are invited to plot the superfunction and the abelfunction for the transfer function τ. This can be done in analogy with the iterates of the power function, presented in Table 3.1. In particular, the half iterate can be written as follows:

$$\tau^{1/2}(z) = -\alpha(-z)^{\sqrt{2}} \quad \text{[logita]} \tag{7.23}$$

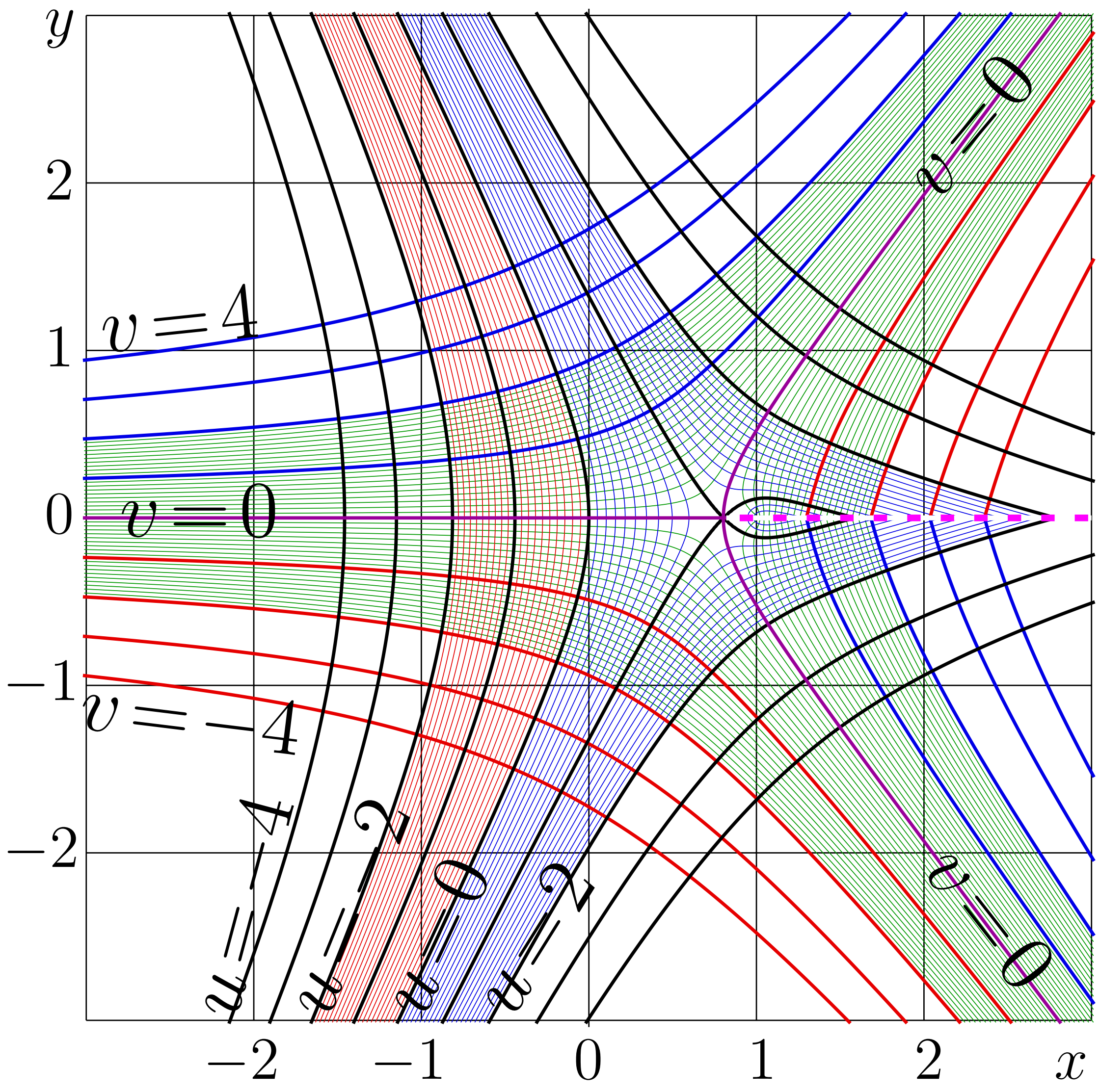

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Figure 7.11: $u+\mathrm{i}v = \mathrm{Elu}_4^{0.5}(x+\mathrm{i}y)$ [logi2b4]

where $\alpha = s^{1/(1+\sqrt{2})}$ is constant, and this constant depends on parameter s slowly.

Figure 7.13 shows the map of function $\tau^{1/2}$ by formula 7.23 at $\alpha = 1.8$; it should be compared to maps 7.10, 7.11 and 7.12 , that correspond to various values of s. At large values x^2+y^2, the only quadratic term in the expansion of the transfer function is important, and all the four maps at figures 7.13, 7.10, 7.11 and 7.12 look similar. In order to make the difference seen, these maps occupy all the width of the page available.

When the non-integer iterate of some function is constructed, it has sense to check the results of iteration of this iterate. For the transfer function Elu_s and its half iterate $h=\mathrm{Elu}_s^{0.5}$, such a check, test is shown

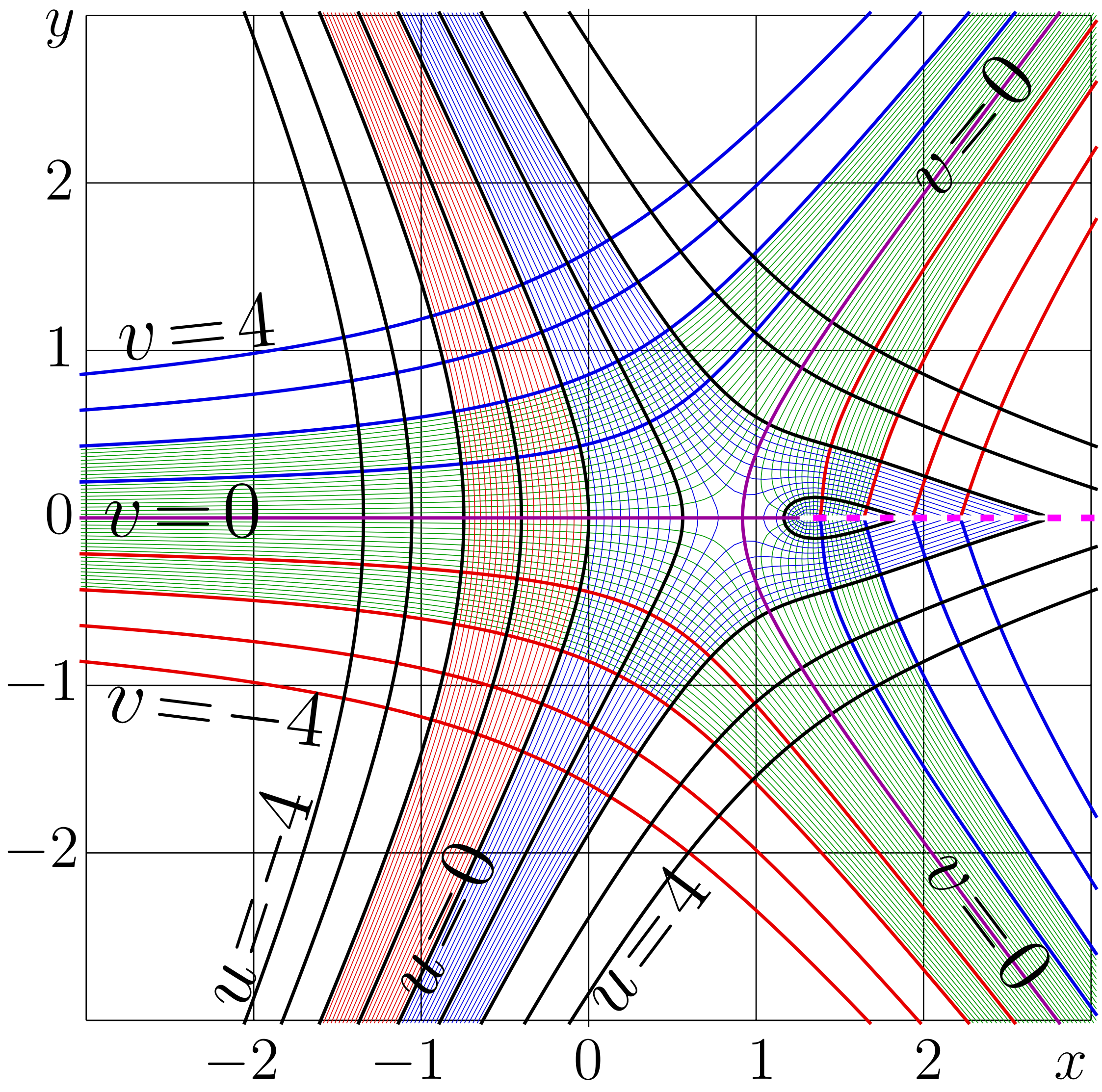

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Figure 7.12: $u+\mathrm{i}v = \mathrm{Elu}_5^{0.5}(x+\mathrm{i}y)$ [logi2b5]

in figure 7.14. This figure shows the map pf function h^2 for $s=3$, $s=4$ and $s = 5$, id est, for the same values of parameter, that are used in figure 7.2.

In the left hand side of maps in figure 7.14, the second iterate of function h coincides with the logistic mapping. The scratched lines show the limit of applicability of relation (7.21).

Relation $h(h(z)) = T(z)$ is valid at least for $\Re(z) < 1/2$. In such a way, the holomorphic extension of the logistic sequence, and also the corresponding non-integer iterates of the logistic operator correspond to the intuitive expectations about these functions.

There is nothing specific in the half iterate of a function, while it is

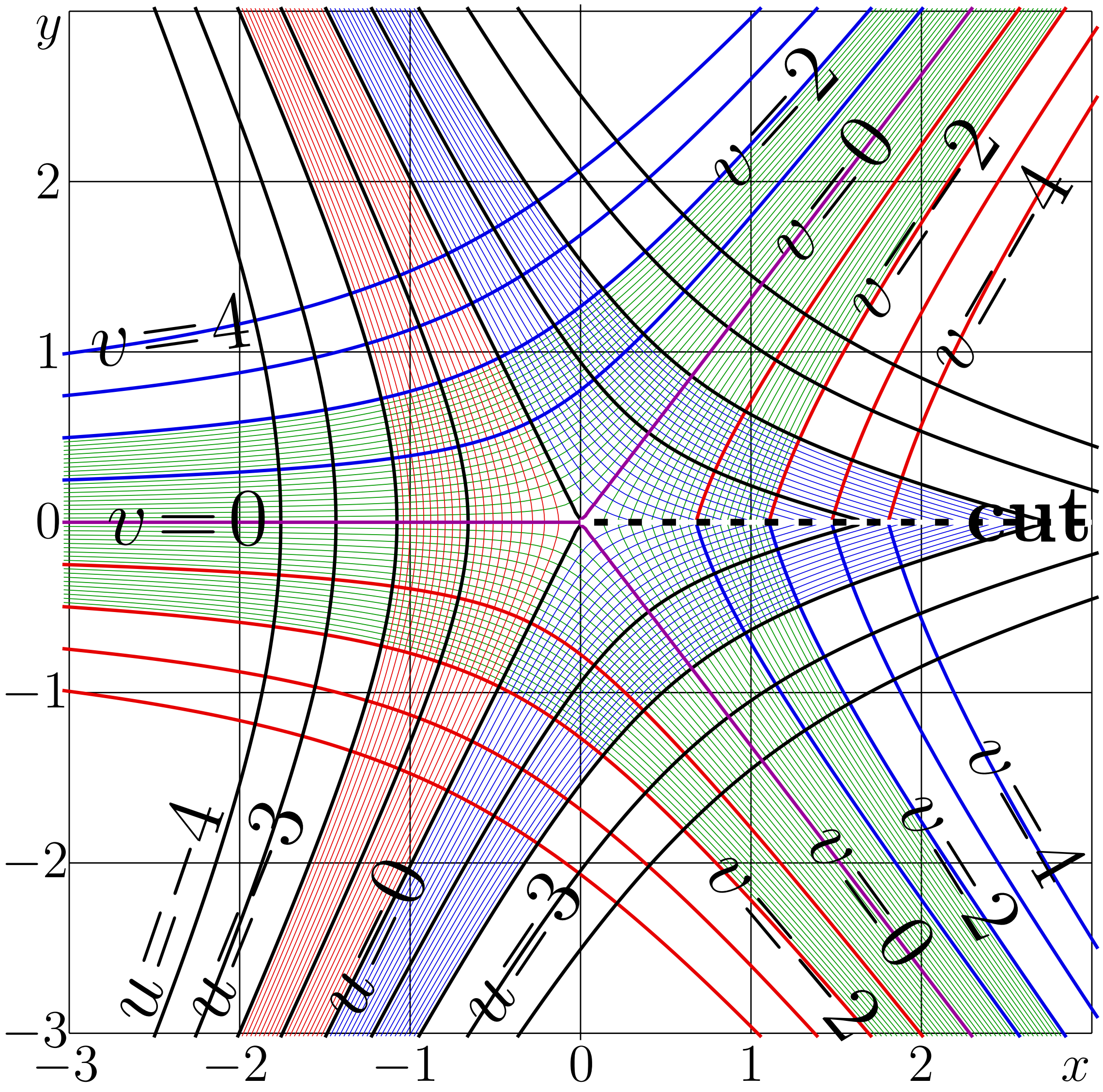

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Figure 7.13: $u+\mathrm{i}v = -1.8(-(x+\mathrm{i}y))^{\sqrt{2}}$

expressed through the superfunction and the abelfunction. It is just non-integer iterate, while number n of iterate is $1/2$. On the other side, in the literature, the iteration half is often considered as something magic; perhaps, because for iterate half of some transfer function, the verification is especially simple; just iterate the halfiterate twice and see the region, where the result coincide with the original transfer function. Figure 7.14 appears as an example of such a verification.

Interest to iterate half, since the half iterate of factorial ($\sqrt{!}$) [36, 37] appears as a tradition. Namely iterate half is mentioned in the titles of publications [10, 92]. So, I follow this tradition and analyse range of validity of representation of a function through its iterate half.

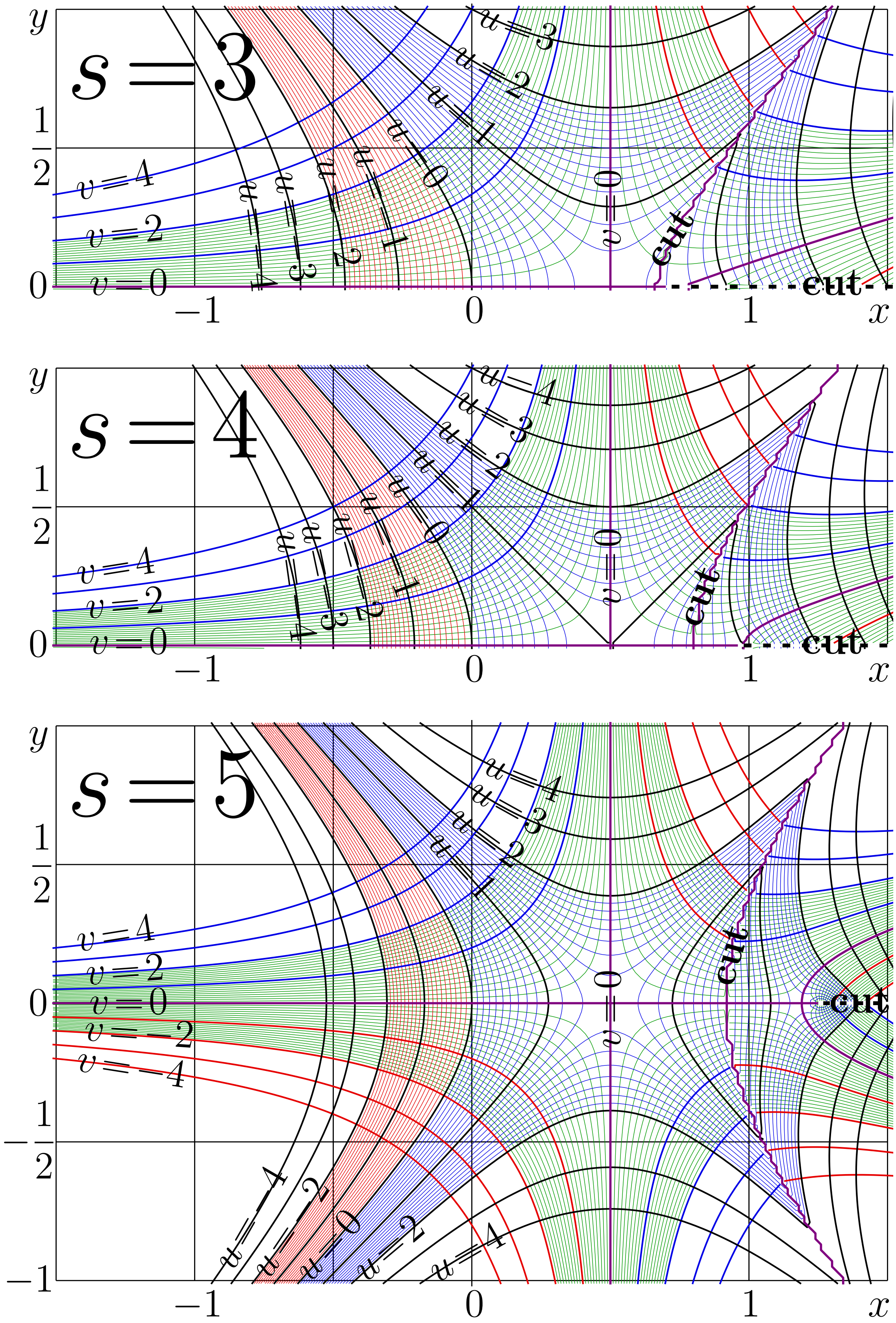

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Figure 7.14: $u+\mathrm{i}v=h(h(x+\mathrm{i}y))$ by (7.20) for $s=3$, $s=4$ and $s=5$

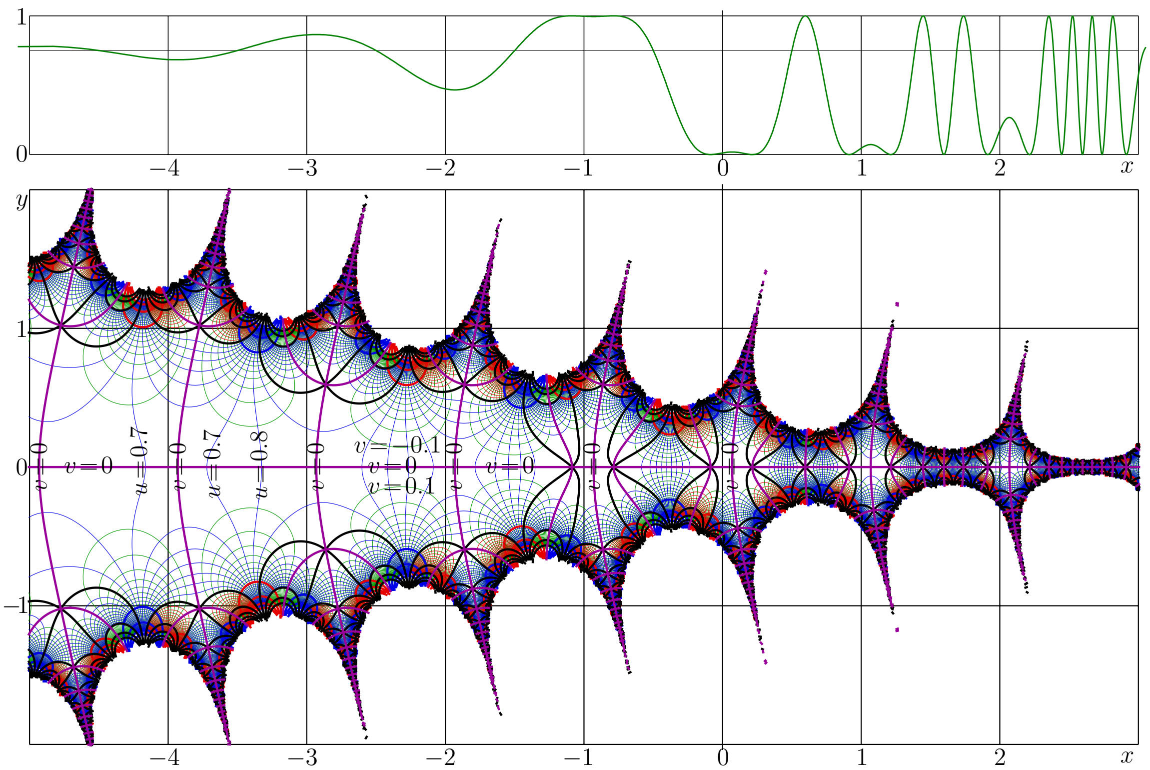

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Figure 7.15: Superfunction $\mathrm{Paw}_4(x)$ by (7.25),(6.12): explicit plot $y = \mathrm{Paw}_4(x)$, top picture, and complex map $u+\mathrm{i}v = \mathrm{Paw}_4(x+\mathrm{i}y)$, bottom picture, by (7.28) [logi5ab4]

5 Another fixed point, $L = 1 - 1/s$

Holomorphic extension F_s of the logistic sequence by formulas (7.4),(6.12) is regular and periodic; the period P_a by (7.10) is pure imaginary and slowly (as logarithm) depends on parameter s. This solution is not unique. In analogy with the solution, that approaches the fixed point $L=0$, one may construct other solutions, that approach the fixed point $L=1-1/s$ at minus infinity. Complex map of one of such solutions is shown in figure 7.15; this superfunction is described in the is section.

The logistic operator, id est, the transfer function T_s by (7.1) has two fixed points, $L=0$ and $L = 1 - 1/s$. Consider the last of these fixed points. By the general way of regular iterations, I construct the superfunction f, that approaches $1-1/s$ at minus infinity, as follows:

$$\tilde{f}(z) = \frac{s-1}{s} + \sum_{n=1}^{N-1} d_n \Big((s-2)^z \cos(\pi z + \varphi) \Big)^n \qquad \text{[logina]} \qquad (7.24)$$

$$f(z) = \mathrm{Paw}_s(z) = \tilde{F}(z) + O\Big((s-2)^z \cos(\pi z + \varphi) \Big)^N \qquad \text{[L11s]}(7.25)$$

where d are real parameters and φ is real constant. Substitution of this expansion into equation (2.12) gives the chain of equations for coefficients d. As in the previous cases, I set $d_1 = 1$; then

$$\begin{aligned} d_2 &= \frac{-s}{(s-1)(s-2)} \\ d_3 &= \frac{-s^2}{(s-1)(s-2)(s-3)} \\ d_4 &= \frac{-(s-7)^3\, s^3}{(s-2)(s-3)(s^3-8s^2+22s-21)} \end{aligned} \qquad \text{[logid2]} \qquad (7.26)$$

The truncated series gives the accurate approximation of function $F(z)$, while the effective parameter of expansion, id est $(s-2)^z \cos(\pi z + \varphi)$ is small. Truncation with 4 coefficients in (7.25) provides of order of 10 significant figures while

$$\pi|\Im(z)| + \ln(s-2)\Re(z) < 4 \qquad (7.27)$$

The range of approximation can be extended with iterations (6.12); this can be used both for definition and algorithm of precise evaluation of the superfunction:

$$f(z) = \mathrm{Paw}_s(z) = \lim_{n\to\infty} \mathrm{Elu}_s^n(\tilde f(z-n)) \qquad \text{[inoelim]} \qquad (7.28)$$

I denote this function with Paw_s, in order to distinguish it from Pav_s in the previous sections; I hope, use of generic name F for different superfunctions will not cause confusions. Representation (7.28) is used to generate figure 7.15.

Superfunction Paw_s by (7.28) is asymptotically–periodic; the asymptotic period

$$P = \frac{2\pi}{\ln(s-2)\,\mathrm{i} + \pi} \qquad \text{[logiquasiP]} \qquad (7.29)$$

in the upped half plane and P^* in the lower half plane.

In contrast with superfunction Pav_s, that is built up at the fixed point zero, for superfunction Paw_s, the choice of the inverse function is not straightforward. One needs to choose, which of the oscillations should be used to return the value of function. For this reason I do not provide the corresponding abelfunction here. The Reader is invited to construct it according to own preferences.

Following the 6th of the TORI axioms, about the simplicity, I consider the superfunction by (7.4),(7.8) as "principal", because it seems to me simpler than that by (7.24). For the logistic operator as transfer function, the regular iteration provides the regular superfunction; however, superfunctions, constructed at different fixed points, may show pretty different behaviour.

Chapter 8

Factorial

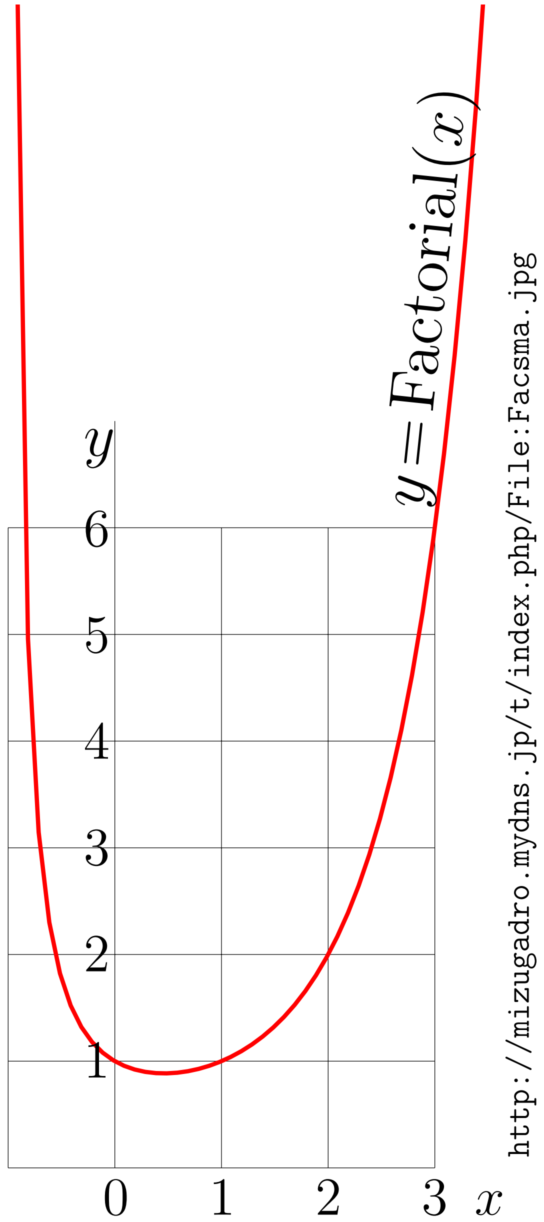


Figure 8.1: $y\!=\!\mathrm{Factorial}(x)$

Factorial is holomorphic solution of equation

$$\mathrm{Factorial}(z\!+\!1) = z\,\mathrm{Factorial}(z) \quad (8.1)$$

Factorial can be expressed with

$$\mathrm{Factorial}(z)\!=\!\int_0^\infty t^z e^{-t}\,\mathrm{d}t \qquad (8.2)$$

I use this notation instead of that with with exclamation, $\mathrm{Factorial}(z) = z!$, in order to simplify indication of the number of iterate in the upper superscript.

Graphic $y = \mathrm{Factorial}(x)$ versus x is shown in figure 8.1. At $x\!>\!2$, factorial shows fast monotonous growth. At $0 < n < 1$, the iterates $\mathrm{Factorial}^n$ should show the similar, but slower growth. These iterates are topic of this chapter.

This chapter describes the superfunction of factorial, denoted below as SuFac, and the Abel function, denoted as AuFac. Then, the iterates of Factorial appear as

$$\mathrm{Factorial}^n(z) = \mathrm{SuFac}\big(n + \mathrm{AuFac}(z)\big) \quad \text{[faciterge]} \quad (8.3)$$

Below, I construct functions SuFac and AuFac and describe their properties. In this chapter, I retell the basic concept of publication in the Moscow University Physics Bulletin, 2010 [65].

1 Physics department

Iterates of factorial and its superfunction and abelfunctions had been reported in 2010 [65], before iterates of the logistic map (Elutin function), described in the precious chapter. In this section I explain, why I consider this function as important.

Past century, during the USSR, my teacher of Quantum Mechanics had asked students to give the physical sense to the operator "square root of factorial". That sign was well known in the USSR as symbol of the Physics Department of the Moscow State University shown in figure 2.4. In analogy with other operators of Quantum Mechanics, "square root of factorial" should be some function h such that its second iteration gives factorial, $h^2 = \text{Factorial}$, id est,

$$h(h(z)) = z! = \text{Factorial}(z) \quad \text{[hhzfac]} \tag{8.4}$$

One could guess, that this function should be real-holomorphic, growing faster than any polynomial but slower than any exponential. That time it was difficult (if at al), to evaluate such a function h: the formalism of superfunctions had not yet been developed.

In principle, the Schroeder function and the scaling function could be used to construct and evaluate the solution h of equation (8.4), instead of superfunctions. That time, for students, dealing with Quantum Mechanics, it was difficult to guess, that for the square root of factorial, the Schroeder (Schröder) equation should be used instead of the widely known Schroedinger (Schrödinger) one [1]. In addition, that time, no algorithms for evaluation of the scaling function and the Schroeder function were available.

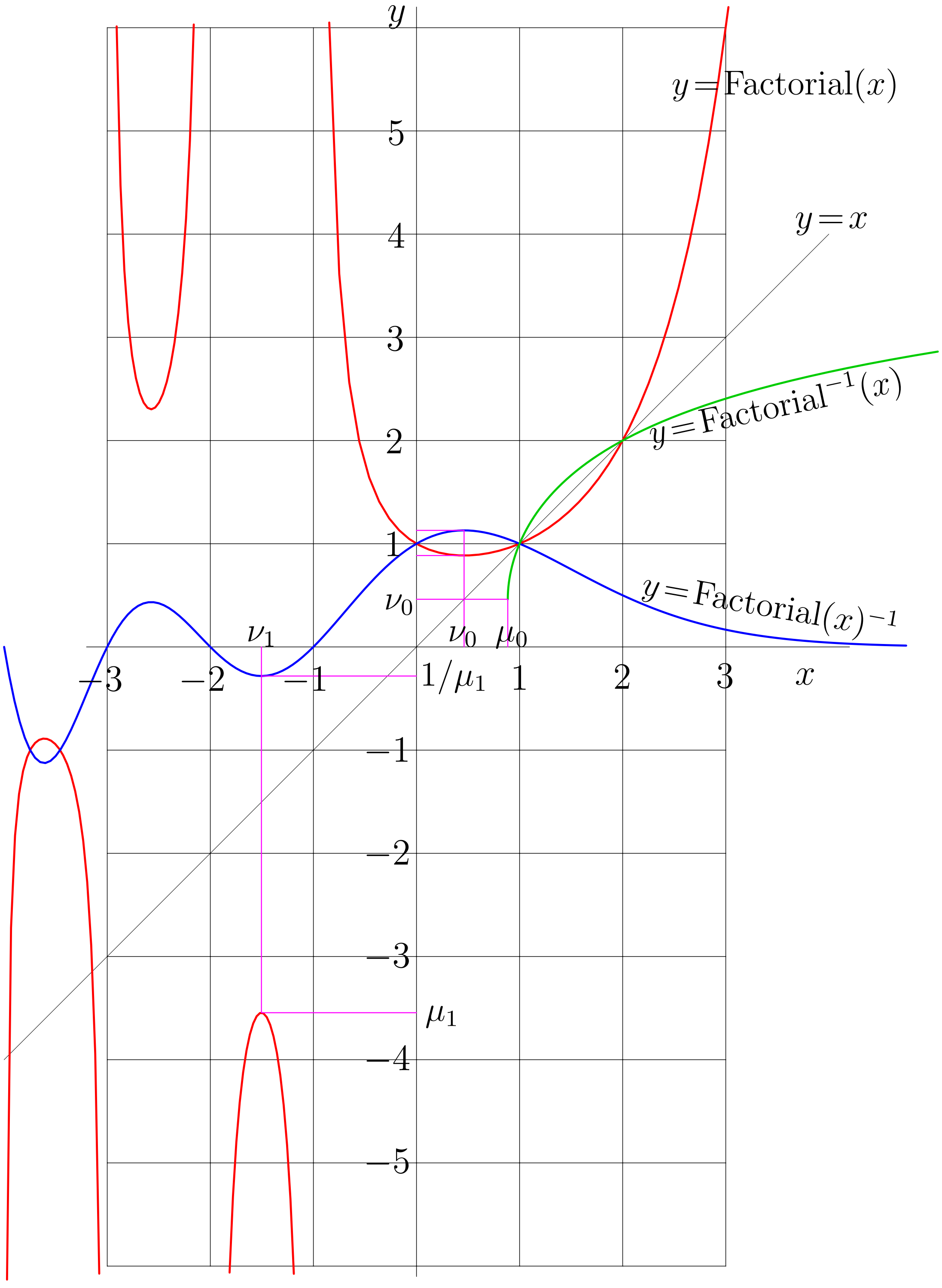
After the successful evalation of tetration to base $\sqrt{2}$, reported in 2009 in journal Mathematics of Computation [61], the problem with iterates of factorial (and, in particular, of the half iterate) happen to be pretty solvable. The solution is published in the Moscow University Physics Bulletin [65] and described below.

I feel, first I should remind the properties of factorial. This is matter of the next section.

[1] Iterates of factorial and $\sqrt{!}$ can be evaluated also with the scaling function f and the Schroeder function g by (6.33)

2 Factorial and its fixed points

For real argument, the explicit plot of factorial is shown in figure 8.1. I extend a little bit that plot in figure 8.2, and I add some other related functions. The complex map of factorial is shown in figure 8.3.



http://mizugadro.mydns.jp/t/index.php/File:FactoReal.jpg

Figure 8.2: Factorial and related functions for real argument [figfac]

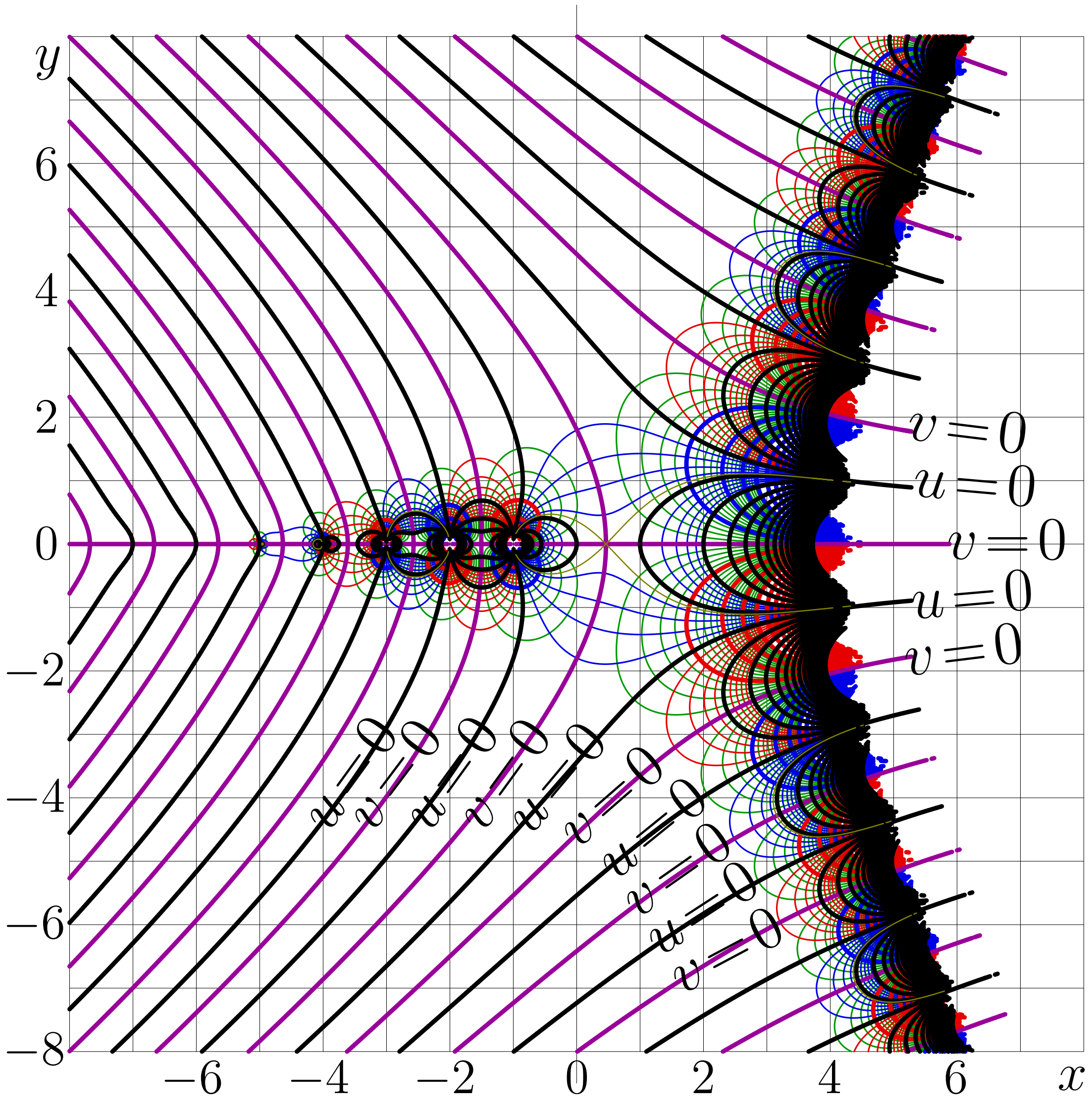


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Figure 8.3: Complex map: $u+\mathrm{i}v = \mathrm{Factorial}(x+\mathrm{i}y)$ [facmap]

For construction of superfunction of any transfer function, there is important question about its fixed points. The fixed points of factorial are solutions L of equation $\mathrm{Factorial}(L) = L$. For real values of argument, the explicit plot of factorial is shown in figure 8.2 with thick curve, $y=\mathrm{Factorial}(x)$. The fixed points correspond to intersections of this curve with line $y=x$, also shown in the figure. For comparison, the thin curves show functions $\mathrm{Factorial}^{-1}$ and $z\mapsto \mathrm{Factorial}(z)^{-1}$. These curves are added in order to remind, that $\mathrm{Factorial}^n(z)$, $\mathrm{Factorial}(z^n)$ and $\mathrm{Factorial}(z)^n$ have pretty different meanings. Complex maps of factorial and ArcFactorial are shown in figures 8.3 и 8.4.

I had found no C++ implementations of factorial and arcfactorial for complex arguments in the literature. Mathematica software allows the evaluation, but does it a little bit slowly; the Maple software happened to be not better [51], see also the appendix, chapter 22, section 2.

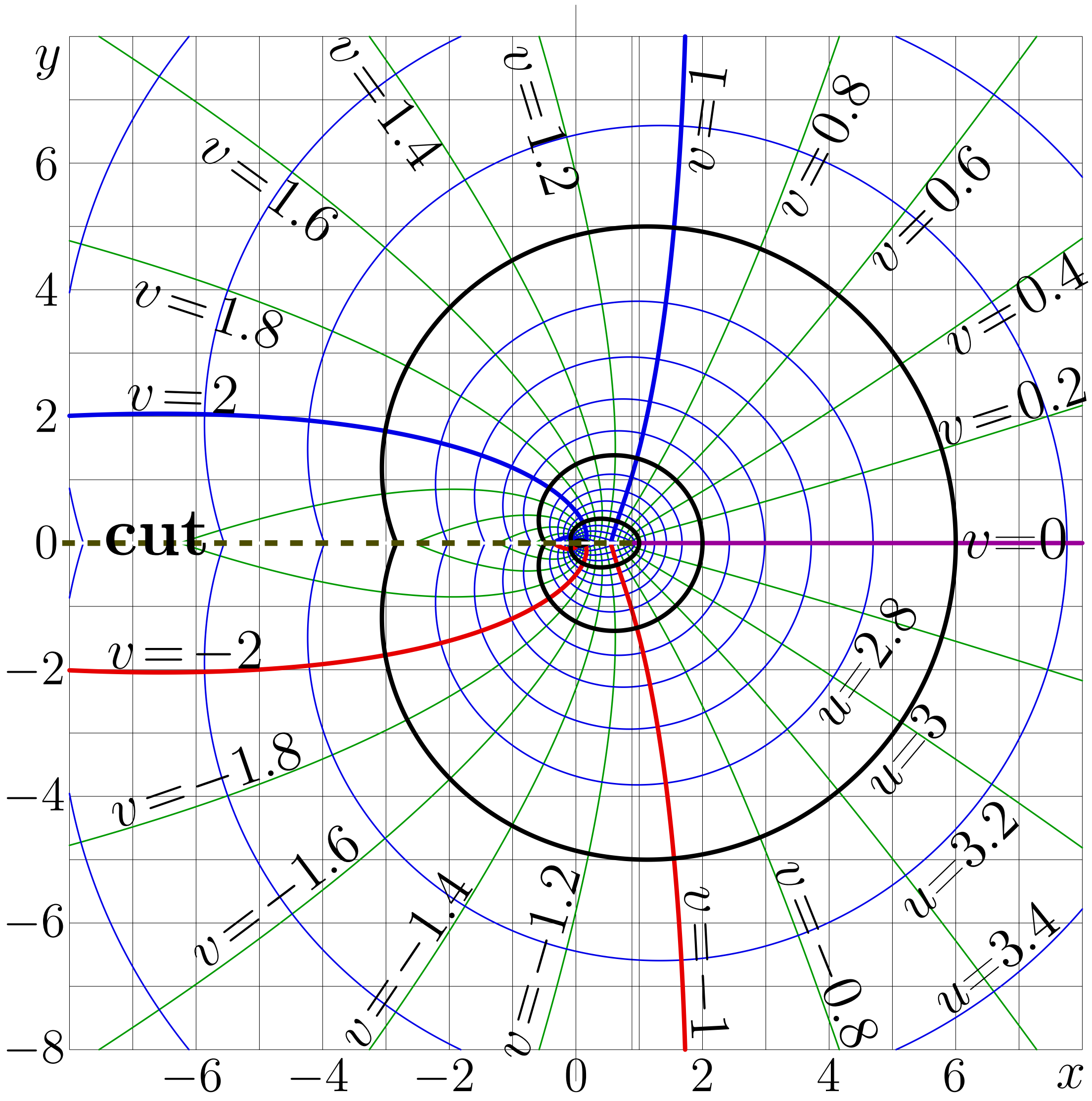

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Figure 8.4: $u+\mathrm{i}v = \mathrm{ArcFactorial}(x+\mathrm{i}y)$ [arcfacmap]

For evaluation of superfunction and the abelfunction, the transfer function and/or its inverse should be evaluated many times, and the efficient (quick and precise) implementation is important. For this reason, the original "complex double" procedures for factorial and arcfactorial are suggested and loaded as
`http://mizugadro.mydns.jp/t/index.php/Fac.cin` and
`http://mizugadro.mydns.jp/t/index.php/Afacc.cin`
These implementations are used to plot figures of this Chapter.

In figure 8.2, I show also line $y=x$ and graphics $y=\mathrm{Factorial}^{-1}(x)$ and $y=\mathrm{Factorial}(x)^{-1}$. Some extremal points of factorial are shown; the local minimum at point $x\!=\!\nu_0$ and local maximum at point $x\!=\!\nu_1$; Values of factorial in these points are denoted as μ_0 и μ_1. Function $y=\mathrm{Factorial}^{-1}(x)$ has the branch point $x=\mu_0$, and ν_0 is its value at this point.

3 Regular iteration for factorial

The asymptotic expansion of superffactorial can be written as follows:

$$\begin{aligned} F(z) &= L + \exp(kx) + a_2 \exp(2kz) + a_3 \exp(3kz) + .. \\ &= L + \varepsilon + a_2\varepsilon^2 + a_3\varepsilon^3 + .. \qquad \text{[sufass]} \end{aligned} \tag{8.5}$$

where $\varepsilon = \exp(kz)$, for some constant increment k and constant coefficients a According to the general formulas (6.9), (6.10), (6.11), substitution of this expansion into the transfer equation

$$\text{Factorial}\big(F(z)\big) = F(z{+}1) \qquad \text{[Tfac]} \tag{8.6}$$

gives the value of the increment. For Factorial's fixed point $L{=}2$, I get

$$k = \ln\big(3{+}2\,\text{Factorial}'(0)\big) = \ln(3{-}2\,\gamma) \approx 0.61278745233070836 \tag{8.7}$$

where $\gamma = -\Gamma'(1) \approx 0.5772156649$ is Euer constant. I set $a_0 = 2$ and $a_1{=}1$; then the partial sum in expansion (8.5) is easy to program. I get

$$a_2 = \frac{\pi^2+6\gamma^2-18\gamma+6}{12(3-5\gamma+2\gamma^2)} \approx 0.79873183517243454158562 1 \tag{8.8}$$

$$\begin{aligned} a_3 &= \Big(-36-39\pi^2-738\gamma^2+324\gamma+99\pi^2\gamma-60\pi^2\gamma^2-\pi^4+24\gamma^5 \\ &\quad +594\gamma^3-120\zeta(3)\gamma+48\zeta(3)\gamma^2+12\gamma^3\pi^2+72\zeta(3)-204\gamma^4\Big)/ \\ &\quad \Big(144(-18+69\gamma-104\gamma^2+77\gamma^3-28\gamma^4+4\gamma^5)\Big) \\ &\approx 0.57788097547648323580 38 \end{aligned} \tag{8.9}$$

In equation (8.9), the Riemann zeta-function appears, $\zeta(z) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$.

We need $\zeta(3) \approx 1.202056903$; values of function ζ at other arguments are not used for evaluation of coefficients a. Approximate values of coefficients in expansion (8.5) are shown in table 8.1.

For factorial, the increment $k > 0$; expansion (8.5) gives good approximation $\tilde F(z)$ at $-\Re(z) \gg 1$. For other z, I use the integer iterates; the superfunction

$$F(z) = \lim_{n\to\infty} \text{Factorial}^n\Big(\tilde F(z{-}n)\Big) \qquad \text{[regufac]} \tag{8.10}$$

For $\Re(z) \sim 1$, it is sufficient to iterate expression (8.10) only few times, in order to get ε of order of 0.1 or less. Then, some 15 terms in the expansion (8.5) provide of order of 14 correct significant figures of $F(z)$.

Let z_3 be is real solution of equation $F(z_3) = 3$; the evaluation gives $z_3 \approx -0.91938596545217788$; then, I define

$$\text{SuFac}(z) = F(z_3{+}z) \qquad \text{[SuFac]} \tag{8.11}$$

Function SuFac is also superfunction of factorial, and $\text{SuFac}(0){=}3$.

Table 8.1: Coefficients a and U in the expansions (8.5) and (8.14)

n	a_n	U_n
2	0.7987318351724345	−0.7987318351724345
3	0.5778809754764832	0.6980641135593670
4	0.3939788096629718	−0.6339640557572815
5	0.2575339580323327	0.5884152357911399
6	0.1629019581037053	−0.5538887519936520
7	0.1002824191713524	0.5265479025985924
8	0.0603184725913977	−0.5041914604280215
9	0.0355544582258062	0.4854529800293392
10	0.0205859954874424	−0.4694346809094714

I have displaced the argument of superfactorial, in order to let it have integer value at zero. This is smallest integer, that is still greater than the fixed point $L=2$.

Graphic of function SuFac of the real argument is shown in figure (8.5), $y=\mathrm{SuFac}(x)$. For comparison, in the same figure, factorial is shown, $y = \mathrm{Factorial}(x)$. Graphic of factorial passes through poins with coordinates (0,1), (1,1), (2,2), (3,6). Graphic of the super factorial passes through points with coordinates (0,3), (1,6), (2,720); the last point is far away from the range of the figure.

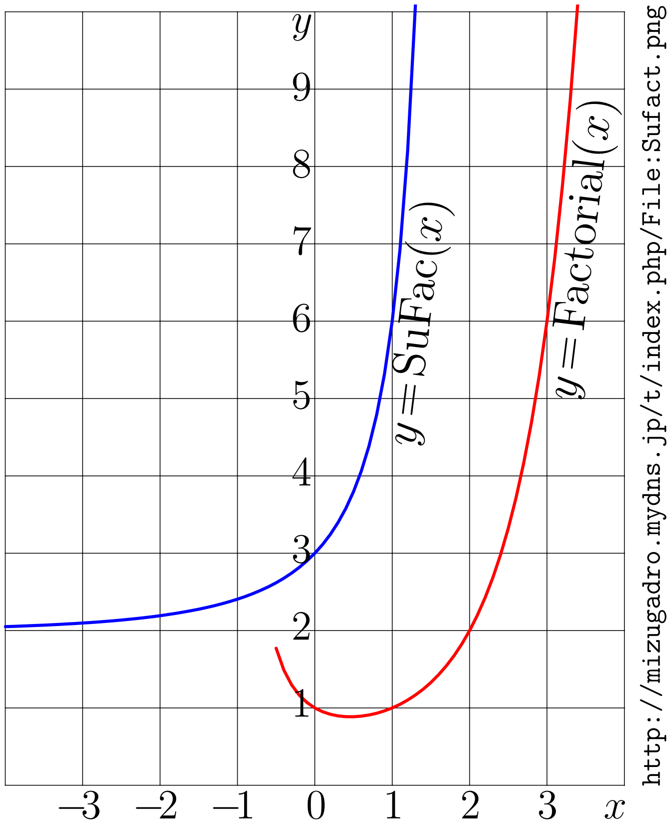

Figure 8.5: $y = x!$ and $y = \mathrm{SuFac}(x)$ by (8.10),(8.11) versus x

Complex map of SuFac is shown in figure 8.6. This function has period

$$P = \frac{2\pi \mathrm{i}}{k} = 2\pi \mathrm{i}\ \ln\!\big(3+2\,\mathrm{Factorial}'(0)\big) \approx 10.253449681156\,\mathrm{i} \qquad (8.12)$$

This period is pure imaginary. A little bit less than two periods fit the height of the upper map in figure 8.6.

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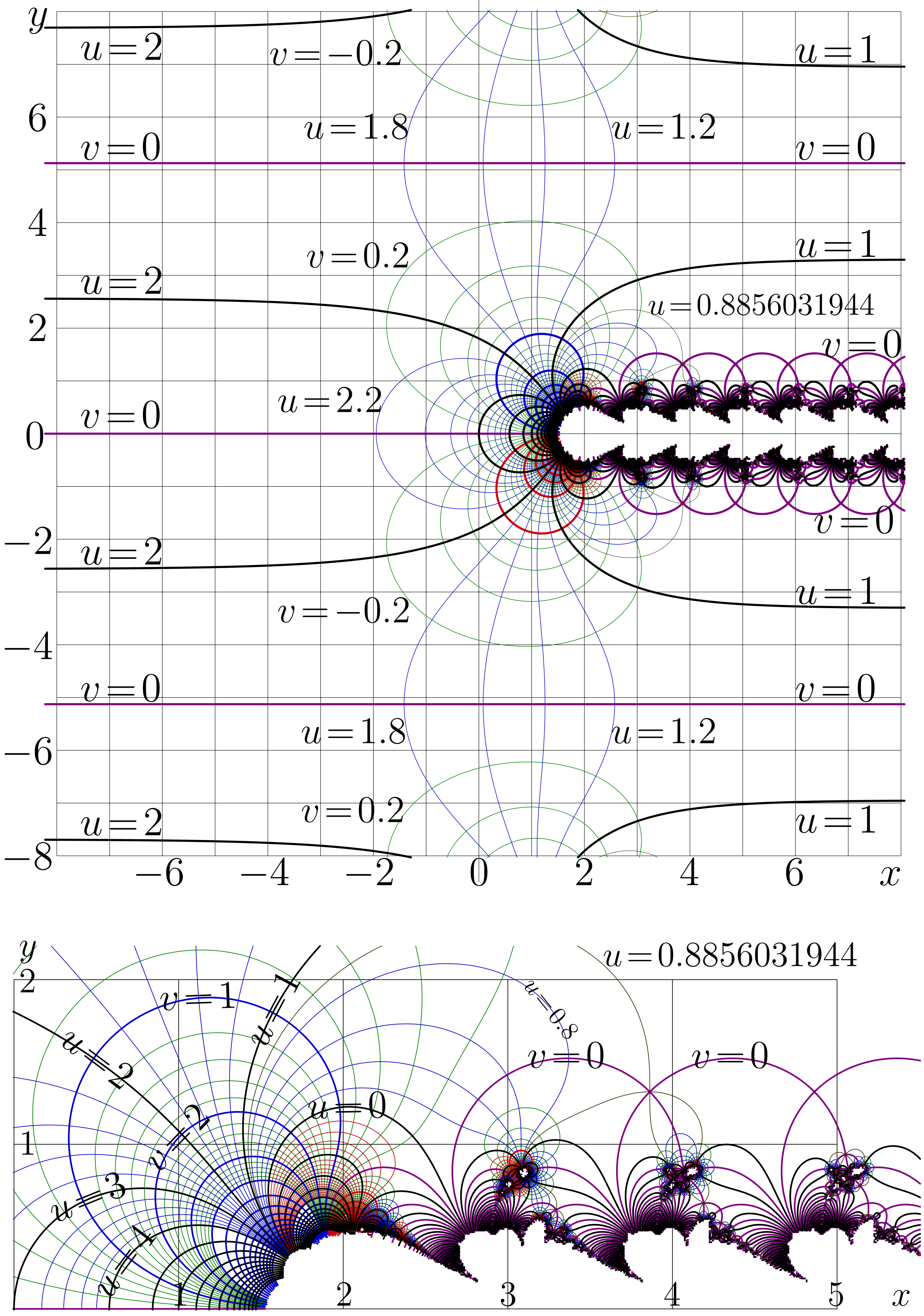

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Figure 8.6: $u+\mathrm{i}v=\mathrm{SuFac}(x+\mathrm{i}y)$ and the zoom-in of the central part

In the half-strips

$$x>2,\ \Big|y+n|P|\Big|<1,\ \text{for integer } n \qquad \text{[sufastrips]} \quad (8.13)$$

SuFac(x + iy) has countable set of singularities. At the map, every negative integer value of $\Im(\mathrm{SuFac}(x+\mathrm{i}y))$, at the translation for unity along the real axis, produces the singularity, due to the transfer equation (8.6). In order to show the structure of these singularities, the zoom-in of the map is shown at the bottom of figure 8.6. Outside the half-strips (8.13), factorial is regular; it approaches to the fixed points L of factorial: $L\!=\!2$ in the left hand side and $L\!=\!1$ in the right hand side of the map.

The fast growth of factorial implies the slow growth of the inverse function, id est, growth of the abelfactorial. I denote this function with symbol AuFac. Complex map of $\mathrm{AuFac}=\mathrm{SuFac}^{-1}$ is shown in figure 8.7; this function is considered in the next section.

4 Abelfactorial

The asymptotic series for super factorial can be inverted. This gives the asymptotic expansion of abelfactorial (arcsuperfactorial) in vicinity of the fixed point $L\!=\!2$:

$$\tilde{G}(z) = \frac{1}{k} \ln\left(\sum_{n=1}^{N-1} U_n (z-2)^n + \mathcal{O}(z-2)^N \right) \qquad \text{[abelFacExp]}(8.14)$$

Parameter k has the same meaning, as in expansion (8.5); its value is determined by (8.7). The first two coefficients U are

$$U_1 \ = \ 1 \qquad (8.15)$$

$$U_2 \ = \ -\frac{\pi^2+6\gamma^2-18\gamma+6}{12(3-5\gamma+2\gamma^2)} \approx 0.7987318 \qquad \text{[UAFac]} \ (8.16)$$

The Mathematica routine **InverseSeries** allow to calculate these coefficients analytically, although the expressions for the highest U are a little bit long. The approximate values of these coefficients are shown in the right column of table 8.1.

For argument close to 2, the truncated series in (8.14) provides the precise evaluation. For other values, the repeated application of the recurrent formula

$$G(z) = G\Big(\mathrm{ArcFactorial}(z)\Big) + 1 \qquad \text{[AbelFacDrugaya]} \quad (8.17)$$

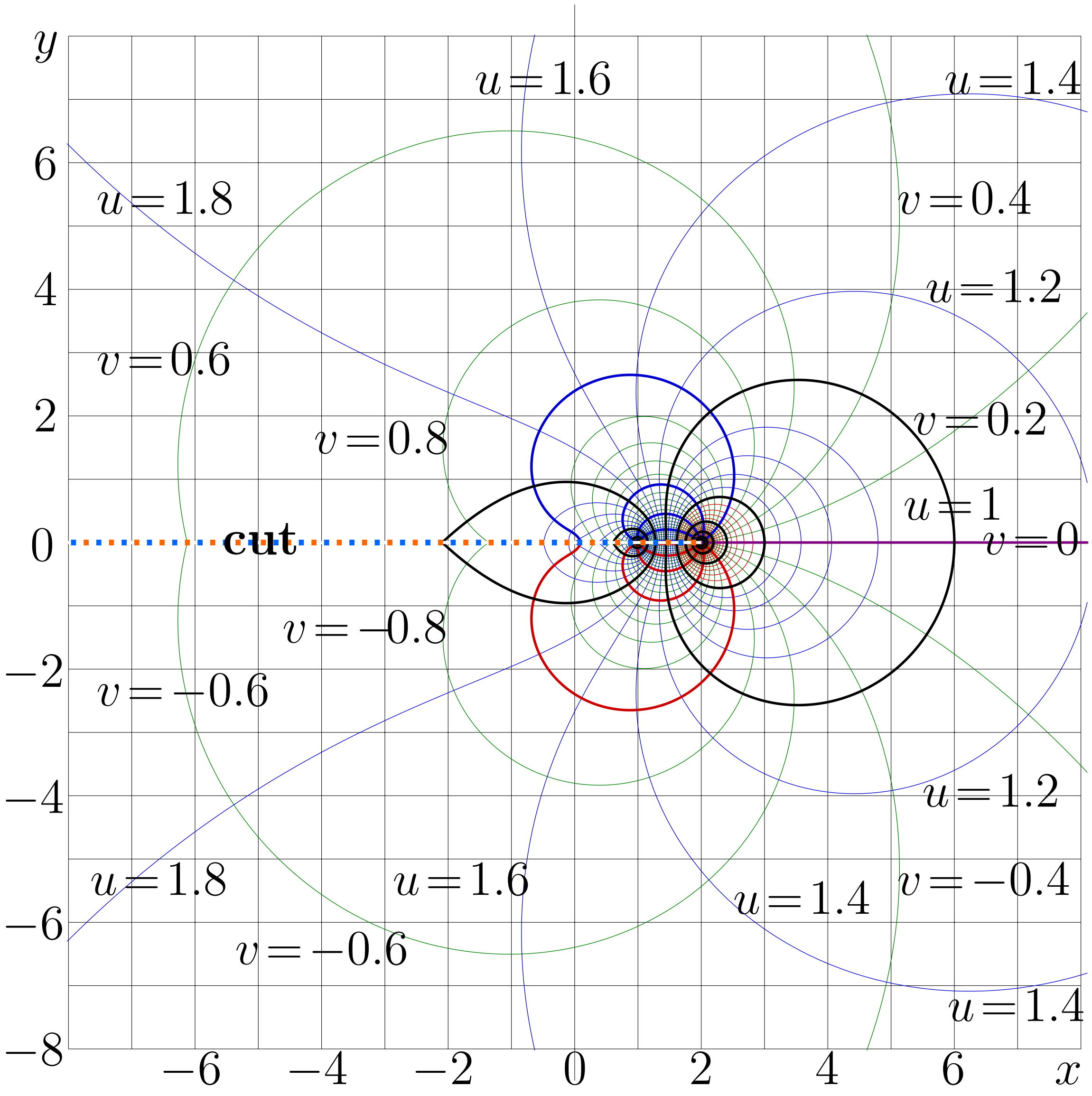

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Figure 8.7: $u+\mathrm{i}v = \mathrm{AbelFactorial}(x+\mathrm{i}y)$. [AbelFactorialMap]

is used, until $|z-2|$ becomes small. This algorithm determines the cut line of the abelfctorial AuFac; the cut goes from 2 to $-\infty$.

The abelfunction G of factorial can be defined with

$$G(z) = \lim_{n\to\infty} \tilde{G}(\mathrm{Factorial}^{-n}(z) + n \quad \text{[facG]} \qquad (8.18)$$

then, AuFac = AbelFactorial = SuFac^{-1}, appears as

$$\mathrm{AuFac}(z) = G(z) - z_3 = G(z) - G(3) \quad \text{[AuFac]} \quad (8.19)$$

where $z_3 = G(3) \approx -0.91938596545218$. This expression automatically provides condition AuFac(3) = 0. In such a way,

$$\mathrm{AuFac}(z) \approx \mathrm{AuFac}_n(z) = \tilde{G}\Big(\mathrm{ArcFactorial}^n(z)\Big) + n - z_3 \quad (8.20)$$

Function AuFac is plotted in figure 8.8. For comparison, graphic of ArcFactorial is also shown. Graphic of ArcFactorial goes through points (1,1), (2,2), (6,3), (24,4), although the last one is already out of range of the figure (and out of range of the page. Graphic of AuFac passes through points (3,0), (6,1), (720,2), and the last one is not only our of range of the page, but also out of room, were this Book is written, and the next point, where AuFac takes the integer value, id est, (720!, 3) is far away from the visible part of our Universe.

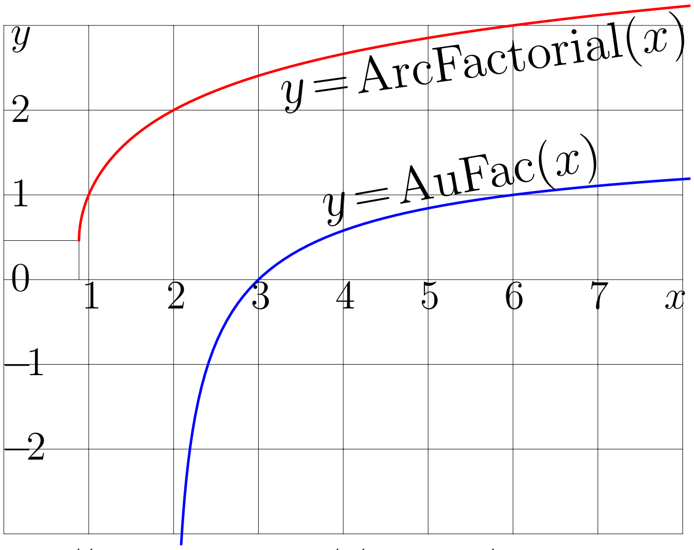

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Figure 8.8: ArcFactorial and AuFac

Function AuFac grows very slowly. If the argument of this function represents some physical quantity (distance, mass, charge, number of atoms, etc.), measured in any reasonable units, then, there is no way to make this quantity so big, that AuFac of it reaches 3.

While superfactorial and abelfactorial, id est, SuFac and AuFac, are already constructed and implemented, one can use them to evaluate the non-integer iterates of factorial. These iterates are matter of the next section.

5 Iterates of factorial

With super factorial and abelfactorial, the nth iterate of factorial can be written as follows:

$$h(z) = \mathrm{Factorial}^n(z) = \mathrm{SuFac}\Big(n + \mathrm{AuFac}(z)\Big) \qquad \text{[Facc]} \quad (8.21)$$

In this representation n has no need to be integer. Figure 8.9 shows graphic $y = \mathrm{Factorial}^n(x)$ versus x for various real n. Figure 8.9 should be compared to figures 4.2, 4.5, 4.6, 4.7, 4.8, 4.13, 5.11, that show iterates of other functions, that can be expressed through the special functions, known at least since century 20. There is no similar representation for the abelfactorial. I hope, in century 21, some of new superfunctions,

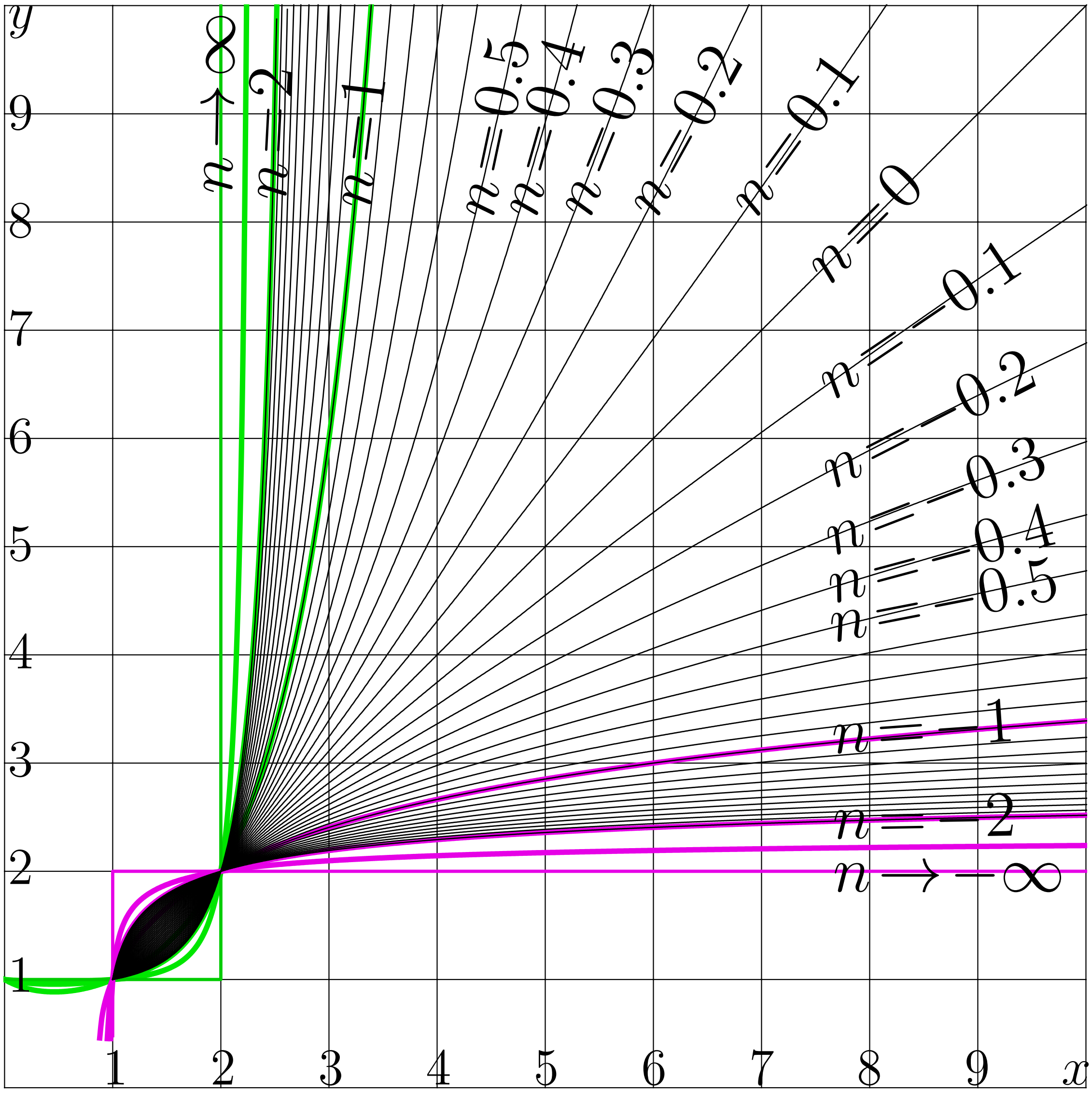

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Figure 8.9: $y\!=\!\mathrm{Factorial}^n(x)$ versus x for various n [facit]

including SuFac, AuFac will be included in the handbooks on the special functions and implemented as built-in routines in the programming languages; then, the difference between iterates of factorial and iterates of other functions, mentioned in the previous chapters, will be even less significant.

At $n = 1/2$, formula (8.21) gives the half iterate of factorial. Map of function $h\!=\!\mathrm{Factorial}^{1/2}$ is shown in figure 8.10. This function can be interpreted as "square root of factorial" $\sqrt{!}$, used as logo of the Phys. Dep. of the Moscow State University, shown in figure 2.4.

Concept of square root of factorial caused confusions and discussions. Some colleagues did not want to see difference between expirissions $\mathrm{Factorial}^{1/2}(z)$ and $\mathrm{Factorial}(z)^{1/2}$, interpreting $\sqrt{\mathrm{Factorial}(z)}$ as equiv-

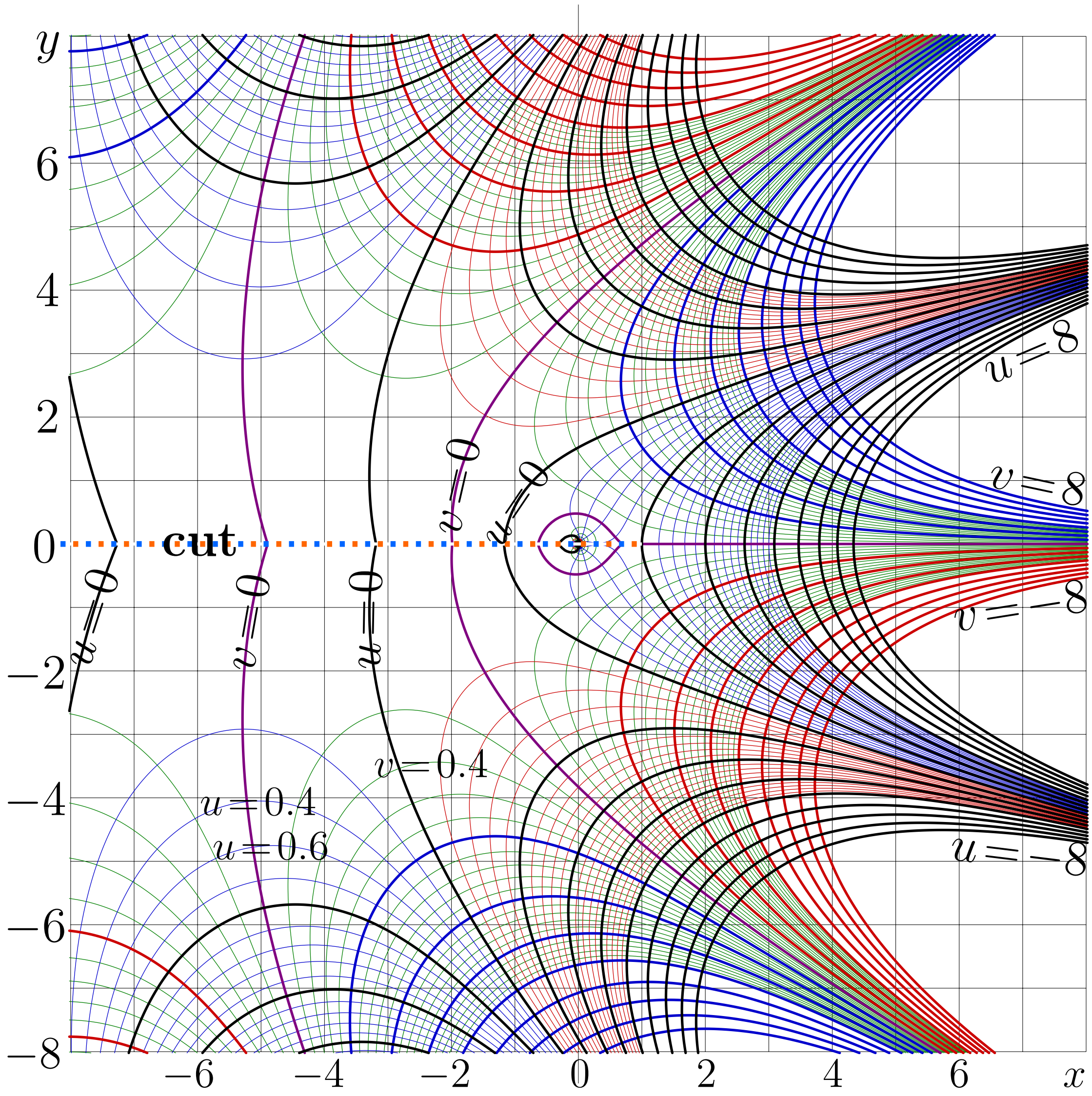

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Figure 8.10: $u+\mathrm{i}v = \mathrm{Factorial}^{0.5}(x+\mathrm{i}y)$ [QFacMap]

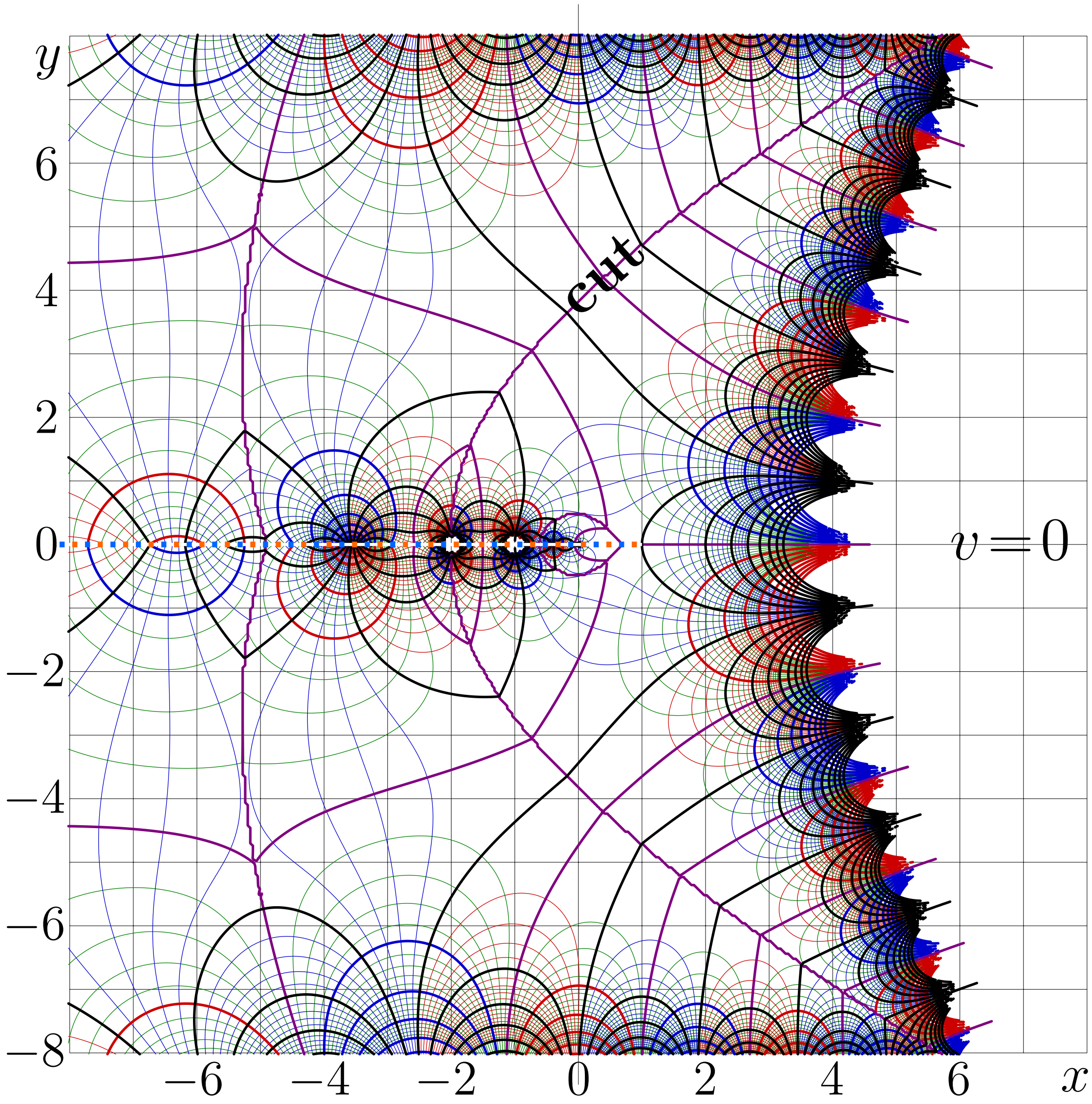
alent of $\sqrt{\mathrm{Factorial}(z)}$. In order to eliminate, en fin, this confusion, function $h\!=\!\mathrm{Factorial}^{1/2}$ is considered here with more details.

The key for the verification of interpretation of function $h = \sqrt{!}$, is analysis of the range of validity of relation

$$h(h(z)) = \mathrm{Factorial}(z) = z! \quad \text{[hh]} \tag{8.22}$$

Map of function h^2 in the left hand side of equation (8.22) is shown in figure 8.11. This figure should be compared to figure 8.3, that represents the complex map of factorial.

In the right hand side of figure 8.11, complex map of the second iterate of function h, id est, $h \circ h = h^2$, coincides with the map of factorial, shown in figure 8.3. At least in the halfstrip $\Re(z)\!>\!1$, $|\Im(z)|\!\le\!4$, relation (8.22) holds. The resulting half iteration of factorial corresponds to the



http://mizugadro.mydns.jp/t/index.php/File:QQFacMT.jpg

Figure 8.11: $u+\mathrm{i}v = h^2(x+\mathrm{i}y)$. [QQFacMap]

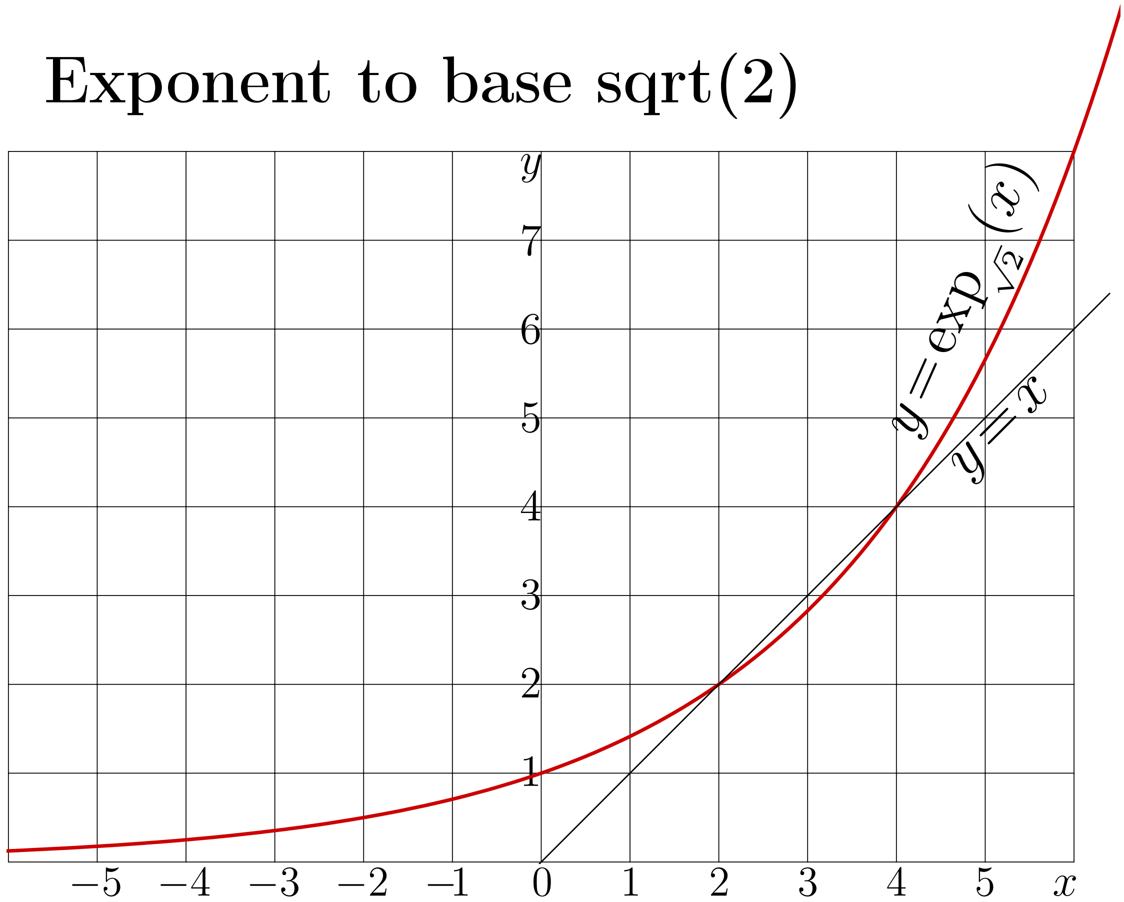
intuitive expectations about this function.

Range of validity of relation (8.22) is limited with cut lines, shown in figure 8.11. Such cutlines are typical for non-integer iterates, if the superfunction cannot be represented through elementary function.

Factorial is not exotic, not an exception. With superfunctions, one can build-up non-integer iterates for other holomorphic functions too. More examples are considered in the following chapters below.

Many transfer functions can be treated in a way, similar to that of this chapter. One of them, namely, the exponential to base $b = \sqrt{2}$, is considered in the next chapter.

Chapter 9



http://mizugadro.mydns.jp/t/index.php/File:ExpQ2plotT.png

Figure 9.1: Exponent to base $b = \sqrt{2}$

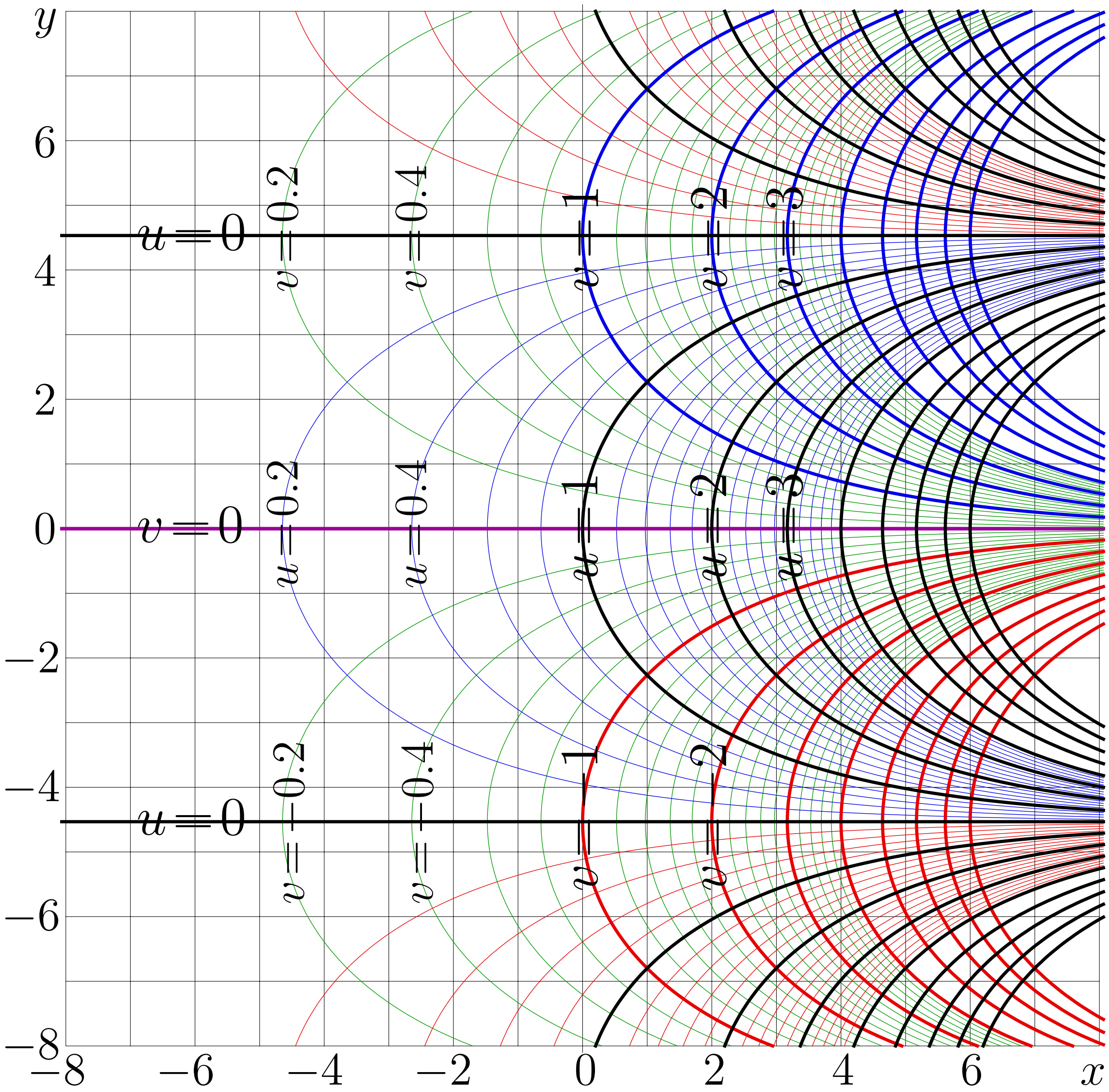
Exponent to base $\sqrt{2}$ happened to be first function treated with the regular iteration [61]. With this example (and for this function), the methods, described in chapter 6, were developed. In this chapter, this exponent to this specific base is considered as transfer function T.

Exponent to base $\sqrt{2}$ can be expressed through the natural exponent:

$$T(z) = \exp_{\sqrt{2}}(z) = \exp\Big(\ln(\sqrt{2})z\Big) = \exp\left(\frac{\ln(2)}{2}\, z\right) \qquad \text{[Texpq2] (9.1)}$$

Explicit plot of this function is shown in figure 9.1.

For exponential to base $b = \sqrt{2}$, the fixed points $L = 2$ and $L = 4$ are natural numbers. In figure 9.1, they correspond to intersections of curve $y = T(x)$ with the straight line $y = x$.



http://mizugadro.mydns.jp/t/index.php/File:ExpQ2mapT.png

Figure 9.2: $u+\mathrm{i}v = \exp_{\sqrt{2}}(x+\mathrm{i}y)$

Complex map of exponent to base $b=\sqrt{2}$ is shown in figure 9.2. This function is periodic; its period

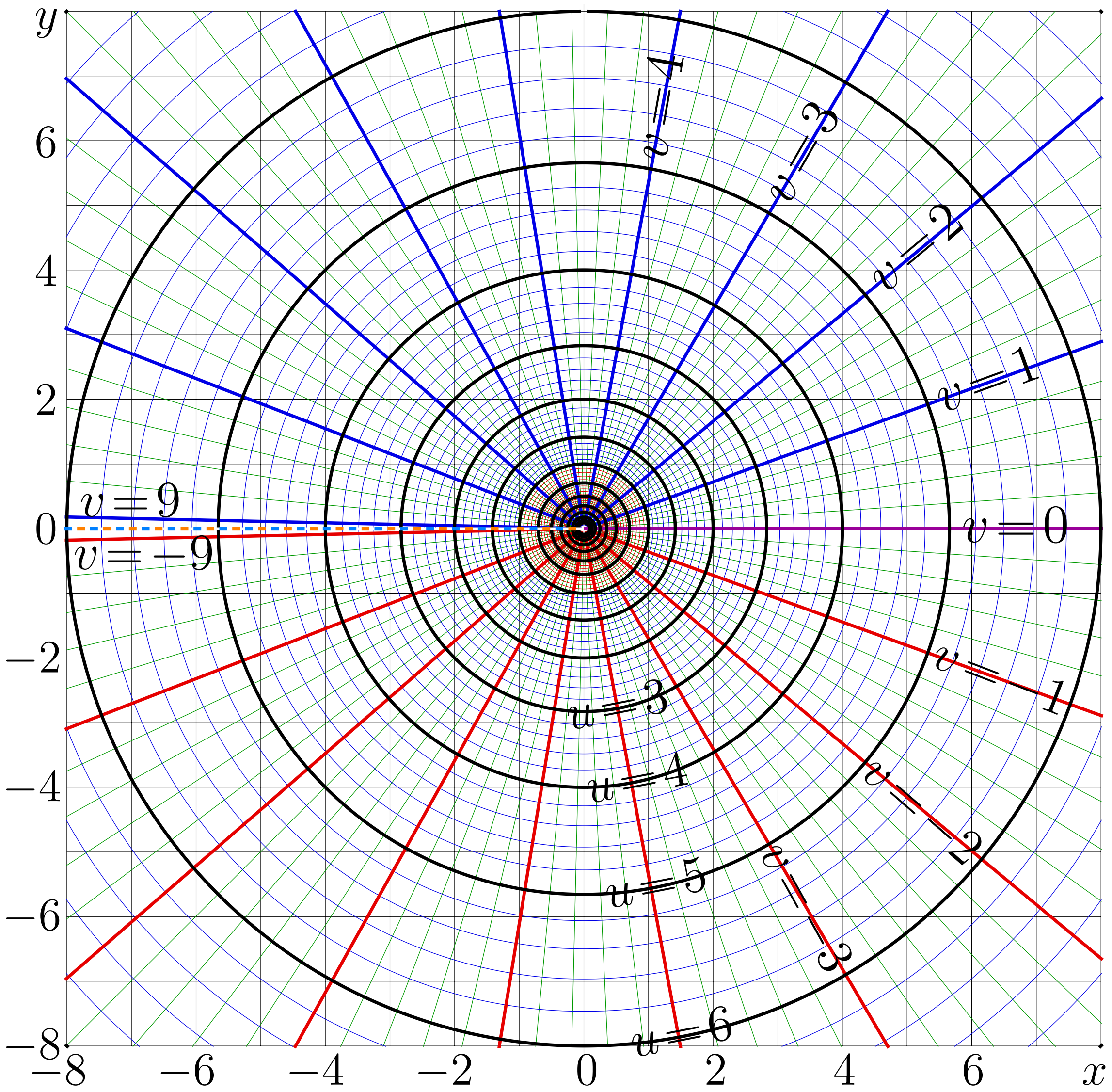
$$P = \frac{2\,\pi\,\mathrm{i}}{\ln(b)} = \frac{2\,\pi\,\mathrm{i}}{\ln\left(\sqrt{2}\right)} \approx 18.1294405673\,\mathrm{i} \quad \text{[Q2P18]} \qquad (9.2)$$

is pure imaginary. A little bit less than one period fits the height of map in figure 9.2.

For the exponential to base $b=\sqrt{2}$, the inverse function is logarithm to base b; this logarithm can be interpreted as minus first iterate:

$$T^{-1}(z) = \log_{\sqrt{2}}(z) = \log_b(z) = \frac{2}{\ln(2)}\ln(z) \quad \text{[logsqrt2]} \quad (9.3)$$

Complex map of logarithm to base $b=\sqrt{2}$ is shown in figure 9.3. The dashed line marks the cut of the range of holomorphism; that cut runs



http://mizugadro.mydns.jp/t/index.php/File:ExpQ2mapT.png

Figure 9.3: $u\!+\!\mathrm{i}v = \log_{\sqrt{2}}(x\!+\mathrm{i}y)$ [LogQ2map]

from zero along the negative part of the real axis. The jump at the cut line is determined by the period (9.2) of the exponent:

$$\log_{\sqrt{2}}(x+\mathrm{i}o) - \log_{\sqrt{2}}(x-\mathrm{i}o) = P \approx 18.1294405673\,\mathrm{i} \qquad (9.4)$$

for $x<0$. At the map, the levels $v=\Im(\log_{\sqrt{2}}(x\!+\!\mathrm{i}y))=-9$ and $v=\Im(\log_{\sqrt{2}}(x\!+\!\mathrm{i}y))\!=\!9$ are seen close to the negative part of the real axis.

Fixed points $L=2$ and $L=4$ of the exponent are also fixed points of the logarithm; these points are seen in figures 9.1, 9.2 and 9.3.

This chapter describes the regular iteration of exponential to base $\sqrt{2}$ at the fixed point $L\!=\!4$. The next section describes the construction of the supedfunction, id est, the growing superexponential to this base.

1 Superfunction at fixed point $L\!=\!4$

Chapter 6 describes the construction of iterates of a transfer function, that are regular in vicinity of its fixed point. Here, that method is used for $T = \exp_{\sqrt{2}}$; the superfunction is constructed, that exponentially approaches the fixed point $L\!=\!4$ at large negative values of the real part of the argument, and grows to infinity at the large positive argument. This function is denoted with $\mathrm{SuExp}_{\sqrt{2},5}$. The last superscript in the name of the function indicates its value at zero, $\mathrm{SuExp}_{\sqrt{2},5}(0) = 5$. In figure 9.4, curve $y\!=\!\mathrm{SuExp}_{\sqrt{2},5}$ is compared to that of $y\!=\!\exp_{\sqrt{2}}(x)$. Below I construct and describe function $\mathrm{SuExp}_{\sqrt{2},5}$.

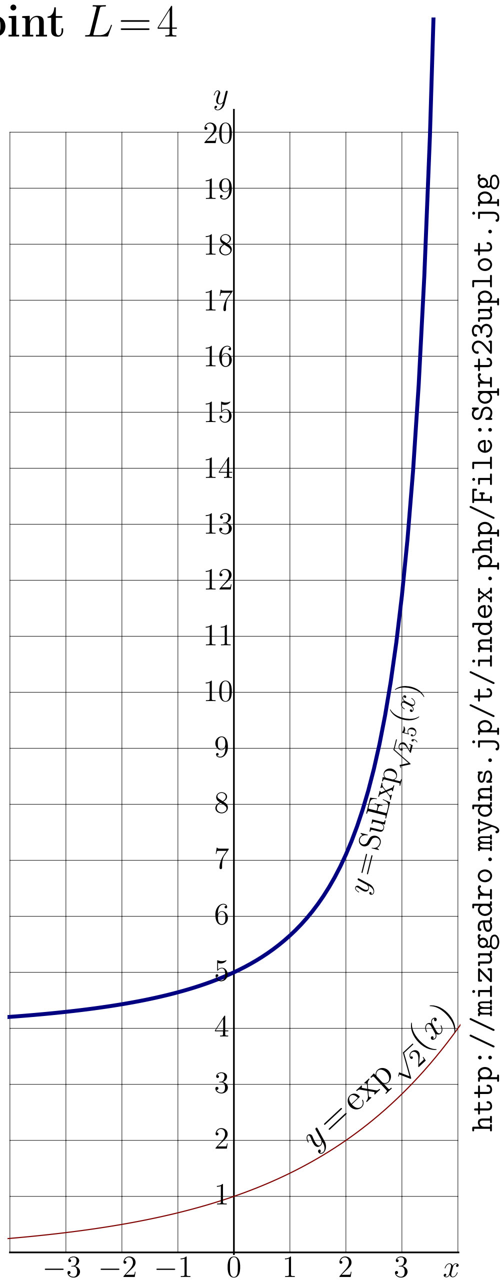


Figure 9.4: $y = \mathrm{SuExp}_{\sqrt{2},5}(x)$ by (9.13) and $y\!=\!\exp_{\sqrt{2},5}(x)$

I use formula (6.2) of the asymptotic expansion of superfuction f:

$$f(z)\!=\!\sum_{n=0}^{N-1} a_n\, \mathrm{e}^{nkz}\!+\!O\big(\mathrm{e}^{Nkz}\big) \qquad (9.5)$$

Here I assume, that $a_0 = L = 4$ and $a_1 = 1$. Coefficient a_1 could be chosen arbitrary, but then other coefficients depend on a_1. Alteration of a_1 is equivalent of scaling of parameter ε and displacement of the argument of the superfunction. Then, formula (6.9) gives

$$k\!=\!\ln\Big(\exp_b{}'(4)\Big)\!=\!\ln\Big(4\ln(\sqrt{2})\Big)\!=\!\ln(2\ln(2))\!\approx\!0.32663425997828 \quad (9.6)$$

This value determines periodicity of the superfunction; its period

$$P = P_4 = \frac{2\,\pi\,\mathrm{i}}{k} = \frac{2\pi\mathrm{i}}{\ln(2\ln(2))} \approx 19.236149042042854712\,\mathrm{i} \qquad (9.7)$$

I use equations (6.10),(6.11) or directly the transfer equation (which gives the same results) to get coefficients a. While the transfer function is exponent to base $\sqrt{2}$, the transfer equation can be written as follows:

$$f(z\!+\!1) = \Big(\sqrt{2}\Big)^{f(z)} \qquad [\text{q2Transfereq}] \qquad (9.8)$$

Table 9.1: Coefficients a and U in expansions (9.5), (9.15)

n	a_n	U_n
1	1.0000000000000000	1.0000000000000000
2	0.4485874311952612	−0.4485874311952612
3	0.1903722467978068	0.2120891200549197
4	0.0778295765369683	−0.1021843675069717
5	0.0309358603057080	0.0496986830373718
6	0.0120221257690659	−0.0243075903261196
7	0.0045849888965617	0.0119330883965109
8	0.0017207423310577	−0.0058736976420089
9	0.0006368109038799	0.0028968672871058
10	0.0002327696003030	−0.0014309081060793
11	0.0000841455118381	0.0007076637148566
12	0.0000301156464937	−0.0003503296158730
13	0.0000106807458130	0.0001735756004664
14	0.0000037565713616	−0.0000860610119291
15	0.0000013111367785	0.0000426959089013
16	0.0000004543791625	−0.0000211930290682
17	0.0000001564298463	0.0000105244225996
18	0.0000000535232764	−0.0000052285174354
19	0.0000000182077863	0.0000025984499916
20	0.0000000061604765	−0.0000012917821121

Substitution of expansion (9.5) into the transfer equation (9.8) determines coefficients a_m for $m > 1$. In particular,

$$a_2 = \frac{\ln(2)/4}{1-2\ln(2)} \approx 0.448587431195261 \qquad (9.9)$$

$$a_3 = \frac{\Big(1-\ln(2)\Big)\ \ln(2)^2/12}{1-2\ln(2)-4\ln(2)^2+8\ln(2)^3} \approx 0.1903722467978067 \qquad (9.10)$$

Similar (but longer) expressions can be written for the other coefficients a. The first column in table 9.1 suggests the approximate values of coefficients a in expansion (9.5).

I use the truncation of the expansion (9.5), taking into account $N=20$

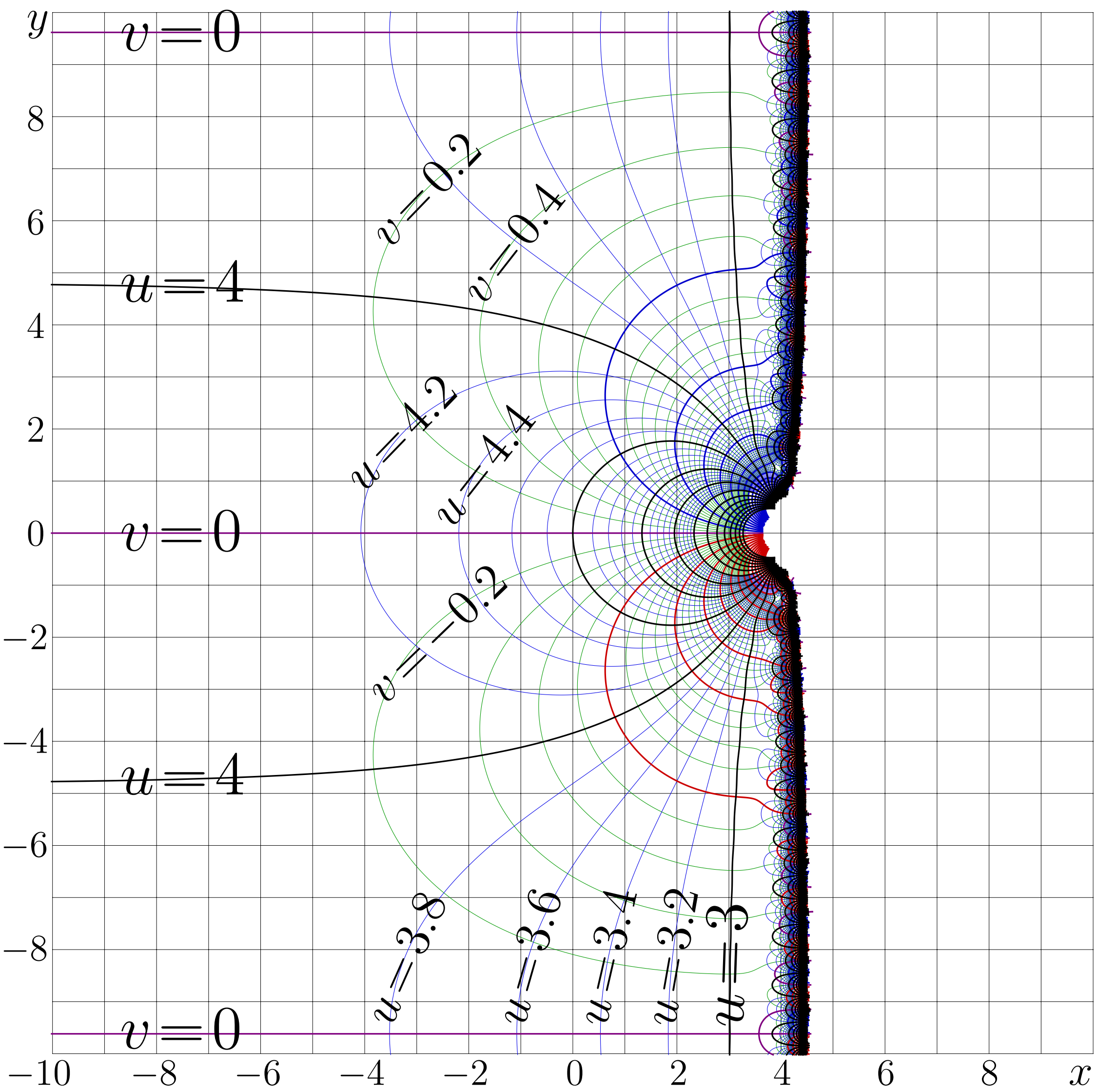


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Figure 9.5: $u+\mathrm{i}v = \tilde f(x-1.115207245131 61 + \mathrm{i}\, y)$; [sqrt2figf45b][mapeq2F4]

terms. This gives the primary approximation $\tilde f$ for superfunction f:

$$\tilde f(z) = \sum_{n=0}^{19} a_n \exp(nkz) \quad \text{[q2Fz4tilde20]} \tag{9.11}$$

Approximation $\tilde f$ is shown in figure 9.5. In order to simplify the comparison with other maps, the argument of function in this figure is displaced for the real constant $x_{45} \approx -1.11520724513161$; this constant provides approximate condition $f(x_{45}) \approx 5$ and the exact equality $\mathrm{SuEx}_{\sqrt{2},5}(0) = 5$ for the superfunction constructed below and shown in figure 9.6.

The primary approximation (9.11) allows to plot the complex map of superfunction in the left hand side of the complex plane. The residual at the substitution of the approximation (9.11) into the transfer equation (9.8) becomes of order of rounding errors at $\Re(z) < -2$. The readers are invited to plot this residual by themselves (preferably), or to look at it in the right hand side of figure 4 of the original publication [61].

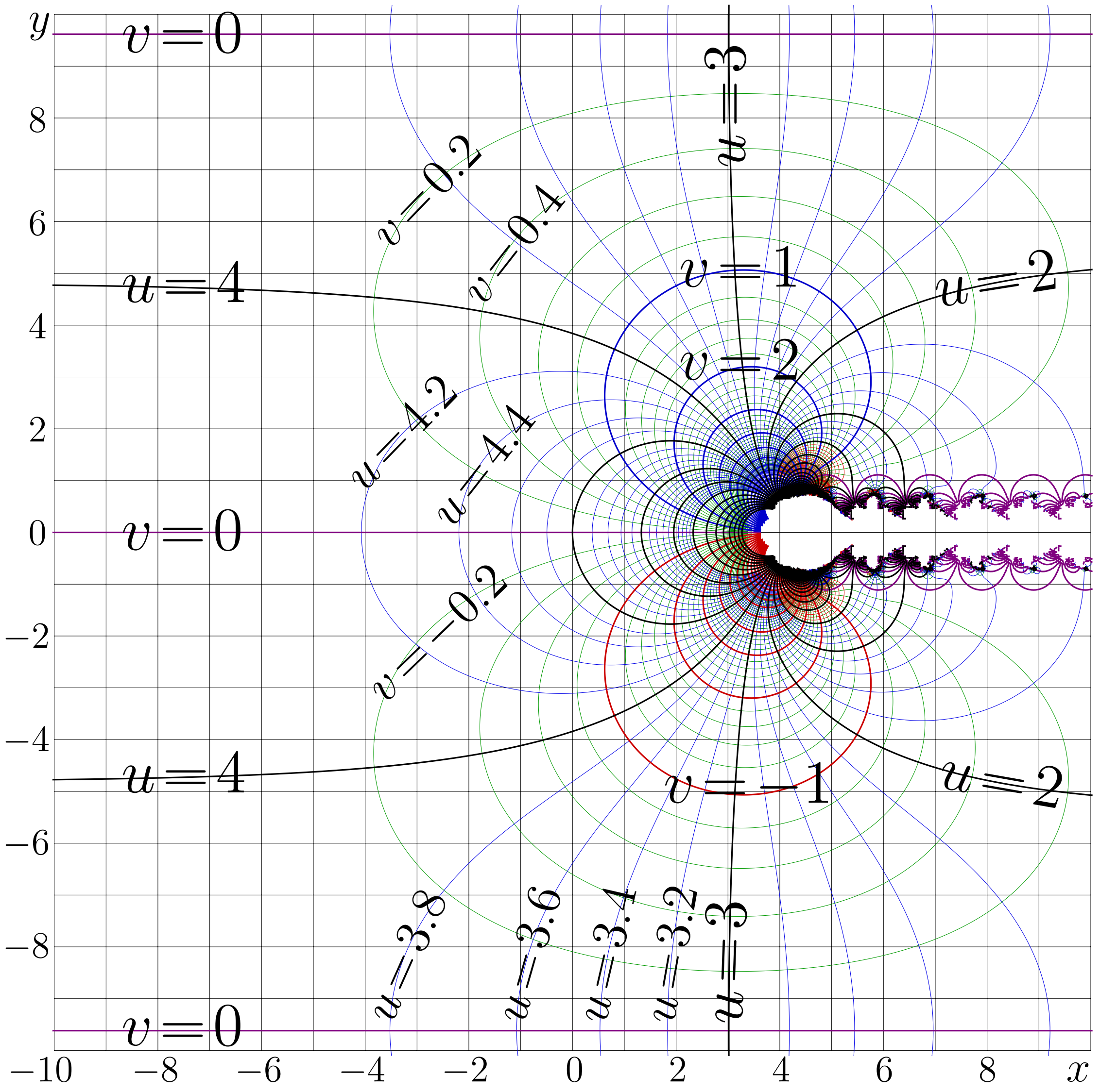

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Figure 9.6: $u+\mathrm{i}v = \mathrm{SuExp}_{\sqrt{2},5}(x+\mathrm{i}y)$ by (9.13) [sqrt2f45map]

Superfunction f appears as limit

$$f(z) = \lim_{n\to\infty} \exp_b{}^n\Big(\tilde f(z-n)\Big) \quad \text{[sqrt2regi45]} \qquad (9.12)$$

This limit does not depend on the number N of terms in the primary approximation (9.11). Instead of 19 terms, one could choose another constant. However, the more terms are taken into account, the faster does the limit converge. Choosing $N=19$, I keep in mind the implementation complex double. For the approximation of f with 14 significant figures, it is sufficient to choose $n>\Re(z)+2$.

It is convenient, when at zero the superfunction takes integer value. I choose this value to be 5. This is smallest integer, that is still greater than the chosen fixed point $L = 4$. I denote this superfunction with symbol $\mathrm{SuExp}_{\sqrt{2},5}$ and define it as follows:

$$\mathrm{SuExp}_{\sqrt{2},5}(z) = F(z) = f(x_{45} + z) \quad \text{[Ff45]} \qquad (9.13)$$

where $x_{45} \approx -1.11520724513161$ is real solution of equation $f(x_{45})\!=\!0$. Namely this constant is used for the displacement of the argument of function $\tilde f$ in figure 9.5 in order of simplify the comparison with map of function $\mathrm{SuExp}_{\sqrt{2},5}$ shown in figure 9.6.

In the left hand side of the complex plane, functions $z\mapsto \tilde f(z\!+\!x_{45})$ and $\mathrm{SuExp}_{\sqrt{2},5}$ practically coincide. In the whole complex plane, the superfunction can be approximated with any arbitrary precision; so, it should be qualified as exact solution. In such a way, the primary approximation $\tilde f$ provides the exact solution for the superfunction $\mathrm{SuExp}_{\sqrt{2},5}$ in the whole complex plane.

For the transfer function as exponential to base $b=\sqrt{2}$, the regular iteration at the fixed point $L\!=\!4$ gives the function $F\!=\!\mathrm{SuExp}_{\sqrt{2},5}$ that is holomorphic in the whole complex plane. Graphic of this function is shown in figure 9.4. Complex map of function $\mathrm{SuExp}_{\sqrt{2},5}$ is shown in figure 9.6.

For function $F=\mathrm{SuExp}_{4,5}$, value at zero is chosen a smallest integer, which is still greater than $L=5$; so, $F(0)=5$. Then, function $F=\mathrm{SuExp}_{4,5}$ can be interpreted as iterate of exponent with initial value 5:

$$F(z) = \mathrm{SuExp}_{\sqrt{2},5}(z) = \exp^z_{\sqrt{2},\mathrm{u}}(5) \qquad \text{[expq2z5]} \quad (9.14)$$

where subscript u indicates, that the regular iteration is built-up at the highest ("upper") fixed point of the exponent. In order to evaluate the non-integer iterates of other argument, the abelfunction is also required, I mean, $G=F^{-1}=\mathrm{AuExp}_{\sqrt{2},5}$. This abelfunction is described in the next section.

2 Abelfunction at fixed point $L\!=\!4$

Superfunction $F\!=\!\mathrm{SuExp}_{\sqrt{2},4}$ of the exponent to base $\sqrt{2}$ is descried in the previous section. This section considers the inverse function, $G=\mathrm{AuExp}_{\sqrt{2},4}\!=\!F^{-1}\!=\!\mathrm{SuExp}^{-1}_{\sqrt{2},4}$. Then, the combination of superfunction F and abelfunction G allows to evaluate the non-integer iterates of the transfer function $T\!=\!\exp_{\sqrt{2}}$ by the general formula (2.14).

Expansion for the abelfunction $g=f^{-1}$ can be obtained, inverting the asymptotics (9.5). This gives for the abelfunction g expansion $\tilde g$ in the following form:

$$\exp(k\tilde g(z)) = \sum_{n=1}^{N-1} U_m\,(z\!-\!L)^n + O(z\!-\!4)^N \quad \text{[q2Gz4]} \qquad (9.15)$$

where $U_0\!=\!0$; this implies, that at the fixed point L, function g becomes infinite. Then, according to expansion (9.5), we get $U_1\!=\!1$ and $U_2\!=\!-a_2$.

While coefficients a are represented with exact expressions (with infinite precision), Matgematica or Maple can calculate of order of 10 coefficients U in expansion (9.15). Routine, that perform this inversion, is called **InverseSeries**. In particular,

$$U_2 = \frac{\ln(2)/4}{1-2\ln(2)} \approx -0.4485874311952612289 \quad (9.16)$$

$$U_3 = \frac{\Big(1+4\ln(2)\Big)\ln(2)^2/24}{1-2\ln(2)-4\ln(2)^2+8\ln(2)^3} \approx 0.2120891200549196975 7 \quad (9.17)$$

Approximate values of coefficients U are shown in the right hand side column of table 9.1. The same coefficients can be obtained also independently on the expansion of superfunction, by substitution to expansion

$$\tilde{g}(z) = \frac{1}{k}\ln\left(\sum_{n=1}^{N-1} U_m\,(z-L)^n + O(z-4)^N\right) \qquad \text{[eq2ga]}\,(9.18)$$

into the Abel equation

$$g(\exp_b(z)) = g(z) + 1 \quad \text{[eq2gass]} \qquad (9.19)$$

In order to fit the with of the screen (at the authomatic computation) or the width of the page (at the calculation with paper and pen), getting $\tilde{g}(z)$ by formyla (9.18) it worth to use the new variable $\zeta = z-L = z-4$. Then, first coefficients U in expansion (9.15) can be found even without computer.

Truncated expansion $\tilde{g}(z)$ approximates $g(z)$ at $|z-4|<2$. For other values, the iterates with the Abel equation (9.19) can be used. Abel-function $G=F^{-1}$, shown in figure 9.7, appears as limit of these iterates:

$$\mathrm{AuExp}_{\sqrt{2},5}(z) = G(z) = \lim_{n\to\infty} \tilde{g}\Big(\log_b(z-n)\Big) + n + x_{45} \qquad \text{[eq2G]}\,(9.20)$$

where $x_{45} \approx -1.11520724513161$ is the same constant, that appears in the definition (9.13) of the superexponent $F = \mathrm{SuExp}_{\sqrt{2},5}$, providing $F(0)=5$.

The inverse function of any non-trivial entire function has cut(s). The abelexponent $G = \mathrm{AuExp}_{\sqrt{2},5}$ is not exception. The cut of the range of its holomorphism is shown in figure 9.7 with dashed line. This cut goes along the real axis from minus infinity to the branch point 4. The abelexponent has logarithmic singularity at this point. This singularity corresponds to the exponential approach of the superesponent to the fixed point, as the real part of the argument goes to minus infinity.

The readers are invited to plot the map of the region, where the relation below holds:

$$\mathrm{SuExp}_{\sqrt{2},5}\Big(\mathrm{AuExp}_{\sqrt{2},5}(z)\Big) = z \quad \text{[SuExpAuExpQ23]}\ (9.21)$$

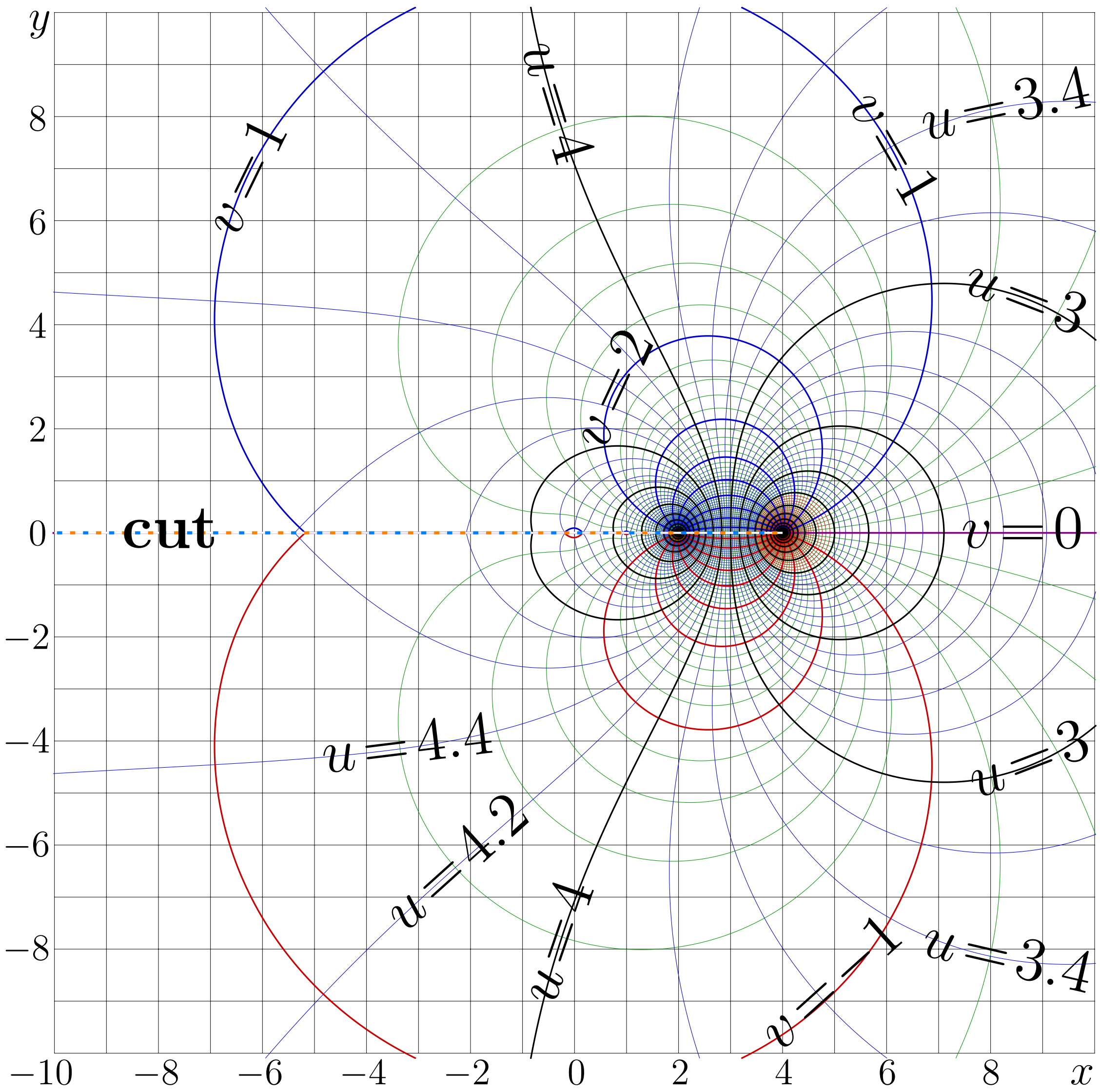

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Figure 9.7: $u+\mathrm{i}v = \mathrm{AuExp}_{\sqrt{2},5}(x+\mathrm{i}y)$ by formula (9.20)

and the map of range of validity of relation

$$\mathrm{AuExp}_{\sqrt{2},5}\Big(\mathrm{SuExp}_{\sqrt{2},5}(z)\Big) = z \tag{9.22}$$

Especially for the exponent to base $\sqrt{2}$, the algorithms of evaluation of the growing superfunction and the abelfunction are loaded as
`http://mizugadro.mydns.jp/t/index.php/F45E.cin` and
`http://mizugadro.mydns.jp/t/index.php/F45L.cin`
They are implemented in C++ as complex double functions of complex double arguments. With these algorithms, the Reader can reproduce figures of this chapter, even without downloading generators of these figures.

3 Notations

The most of notations, used in this Book, are already introduced. This section suggests some kind of short overview of these notations. for the transfer function (practically, it is any holomorphic function, for which the superfunction and abelfunctions are considered): F for the superfunction, and $G\!=\!F^{-1}$ for the corresponding abelfunction. These notations are convenient, while it is clear, which transfer function is kept in mind, and which of its superfunctions is denoted with F and which of its inverse G is assumed.

However, these notations may cause confusions, if cited from other chapters. Functions T, F and G may have (and actually have) different meanings in different chapters. For this reason, in this chapter, I introduce also long names $\mathrm{SuExp}_{\sqrt{2},5}\!=\!F$ and $\mathrm{AuExp}_{\sqrt{2},5}\!=\!G$. The names above have simple mnemonics. The first letter indicates, that it refers to Superfunction or to Abelfunction. This letter is capitalised, following the tradition of language Mathematica. The second letter indicates, that the biggest, "upper" fixed point of the transfer function is used. This have sense, if the transfer function T is real-holomorphic, and has real fixed point(s), and one of them can be qualified as maximal, upper. This takes place for the exponent to base $\sqrt{2}$. The following three characters refer to the name of the transfer function, id est, exp; again, its first letter is capitalised. The subscript indicates the base of this exponent, $b\!=\!\sqrt{2}$ and value of this superexponent at zero; $F(0)\!=\!5$ and $G(5)\!=\!0$.

4 Iterates of exponent to base $b = \sqrt{2}$

With superfunction $F\!=\!\mathrm{SuExp}_{\sqrt{2},5}$ and abefunction $G\!=\!\mathrm{AuExp}_{\sqrt{2},5}$, the nth iterate of transfer function $T\!=\!\exp_{\sqrt{2}}$ can be expressed as follows:

$$T^n(z) = \exp^{\,n}_{\sqrt{2},\mathrm{u}}(z) = \mathrm{SuExp}_{\sqrt{2},5}\Big(n\!+\!\mathrm{AuExp}_{\sqrt{2},5}(z)\Big) \qquad [\mathrm{q2Tn5}] \quad (9.23)$$

This formula is valid at least for $\Re(z)\!>\!4$. If one choose the upper (or lower) border at the cut of abelfuntion $G\!=\!\mathrm{AuExp}_{\sqrt{2},5}$ between 2 and 4, the resulting iterate $T^n(z)$ is holomorphic in the whole compilex plane except the cut along line $z\!\le\!2$. In figure 9.8, this $T^n(x)$ is shown versus x for various real values of n. These iterates are defined at least for $x\!>\!2$. For integer n, the curves can be extended also to $x\!\le\!2$.

For $n\!=\!1/2$, the complex map of function $T^n(z)$ by (9.23) is shown in figure 9.9. This is half iterate of exponential to base $\sqrt{2}$:

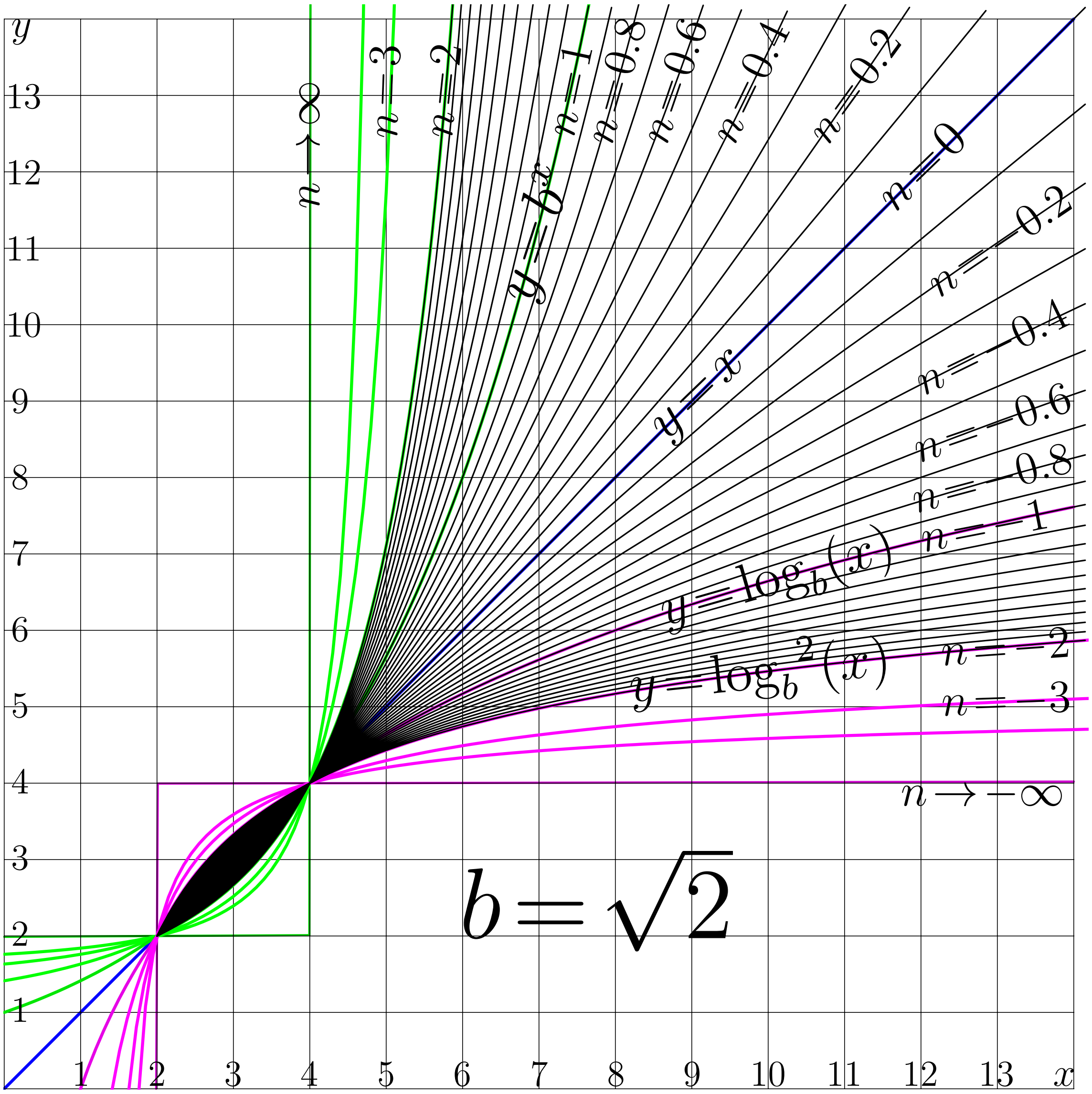

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Figure 9.8: $y=\exp^{n}_{\sqrt{2},\mathrm{u}}(x)$ by (9.23) for various n. [iterEq2plot]

$$T^{1/2}(z)=\exp^{1/2}_{\sqrt{2},\mathrm{u}}=F\left(\frac{1}{2}+G(z)\right)=\mathrm{SuExp}_{\sqrt{2},5}\left(\frac{1}{2}+\mathrm{AuExp}_{\sqrt{2},5}(z)\right) \quad (9.24)$$

I remind, that symbol "u" in the subscript indicates, that for the regular iterate, the highest ("upper") fixed point is used as asymptotics of superfunction at infinity.

Abelfunction G, built up at the fixed point $L=4$, has logarithmic singularity at this point, and the corresponding cut; the jump at this cut is constant; and this constant is just period of the superfunction F. For this reason, the iterates, even non-initeger, and in particular, the half iterate by equation (9.24), are regular at the point $L = 4$. However, the non-integer iterates have branch point at another fixed point of the transfer function, namely, $L=2$. In figure 9.9, the corresponding cut is marked with dashed line.

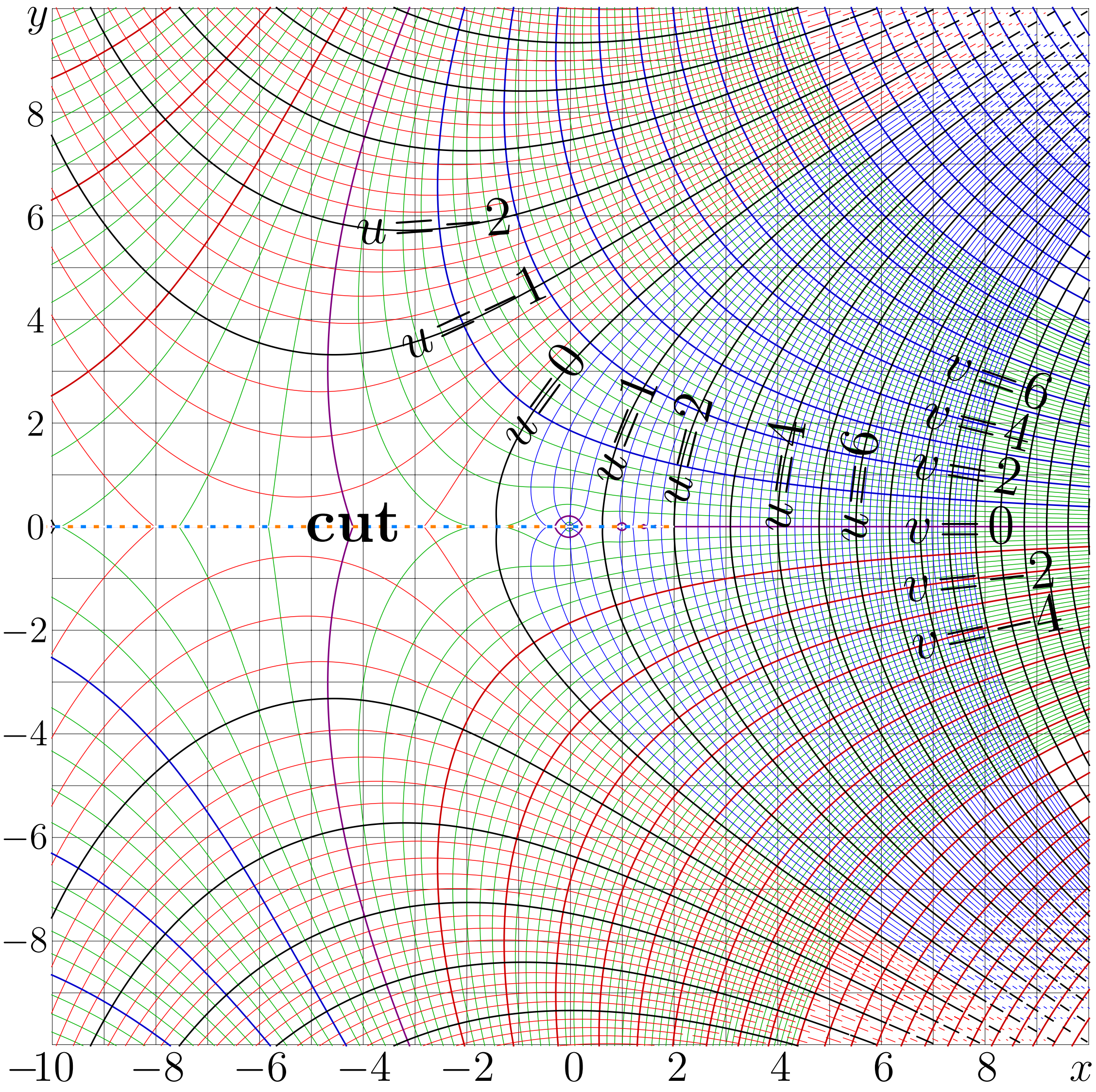

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Figure 9.9: $u+\mathrm{i}v=T^{1/2}(x+\mathrm{i}y)$ by formula (9.24). [iterEq2map]

5 Intermediate finish

Several results and tools for superfunctions abelfunctions and noninteger iterates are already presented above. Here I make a view on the way already past and announce, what will be in the future way, as it is shown in figure 9.10.

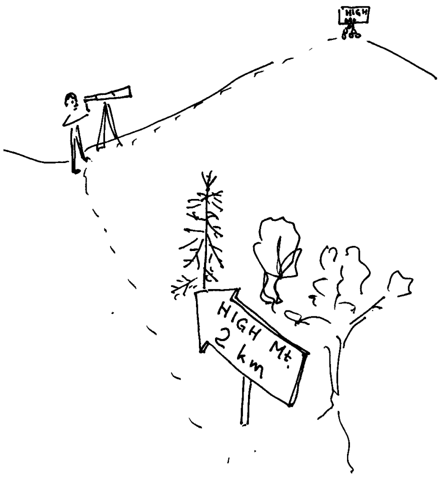

Figure 9.10: View from the saddle point

This chapter, above, deals with transfer function $T(z)=\left(\sqrt{2}\right)^z=\exp_{\sqrt{2}}(z)$; superfunction F, abelfuncton G and iterates of the transfer function T are described:

$$F = \mathrm{SuExp}_{\sqrt{2},5} \tag{9.25}$$

$$G = \mathrm{AuExp}_{\sqrt{2},5} \tag{9.26}$$

$$T^n(z) = F(n+G(z)) = \exp_{\sqrt{2},\mathrm{u}}^{\,n}(z) \qquad \text{[expbc]} \tag{9.27}$$

Iterates by equation (9.27) are shown in figure 9.8. These iterates look similar to iterates of other growing functions, in particular, those shown in figures 4.13 for iterates of the power function and 8.9 for iterates of factorial. One may expect, at least for the real-holomorphic growing functions, the similar iterates can be constructed in the similar way, even if the superfunction and the abelfunction cannot be expressed through the special functions described in the textbooks of century 20. I mean, with the regular iteration,

With the examples above, the Reader already understands, how can one iterate a holomorphic function in such a way, that the number to iterates has no need to be integer. These iterates greatly extends the tools, available for scientific description of various processes, for approximation (fitting) of physical (and, perhaps, not only physical) dependences. The non-integer iterates of exponential can be used, and, in particular, those to base $\sqrt{2}$. The example of such a non-integer iterates is shown in figure 9.9. I believe, the Reader can easy plot other examples of the iterates of the exponential to this base, and also for some other values of base.

The results presented above may cause impression, that they finish the investigation of superfunctions, abelfunctions and non-integer interates, and that, in future, one needs just to apply the regular iteration to various special functions. Actually, it is not the case.

For some functions, the regular iterates with asymptotic (6.2),(6.3) cannot be constructed, as the equations for the coefficients of the expansion have no solution. One of these cases is considered in the the next chapter and qualified as “exotic”.

Chapter 10

Exotic iteration and exp to base $\mathrm{e}^{1/\mathrm{e}}$

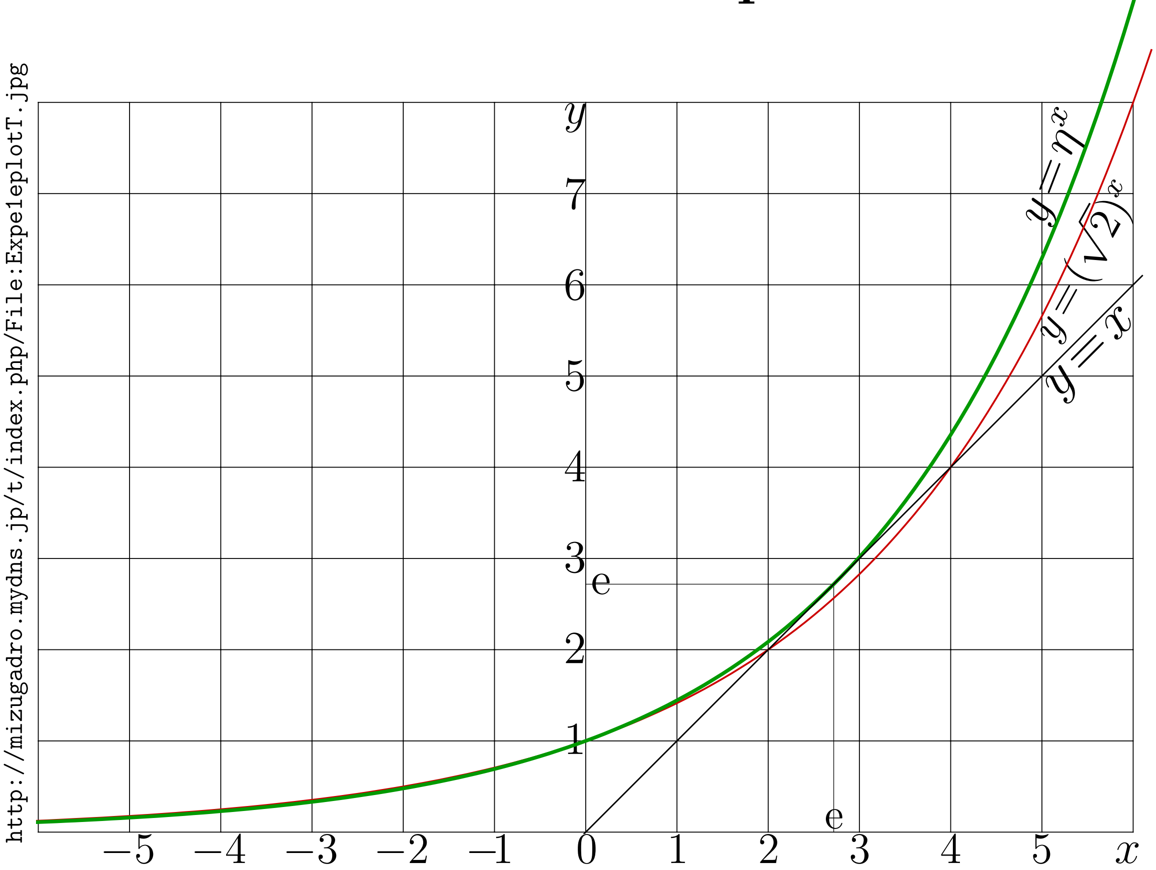

Figure 10.1: $y\!=\!b^x$ for $b\!=\!\eta\!=\!\mathrm{e}^{1/\mathrm{e}}\!\approx\!1.44466786$ and $b\!=\!\sqrt{2}\!\approx\!1.41421356$

The regular iteration by (6.2)-(6.12) looks as general method of construction of superfunctions. However, in some cases, it cannot be applied as is. In particular, the asymptotic expansion (6.2) becomes invalid, if the derivative of the transfer function at the fixed point is unity:

$$T(L) = L \quad , \quad T'(L) = 1 \quad \text{[Tprimeu]} \qquad (10.1)$$

In this case, expression $T'-1 = T'(L)-1$ in denominator of fraction in the right hand side of equation (6.10) becomes zero; and the coefficients in expansion (6.2) loss their meaning. Exponential to base $b\!=\!\mathrm{e}^{1/\mathrm{e}}\!\approx\!1.44466786$ is an example of such an exotic transfer function; construction of superfunction for such a case is matter of this chapter.

1 Exotic iterate

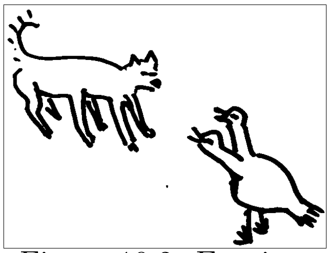

Figure 10.2: Exotics

The coincidence, that the derivative in equation (10.1) becomes unity, can be qualified as exotic. This determines the title of the chapter and that if this section. Condition (10.1) is not only exotic possible, (see figure 10.2), but, first, I consider namely case (10.1).

The general formulas of chapter 6 fail for the special case $T'(L)=1$, and Henryk Trappmann expected, that for this case the precise evaluation of superfunction is very difficult, if at all. In order to convince him, that it is doable, we had to write the paper [79]. Part of that publication is described below.
This section considers the case, when the derivative of the transfer function T in the fixed point L is unity. Then, formulas of section 6 for the regular iteration cannot be applied as is.

Calculation of iterates of a function is simpler, if its fixed point is zero. If the fixed point L of transfer function T is not zero, then, the superfunction F can be represented in the following form:

$$F(z) = f(z) + L \quad \text{[FfL]} \tag{10.2}$$

Then, for function f, we get

$$f(z+1) = F(z+1) - L = T(F(z)) - L = T(L+f(z)) - L \tag{10.3}$$

We may define

$$T_{\text{new}}(z) = T(L + z) - L \quad \text{[TnewTLzL]} \tag{10.4}$$

and interpret this T_{new} as new transfer function, and f is superfunction for it. Below, the subscript $_{\text{new}}$ is omitted. This is equivalent to assumption

$$T(0) = 0 \tag{10.5}$$

Let the Taylor expansion of the transfer function have the following form:

$$T(z) = z + vz^2 + wz^3 + .. \quad \text{[TExpan]} \tag{10.6}$$

where $v \neq 0$. For such a transfer function, it is difficult (and, perhaps, impossible) to built-up any superfunction, that exponentially approaches zero at infinity. But it is possible to construct the superfunction, that decays, roughly, as the inverse proportional function. Let

$$f(z) = \frac{a}{z} + \frac{b\ell}{z^2} + \frac{\alpha\ell^2\!+\!\beta\ell\!+\!\gamma}{z^3} + .. \quad \text{[FExpan]} \qquad (10.7)$$

where a, b, α, β and γ are constants, and $\ell = \ln(z)$. Below I assume, that coefficients v and w of the expansion (10.6) are known, and show, how to calculate coefficients a and b of the asymptotic (10.7).

For the displacement of the argument $z \mapsto z+1$, expressions in the asymptotic representation (10.7) are transformed in the following way:

$$\frac{1}{z} \;\mapsto\; \frac{1}{z\!+\!1} = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1} = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} + .. \qquad (10.8)$$

$$\frac{1}{z^2} \;\mapsto\; \frac{1}{(z\!+\!1)^2} = \frac{1}{z^2}\left(1+\frac{1}{z}\right)^{-2} = \frac{1}{z^2} - \frac{2}{z^3} + \frac{3}{z^4} + .. \qquad (10.9)$$

$$\ell\!=\!\ln(z) \;\mapsto\; \ln(z\!+\!1) = \ln\!\left(z\cdot\left(1+\frac{1}{z}\right)\right)$$
$$= \ln(z) + \ln\!\left(1+\frac{1}{z}\right) = \ell + \frac{1}{z} - \frac{1}{2z^2} + .. \qquad (10.10)$$

Using these preparations, the left hand side of the transfer equation

$$f(z\!+\!1) = T(f(z)) \quad \text{[transff]} \qquad (10.11)$$

can be written as follows:

$$f(z\!+\!1) = \frac{a}{z} - \frac{a}{z^2} + \frac{a}{z^3} + \frac{b\ell + b/z}{z^2}\left(1-\frac{2}{z}\right) + \frac{\alpha\ell^2\!+\!\beta\ell\!+\!\gamma}{z^3} + ..$$
$$= \frac{a}{z} + \frac{1}{z^2}\Big(-a\!+\!b\ell\Big) + \frac{1}{z^3}\Big(a\!+\!b - 2b\ell + \alpha\ell^2\!+\!\beta\ell\!+\!\gamma\Big) + .. \; (10.12)$$

The right hand side of equation (10.11) becomes

$$T(f(z)) = \frac{a}{z} + \frac{b\ell}{z^2} + \frac{\alpha\ell^2\!+\!\beta\ell\!+\!\gamma}{z^3} + v\cdot\left(\frac{a}{z} + \frac{b\ell}{z^2} + ..\right)^2 + w\left(\frac{a}{z} + ..\right)^3 ..$$
$$= \frac{a}{z} + \frac{1}{z^2}\Big(b\ell\!+\!va^2\Big) + \frac{1}{z^3}\Big(\alpha\ell^2\!+\!\beta\ell\!+\!\gamma\!+\!2vab\ell\!+\!wa^3\Big) + .. \qquad (10.13)$$

We should equalize the right hand sites of equations (10.12) and (10.13). Coefficients at $\frac{1}{z}$ coincide automatically. Coefficients at $\frac{1}{z^2}$ gives

$$-a = va^2 \quad \text{[ava2]} \qquad (10.14)$$

Equalisation of coefficients at $\frac{1}{z^3}$ gives

$$a+b-2b\ell = 2vab\ell + wa^3 \quad \text{[f1tttttt]} \tag{10.15}$$

Variable $\ell=\ln(z)$ depends on z, but equation (10.15) should be valid for various z. So, we get the two equations:

$$-2b = 2vab \quad \text{[again]} \tag{10.16}$$

which, at $b\neq 0$, gives the same, as (10.14), and also

$$a+b = wa^3 \quad \text{[f1tttt]} \tag{10.17}$$

Solving these two equations, I get

$$a=\frac{-1}{v} \quad , \quad b=\frac{-w}{v^3}+\frac{1}{v} \quad \text{[avbwv]} \tag{10.18}$$

One can add more terms in the expansion (10.7), and, in the similar way, calculate coefficients α, β, γ and even higher, assuming, that the expansion (10.6) of the transfer function is known. In such away, the coefficients of the asymptotic expansion (10.7) of the superfunction f are determined by the coefficients of expansion of transfer function T.

The asymptotic solution (10.7) allows evaluation of the transfer function at large z. Then, as in the case of the regular iteration, the asymptotic solution can be extended to the whole complex plane (except the cut lines) with the transfer equaltion, applying one of the two formulas below:

$$f(z)=T^n\Big(f(z-n)\Big) \quad \text{[iterA]} \tag{10.19}$$

or

$$f(z)=T^{-n}\Big(f(z+n)\Big) \quad \text{[iterB]} \tag{10.20}$$

in order to make the argument of the superfunction large, bringing it to the range, where the truncated asymptotic expansion (10.7) provides the required precision. The choice of one of the formulas (10.19) or (10.20) determines the position of the cut lines of the resulting superfunction. The direction of the cut lines can be changed also with replacement of $\ell=\ln(z)$ to $\ell=\ln(-z)$. In order to get unique solution, we need to choose (and postulate) the asymptotic behavior of the function we want to construct. In the following section, the method above is applied to exponential to base $e^{1/e}$, and then, in the following chapters, to other functions.

In principle, even with two terms in equation (10.7), one can get the camera-ready maps of the superfunction and related functions. Perhaps, this is sufficient for the application in physics and other sciences, where the precision of measurements is usually less than 14 decimal digits. The precision can be infinitely improved with modification of the argument of the primary approximation f with equations (10.19) or (10.20), increasing the number n of iterates.

However, for the numerical tests, it worth to calculate several coefficients of expansion of superfunction. This reduces the time of evaluation and improves the precision. For various transfer functions, that are holomorphic in vicinity of the fixed point and have expansion beginning with (10.6), some tens of coefficients of the expansion (10.7) can be calculated. Examples of the numerical tests for such expansions are presented below.

2 Exponent to base $\exp(1/\mathrm{e})$

Let us apply the formalism of exotic iterations above to the exponential to base $b\!=\!\eta\!=\!\exp(1/\mathrm{e})\!\approx\!1.44466786$. This section deals with function $\mathcal{T}$ by

$$\mathcal{T}(z) = \eta^z = \exp_\eta(z) = \exp(\ln(\eta)\, z) = \exp(z/\mathrm{e}) \quad [\text{etaz}](10.21)$$

Explicit plot of this transfer function is shown in figure 10.1 with thick curve. For comparison, the thin curve shows the exponent to base $\sqrt{2}$, considered above in chapter 9; this curve is borrowed from figure 9.1. The curves at this figure look close; at small and negative values of the argument, they almost overlap. However, the deviation is significant at large positive values of the argument, and this determines the pretty different behaviour of the corresponding superfunctions.

Complex map of the transfer function $\exp_\eta$ is shown in figure 10.3. This map looks similar to the map at figure 9.2; the only period is slightly different. For the base η, this period

$$P_{\exp_\eta} = 2\,\pi\,\mathrm{i}\,\mathrm{e} \approx 17.0794684453\,\mathrm{i} \quad [\text{Pexpeta}] \qquad (10.22)$$

It is amassing, that the simple formula (10.22) combines the 3 fundamental constants, : π, i and e. In addition, the Henryk base $\eta\!=\!\exp^2(-1)\!=$

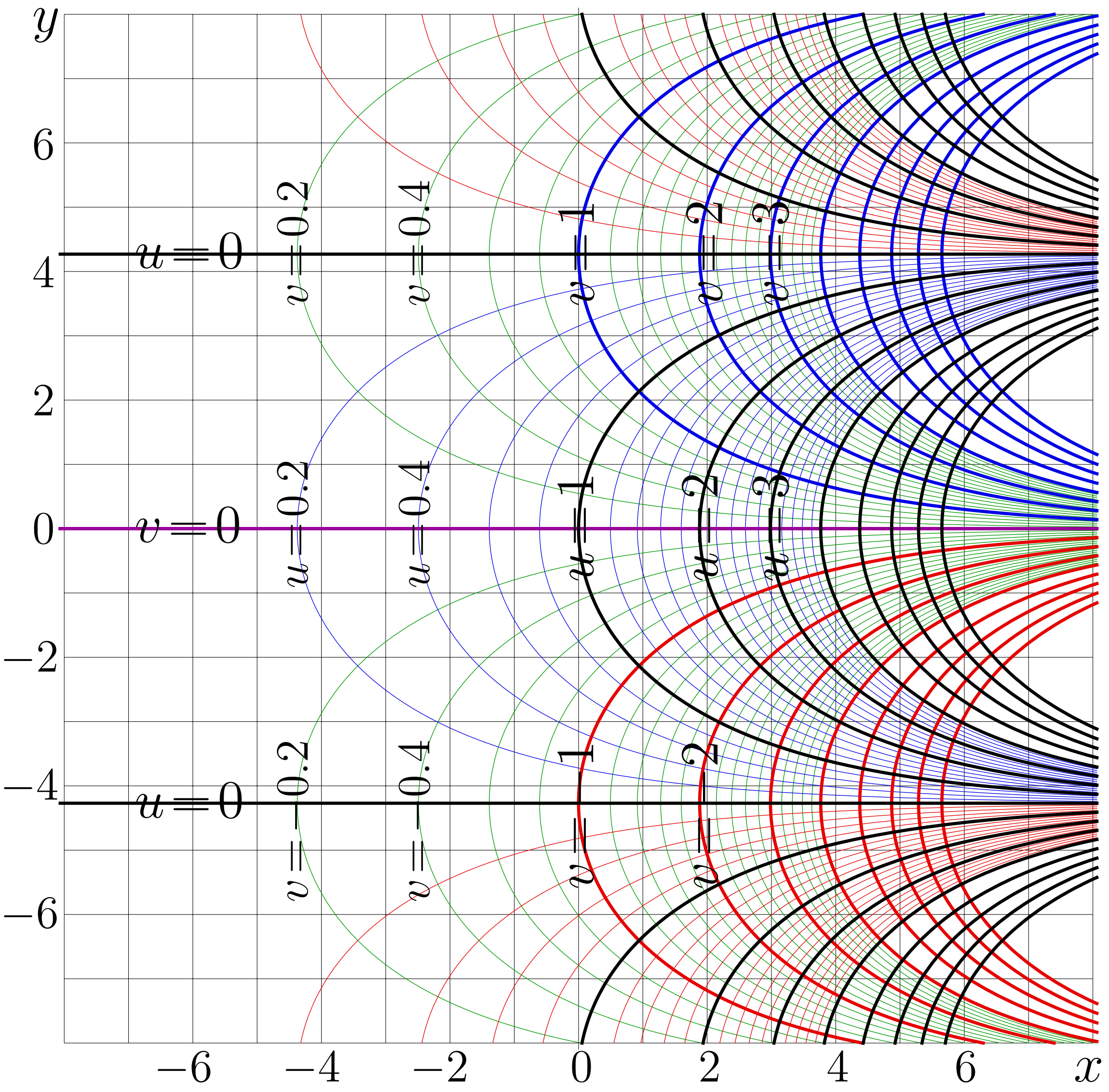

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Figure 10.3: $u+\mathrm{i}v = \exp_\eta(x+\mathrm{i}y)$ [expe1emap]

$\exp(1/\mathrm{e}) \approx 1.44466786101$ also is mathematical constant. A little bit leas than one period $P_{\exp_\eta}$ fits the range of map in figure 10.3.

For transfer function $\mathcal{T} = \exp_\eta$, the inverse function is $\mathcal{T}^{-1} = \exp_\eta^{-1} = \log_\eta$. Complex map of this function is shown in figure 10.4. This map looks similar to the map of logarithm to base $\sqrt{2}$, shown in chapter 9 at figure 9.3. At that map, the mesh of isolines is a little bit more dense; for example, all the line $u=6$ happen to be in the range of the map, while at the map at figure 10.4 the only part of this line is seen; as for level $v=\pm 9$ it happen to be beyond the cut line and, therefore, not seen in the map.

Value $b = \eta$ is maximal real base, at which the exponent still has at least one real fixed point. Below, for this exponent, the superfunction is constructed.

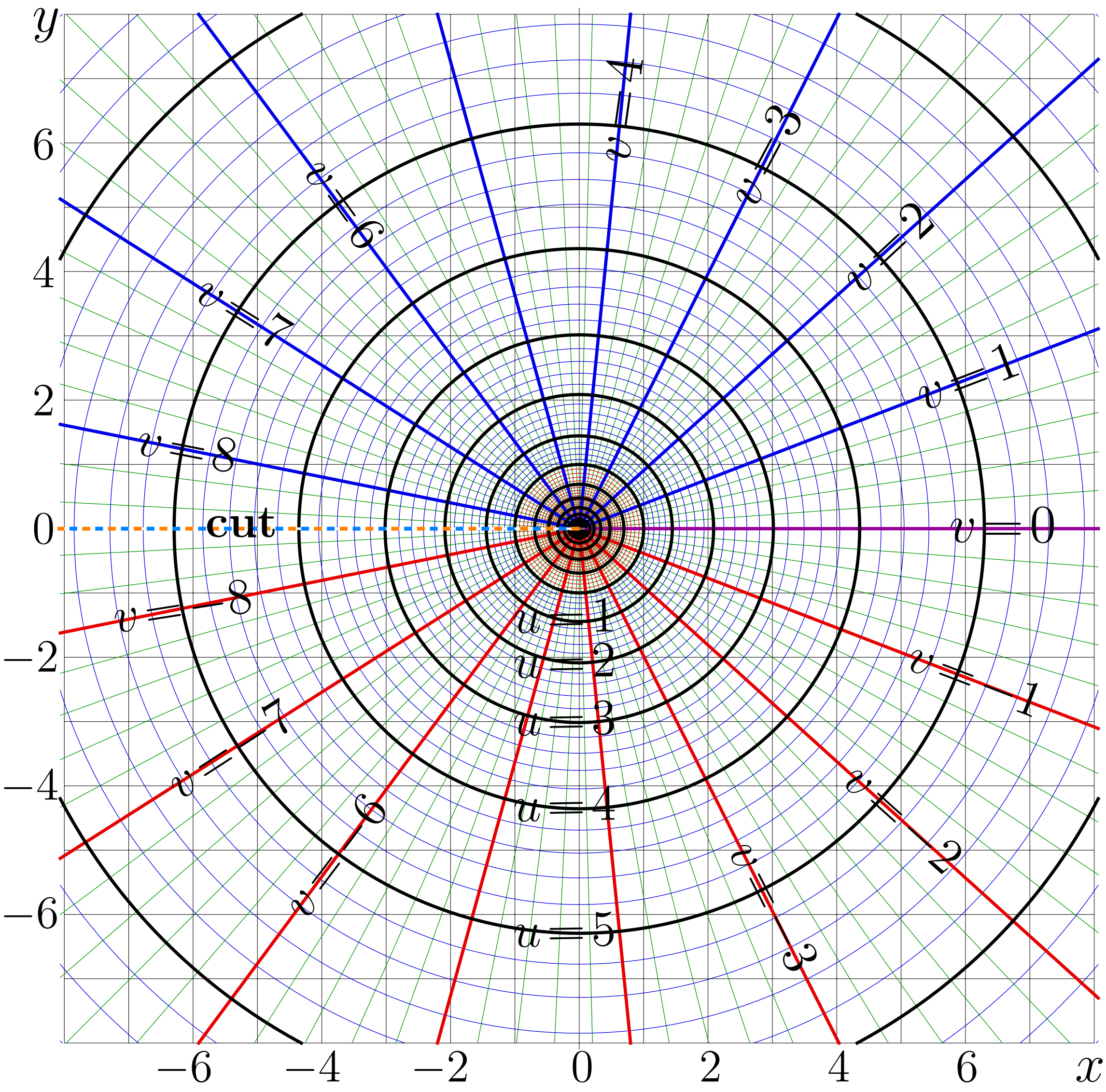

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Figure 10.4: $u+\mathrm{i}v=\log_\eta(x+\mathrm{i}y)$ [loge1emap]

3 Superexponent to base $\eta=\mathrm{e}^{1/\mathrm{e}}=\exp^2(-1)$

This case had been considered by request from Henryk Trappmann. He believed, that for the exponential to base

$$\eta = \exp(1/\mathrm{e}) = \exp^2(-1) \approx 1.44466786101 \quad \text{[eta]} \quad (10.23)$$

I cannot construct the efficient (fast and precise) algorithm of evaluation. The similar opinion had been expressed in century 20 by Peter L. Walker [24].

In order to convince Henryk, we had to write the special article for the journal Mathematics of Computation [79]. Some formulas and figures from that publication are repeated below.

Let the transfer function $\mathcal{T}$ be defined with equation 10.21 of the precious section; let

$$\mathcal{T}(z)=\eta^z=\exp_\eta(z)=\exp\big(\ln(\eta)\,z\big)=\exp(z/\mathrm{e}) \tag{10.24}$$

In order to apply the formalism of exotic iteration from the previous section, I define the new, "displaced" transfer function, as it is suggested by equation (10.4); for the initial transfer function $\mathcal{T}$ by (10.24). To distinguish these transfer functions, I use slightly different fonts for the letter T in the names. The new (displaced) transfer function

$$T(z)=\mathcal{T}(z+\mathrm{e})-\mathrm{e}=\exp\big((z+\mathrm{e})/\mathrm{e}\big)-\mathrm{e}=\exp(z/\mathrm{e}+1)-\mathrm{e} \tag{10.25}$$

For this transfer function,

$$\begin{aligned}
T(0) &= 0 && (10.26)\\
T'(0) &= 1 && (10.27)\\
T''(0) &= 2\,v \;=\; 1/\mathrm{e} && (10.28)\\
T'''(0) &= 6\,w \;=\; 1/\mathrm{e}^2 && (10.29)\\
\dots & && (10.30)\\
T^{(n)}(0) &= 1/\mathrm{e}^{n-1} && (10.31)
\end{aligned}$$

In the last formula, in the left hand side, the parenthesis in the superscript indicates not a number if iterate, but number of derivative; function T is differentiated n times.

For this case, the formalism of exotic iteration, described in the precious section, can be applied as is. Expansion (10.6) can be written as follows:

$$T(z)=z+\frac{1}{2\mathrm{e}}z^2+\frac{1}{6\mathrm{e}}z^3+.. \tag{10.32}$$

For $v=\dfrac{1}{2\mathrm{e}}$ and $w=\dfrac{1}{6\mathrm{e}^2}$, formulas (10.18) give

$$\begin{aligned}
a &= -2\mathrm{e}\approx -5.43656365691809 && (10.33)\\
b &= -\frac{\mathrm{e}}{3}+\frac{1}{8\mathrm{e}^3}\approx 1.604598172578777 && (10.34)
\end{aligned}$$

Then, the primary approximation f for superfunction (10.7) has the following form:

$$\tilde f(z) \;=\; -\frac{1}{z}+\Big(1+\frac{1}{\mathrm{e}}\Big)\frac{\ln(\pm z)}{z^2}+.. \quad \text{[fe1ez2]} \tag{10.35}$$

More terms of this expansion can be calculated analytically, especially, if some Mathematica or Maple are used. In principle, the truncated series in (10.35), even with two terms, approximates the superfunction and can be used for its definition. For $|z|>100$, such a representation gives few correct decimal digits and can be used to plot the complex maps.

For the accurate representation of superfunction, more terms in the expansion, similar to (10.35), should be calculated. The asymptotic expansion can be written as follows:

$$\tilde{F}(z) = \frac{-2\mathrm{e}}{z}\left(1+\sum_{m=1}^{M}\frac{P_m\big(-\ln(\pm z)\big)}{(3z)^m}+\mathcal{O}\Big(\ln(\pm z)/z\Big)^{M+1}\right) \quad (10.36)$$

where

$$P_m(x)=\sum_{n=0}^{m} c_{m,n}x^n \quad \text{[e1ePm]} \quad (10.37)$$

One may choose the first coefficient $c_{0,0}=1$; then, $P_0(t)=1$. Other coefficients c are determined by the substitution of the expansion (10.36) into the transfer equation

$$\tilde{F}(z+1)=\exp\Big(\tilde{F}(z)/\mathrm{e}+1\Big)-\mathrm{e}$$

The first 5 polynomials P are shown below:

$$P_1(t) = t \quad (10.38)$$

$$P_2(t) = t^2+t+1/2 \quad (10.39)$$

$$P_3(t) = t^3+\frac{5}{2}t^2+\frac{5}{2}t+\frac{7}{10} \quad (10.40)$$

$$P_4(t) = t^4+\frac{13}{3}t^3+\frac{45}{6}t^2+\frac{53}{10}t+\frac{67}{60} \quad (10.41)$$

$$P_5(t) = t^5+\frac{77}{12}t^4+\frac{101}{6}t^3+\frac{83}{4}t^2+\frac{653}{60}t+\frac{2701}{1680} \quad (10.42)$$

The superfunction F appears as limit

$$F(z)=\lim_{n\to\pm\infty} T^n(f(z-n)) \quad \text{[e1elim]} \quad (10.43)$$

where function f is just truncation of the asymptotic expansion in (10.36) at some positive integer M, and n is chosen positive or negative, dependently on the sign in the argument of logarithm in formula (10.36).

It is convenient to deal with functions, that have integer value at zero; so, I use the translation of the argument in order to define iteration tet_η and the growing sperexponential $\mathrm{SuExp}_{\eta,3}$ with the following formulas:

$$\mathrm{tet}_\eta(z) = F_1(z) = f(z+x_1) \quad \text{[e1etet]} \qquad (10.44)$$

$$\mathrm{SuExp}_{\eta,3}(z) = F_3(z) = f(z+x_3) \quad \text{[e1eSuExp]} \qquad (10.45)$$

where constants $x_1 \approx 2.798248154231454$ and $x_3 \approx -20.28740458994004$ are chosen to provide relations $F_1(0)=1$ and $F_3(0)=3$.

Function tet_η corresponds to the upper sign go the $\pm$ in formuas (10.36), (10.43), while finction $\mathrm{SuExp}_{\eta,3}$ by (10.45) refers to the lowerst sign. In such a way, F_1 and F_3 are pretty different functions, and it is difficult to express one of them through another.

In formulas (10.44) and (10.45), for the two superfunctions F_1 and F_3, a little bit longer names $\mathrm{tet}_{\exp(1/\mathrm{e})}$ and $\mathrm{SuExp}_{\exp(1/\mathrm{e})}$ are suggested. These long names simplify the identification at the use from other chapters of this Book (and also from other publications), where the base b may have various values, and not necessary $b=\eta$.

I repeat the meanings of the names suggested. tet_η, indicates, that this refers to tetration. For any tetration (to any base b), I assume condition $\mathrm{tet}_b(0)=1$.

The shoice of value at zero of the growing super exponential F_3 is not so obvious. In publications [61, 79] (and not only there), the value at zero is chosen as minimal integer, that is still larger, than the real fixed point. Such a choice has sense, while the only one transfer function without parameters is considered, or if value of its parameter is fixed, and no continuity properties with respect to this parameter are analysed. However, if we treat base b as parameter, then, the choice of integer value at zero leads to discontinuity with respect to base b. For the case, if such a dependence will be requested, the notation $\mathrm{SuExp}_{\eta,3}$; is used; the value at zero is indicated as the additional subscript.

For the real argument, functions F_3 and F_1 are shown in figure 10.5. Complex maps of these superfunctions are shown in figure 10.6. These functions are pretty different. In order to release the notation F with subscripts for other functions, used in other chapters (and in other publications), I give them the special names $\mathrm{SuExp}_{\eta,3} = F_3$ and $\mathrm{tet}_\eta = F_1$.

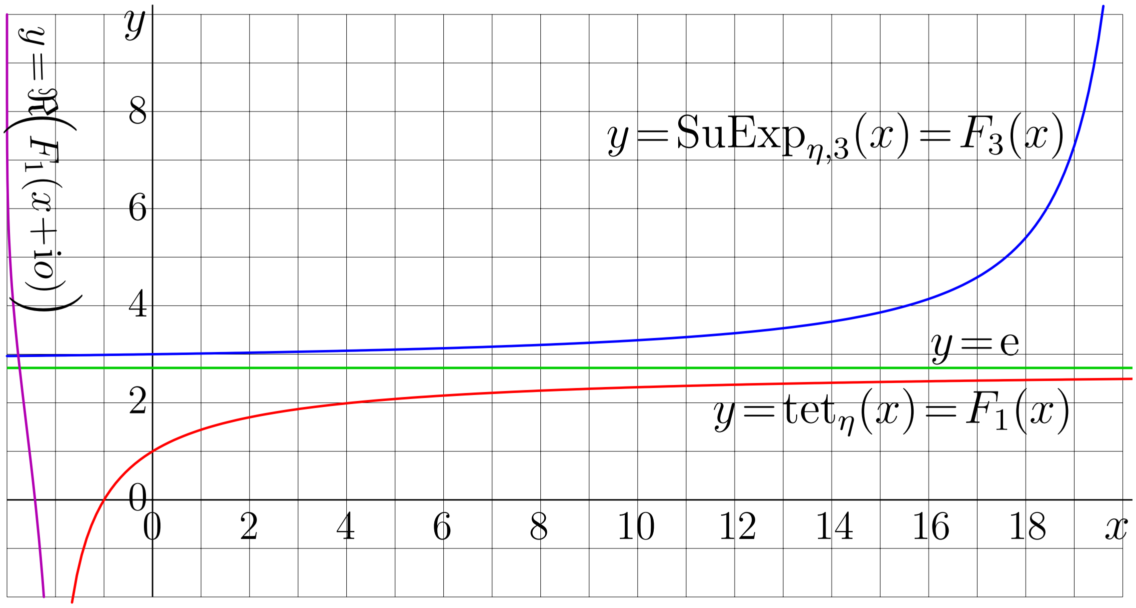

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Figure 10.5: Two super-exponentials to base $b = \exp(1/\mathrm{e})$ [e1eplot]

Symbol tet refers to tetration; in the following chapters, it is defined also for other values of base.

Along the real axis, function $\mathrm{SuExp}_{\eta,3}$ infinitely grows at the positive values of the argument, and approaches its limiting value e at large negative values. The function approaches to the same value in the most of the complex plane, except the strip along the positive direction od the real axis.

Function tet_η, at large values of the argument, approaches its limiting value η almost everywhere, but has logarithmic singularity at -2, going to infinity at this point. Tetration to any base has this singularity; it follows from the additional condition $\mathrm{tet}_b(0) = 1$ and the transfer equation, that in vicinity of the real axis can be rewritten in the following form:

$$\mathrm{tet}_b(z) = \log_b\Big(\mathrm{tet}_b(z+1)\Big) \tag{10.46}$$

For iterates of exponent to base η, we need not only the superfunctions, but also the corresponding abelfunctions. These abelfunstions are considered in the next section.

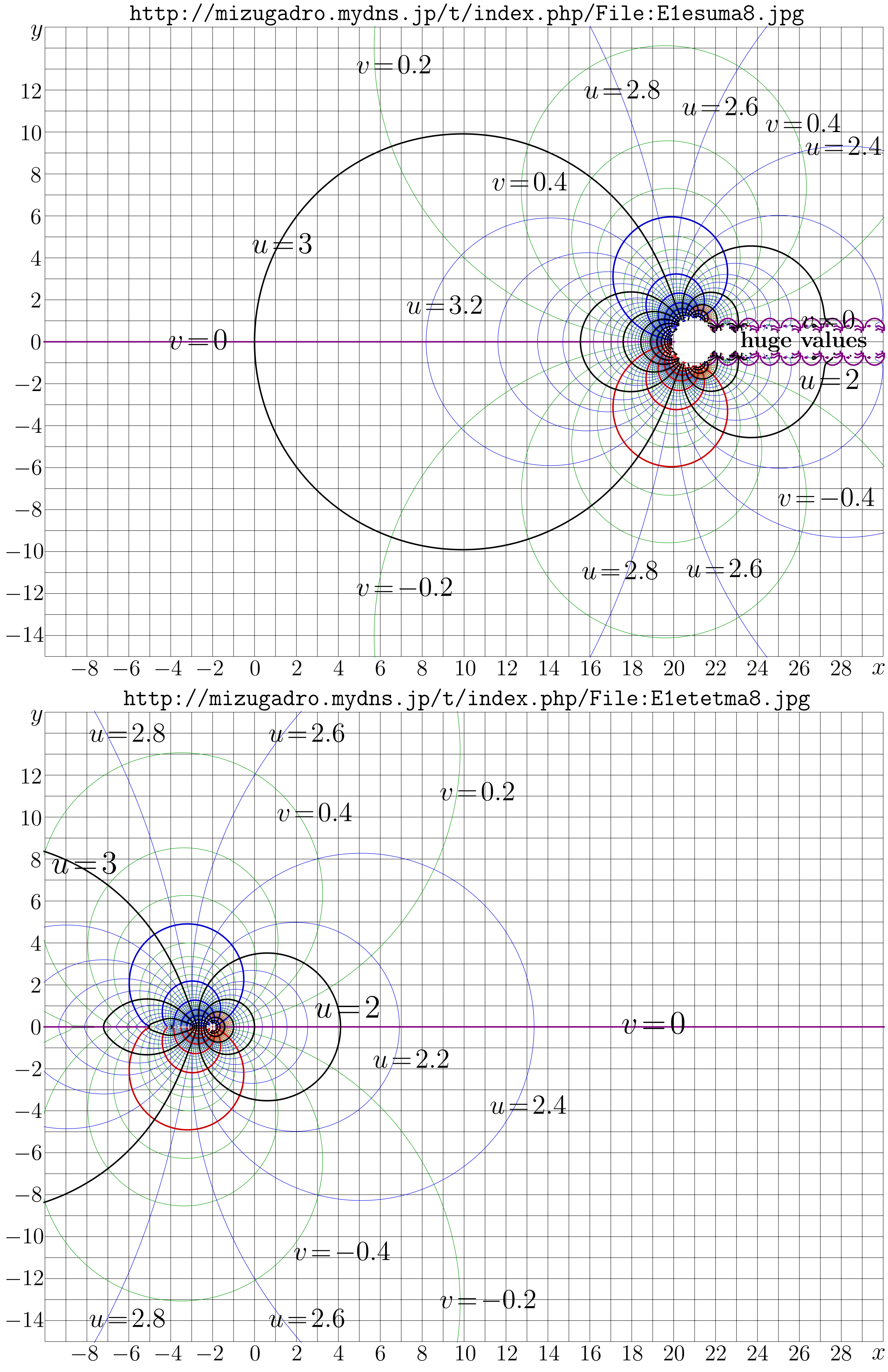


Figure 10.6: $u+\mathrm{i}v = \mathrm{SuExp}_{\eta,3}(x+\mathrm{i}y) = F_3(x+\mathrm{i}y)$ by (10.45), at the top, and $u+\mathrm{i}v = \mathrm{tet}_\eta(x+\mathrm{i}y) = F_1(x+\mathrm{i}y)$ by (10.44), at the bottom. [e1efig2]

4 Abelexponent to Hernyk base

Historically, the exponent to base $\eta = \exp(2/\mathrm{e})$ is first function, for which the exotic iteration had been applied [79]. The two superfunctions for this exponent are specified in the previous section. Here, I describe the inverse functions, id est, the two abelfunctions of the exponent to base η.

Let $G_1 = F_1^{-1}$ and $G_3 = F_3^{-1}$. Maps of these functions are shown in figure 10.7. For evaluation of functions G_1 and G_3, the asymptotic representations are used:

$$g(z) = g_{\pm}(z) = \frac{\ln(\pm t)}{3} + \frac{2}{t} + \sum_{n=1}^{15} c_n t^n + O(t^{16}) \quad \text{[e1egpm]} \quad (10.47)$$

where $t = (z - \mathrm{e})/\mathrm{e}$. The coefficients c can be found inverting the expansion for the superfunction, and also by substituting expansion (10.47) into the Abel equation

$$g(z) + 1 = g\Big(\exp\big(g(z)/\mathrm{e}\big)\Big) \quad \text{[e1eAbeleq]} \qquad (10.48)$$

The truncation of the series, id est, omitting of O in the right hand side of (10.47), provides the algorithm for evaluation of the abelfunction g with at least 15 significant figures, while $|t| < 1/2$. For larger values, the argument of g should be transformed, using the equation (10.48).

Then the inverse functions $G_1 = F_1^{-1}$ and $G_3 = F_3^{-1}$ can be defined, adding the corresponding constants,

$$\mathrm{ate}_\eta(z) = G_1(z) = g(z) - g(1) \approx g(z) - 3.029297214418 \qquad (10.49)$$

where the upper sign in (10.47) is used, and

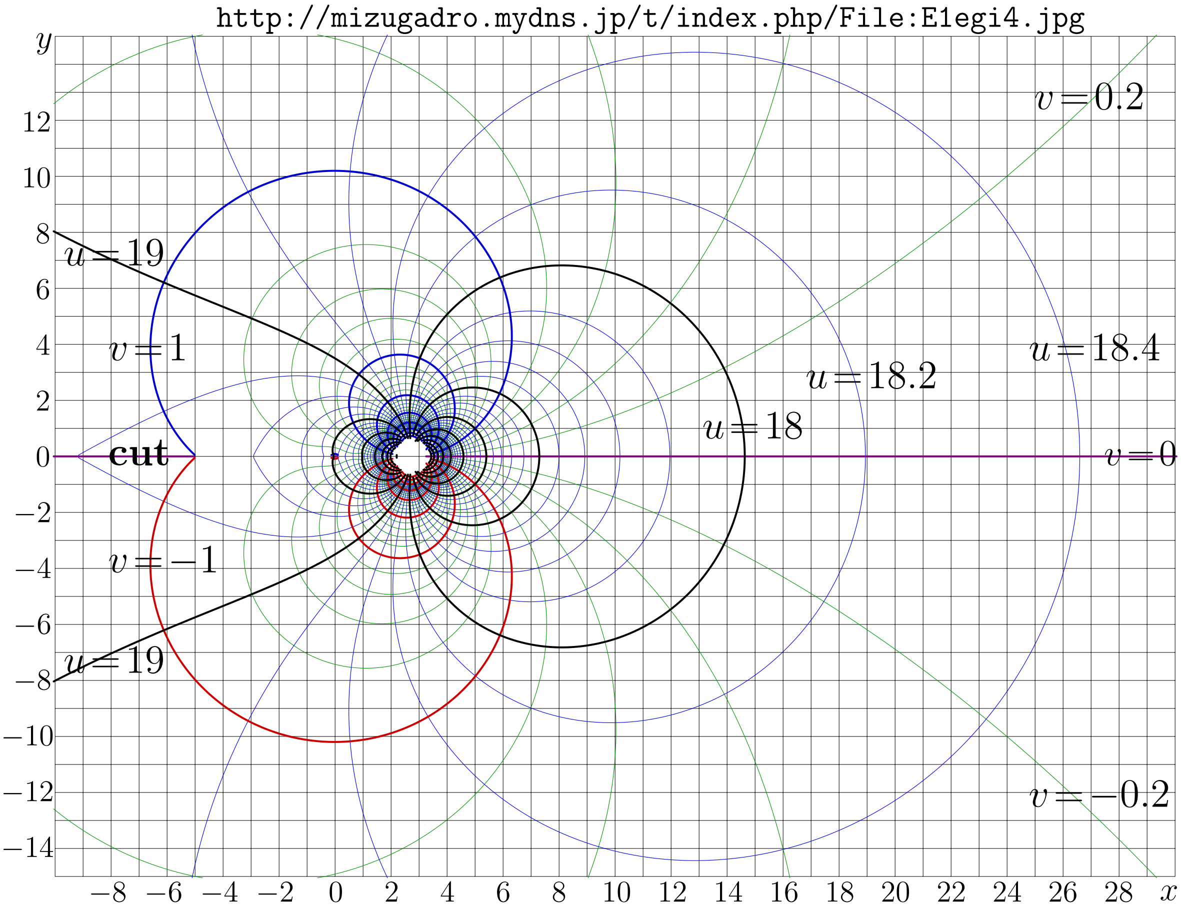
$$\mathrm{AuExp}_{\eta,3} = G_3(z) = g(z) - g(3) \approx g(z) + 20.0563555297533789 \qquad (10.50)$$

where the lower sign in (10.47) is used.

The additional names ate and AuExp are introduced in order to simplify identification and referencing to these functions from other chapters.

Representations (10.49) and (10.50) provide relations $G_1(1) = 0$ and $G_3(3) = 0$, as it is supposed to be for the inverse functions of F_1 and F_3. The numerical tests confirm, that in wide ranges of values z, the relations

$$F_1(G_1(z)) = z\,,\ F_3(G_3(z)) = z\,,\ F_1(G_1(z)) = z\,,\ G_3(F_3(z)) = z \qquad (10.51)$$



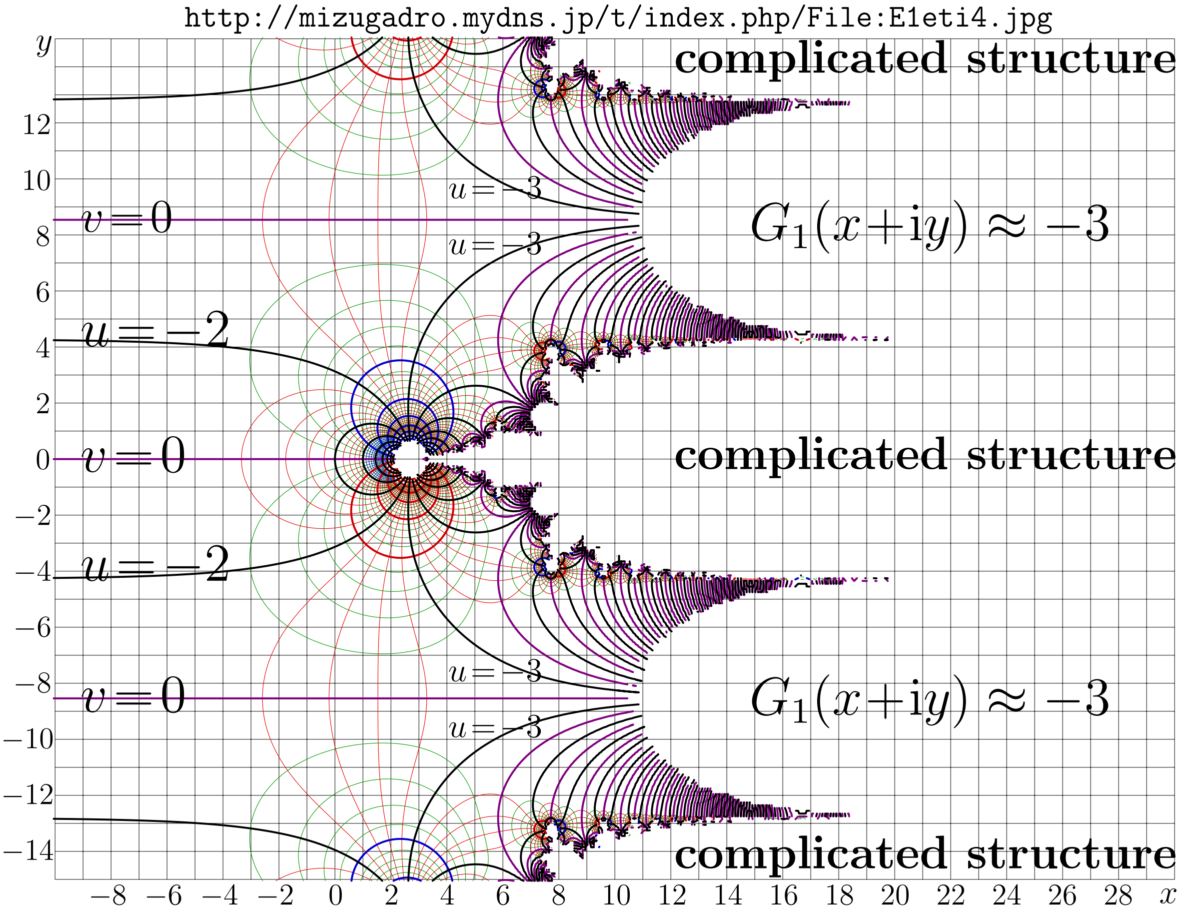


Figure 10.7: $u+\mathrm{i}v = G_3(x+\mathrm{i}y) = \mathrm{AuExp}_{\eta,3}(x+\mathrm{i}y)$ by (10.50), top map, and $u+\mathrm{i}v = G_1(x+\mathrm{i}y) = \mathrm{ate}_\eta(x+\mathrm{i}y)$ by (10.49), map at the bottom [e1efig3]

hold with 14 significant figures. The readers are invited to plot the map of the agreement function

$$A(z) = -\lg\left(\frac{|F(G(z)) - z|}{|F(G(z))| + |z|}\right) \quad \text{[e1eA]} \qquad (10.52)$$

adding appropriate subscripts and exchanging $F \leftrightarrow G$. as numerical check of equalities in statement (10.51). Levels of $A(x+\mathrm{i}y)$ in the x, y plane indicate, where the algorithm works almost without loss of precision, id est, close to the maximal precision allowed for the complex double variables.

5 Iterates

With function F_3 and G_3, one can express the iterates of exponential to base exp(1/e). For $T(z) = \exp(z/\mathrm{e})$,

$$T_{\mathrm{u}}^{n}(z) = \exp_{\eta,\mathrm{u}}^{n}(z) = F_3(n + G_3(z)) = \mathrm{SuExp}_{\eta,3}\Big(n + \mathrm{AuExp}_{\eta,3}(z)\Big) \quad (10.53)$$

Here, the subscript u indicates, that the this iterate is holomorphic at the highest, upper range of the real values of the argument, and, in particular, in the vicinity of the half-line $z > \mathrm{e}$.

In the representation (10.53), neither argument z nor number of iterate has need to be real. For $n = 1/2$, the complex map of the half iteration of exponential to base exp(1/e) is shown in figure (10.9).

For real values of argument $T_{\mathrm{u}}^{n}(x)$ versus x for various values of number n of iteration is shown in figure 10.8. The thick curves correspond to the integer values of n. The non-integer iterates are holomorphic at least in some vicinity of falf-line $x > \mathrm{e}$. The integer iterates can be extended to $-\infty$ for positive b or to the closest vertical asymptotic of the corresponding logarithm for negative number of iteration. The bissectrisse of the First quadrant of the coordinate plane corresponds to the identity function.

The non integer iterates by (10.53) are shown in figure 10.8 with thin lines. These lines cannot be extended beyond the fixed point e; this is branch point of the non-integer iterates. These lines remain in the range $x > \eta$, $y > \eta$; in particular, they cannot be holomorphically extended into the third quadrant.

The non-integer iterates by formula (10.53) have cut line along the real axis from $-\infty$ to $\mathrm{e} \approx 2.71$; in order to shown this cut, figure 10.9

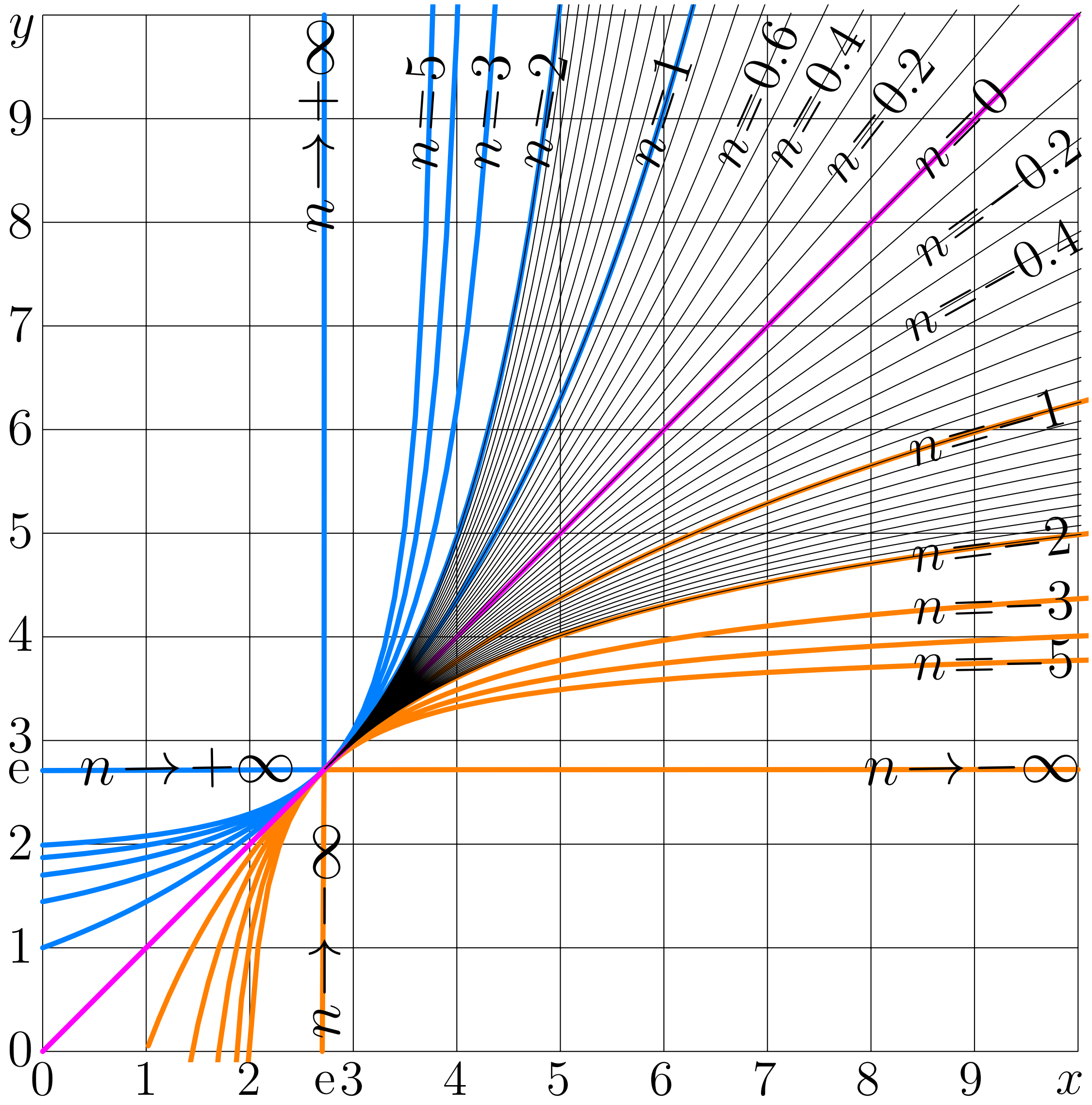


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Figure 10.8: $y=\exp_{\eta,\mathrm{u}}^{n}(x)$ by (10.53) versus x for various n [e1eiter]

represent the complex map of the half iterate, $n=1/2$. This cut is traced with dashed line and labeled with symbol **cut**.

As other complex maps of other iterates of a real-holomorphic function, the map at figure 10.9 is symmetric with respect to reflection from the real axis,

$$\exp_{\eta,\mathrm{u}}^{1/2}(z^*)=\exp_{\eta,\mathrm{u}}^{1/2}(z)^* \tag{10.54}$$

In order to remind this symmetry, I plot the maps, placing the real axis at the centre of the figure. However, sometimes, there is not enough space at the page, then I plot the only upper part of the map. I hope, the Reader has enough imagination, that allows him or her to consider the imaginary mirror, and imaginate in this imaginary mirror the function with inverted sign of the its imaginary part. With hopes for such imagination, in Figure 10.10, I show the iterates of the exponent to base

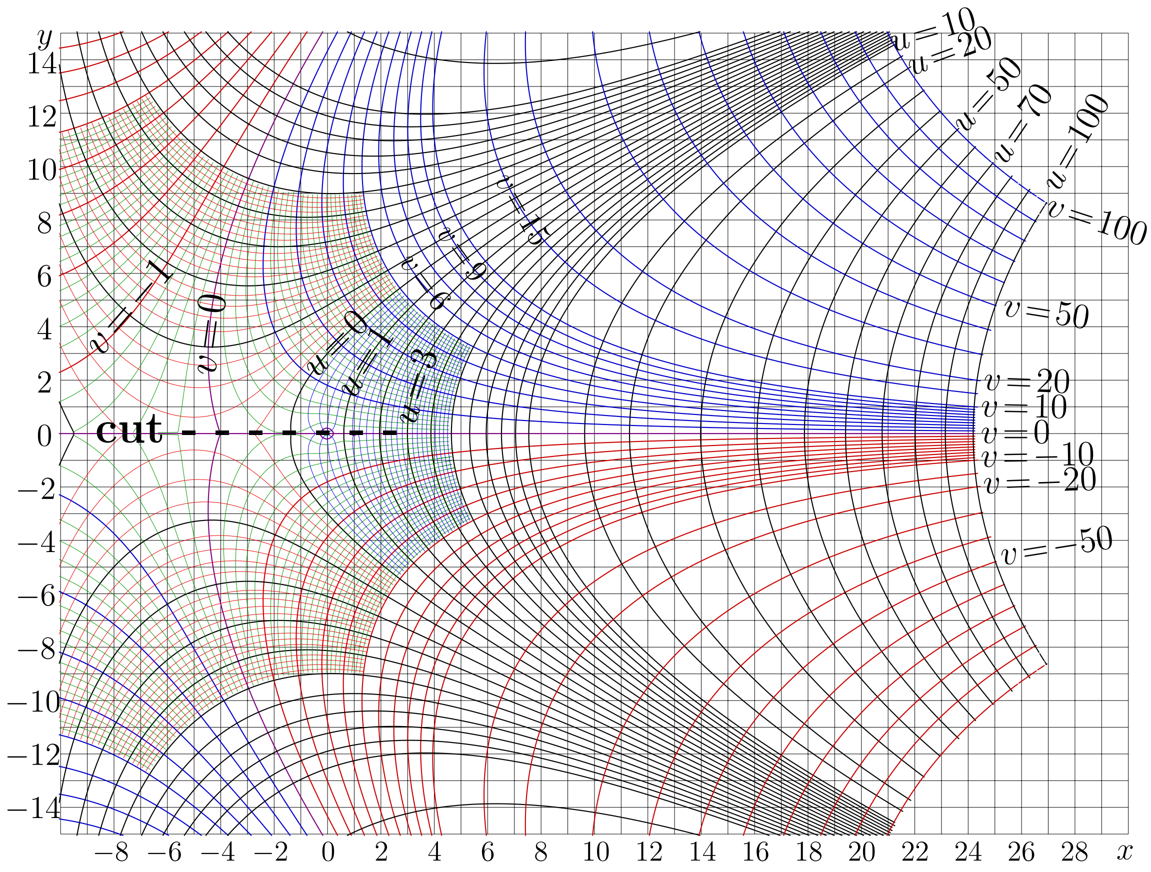

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Figure 10.9: $u+\mathrm{i}v = \exp_{\eta,\mathrm{u}}^{1/2}(x+\mathrm{i}y)$ by (10.53) for $n=1/2$ [e1eghalf]

η for various number of iterate n in a similar way, as the case $n=1/2$ is shown in Figure 10.9. However, the only halts of the maps are shown in Figure 10.10.

6 Not only exp

Exponential to base $\eta = \exp(1/\mathrm{e}) = \exp^2(-1)$ is not only transfer function, that have derivative unity at its fixed point, $T'(L) = 1$. Some of such transfer functions can be treated with the method of exotic iteration, described in this chapter.

One more example is considered in the next chapter, referring to the elementary transfer function $T(x) = \mathrm{zex}(x) = z\exp(z)$. It is treated in a pretty similar way, as $\exp_\eta$.

Then, in the following chapters, even more exotic cases are treated, when $T'(L) = 1$ and $T''(L) = 0$ also can be treated. However, I try to go step by step. So, open the next chapter and read about iterates of zex.

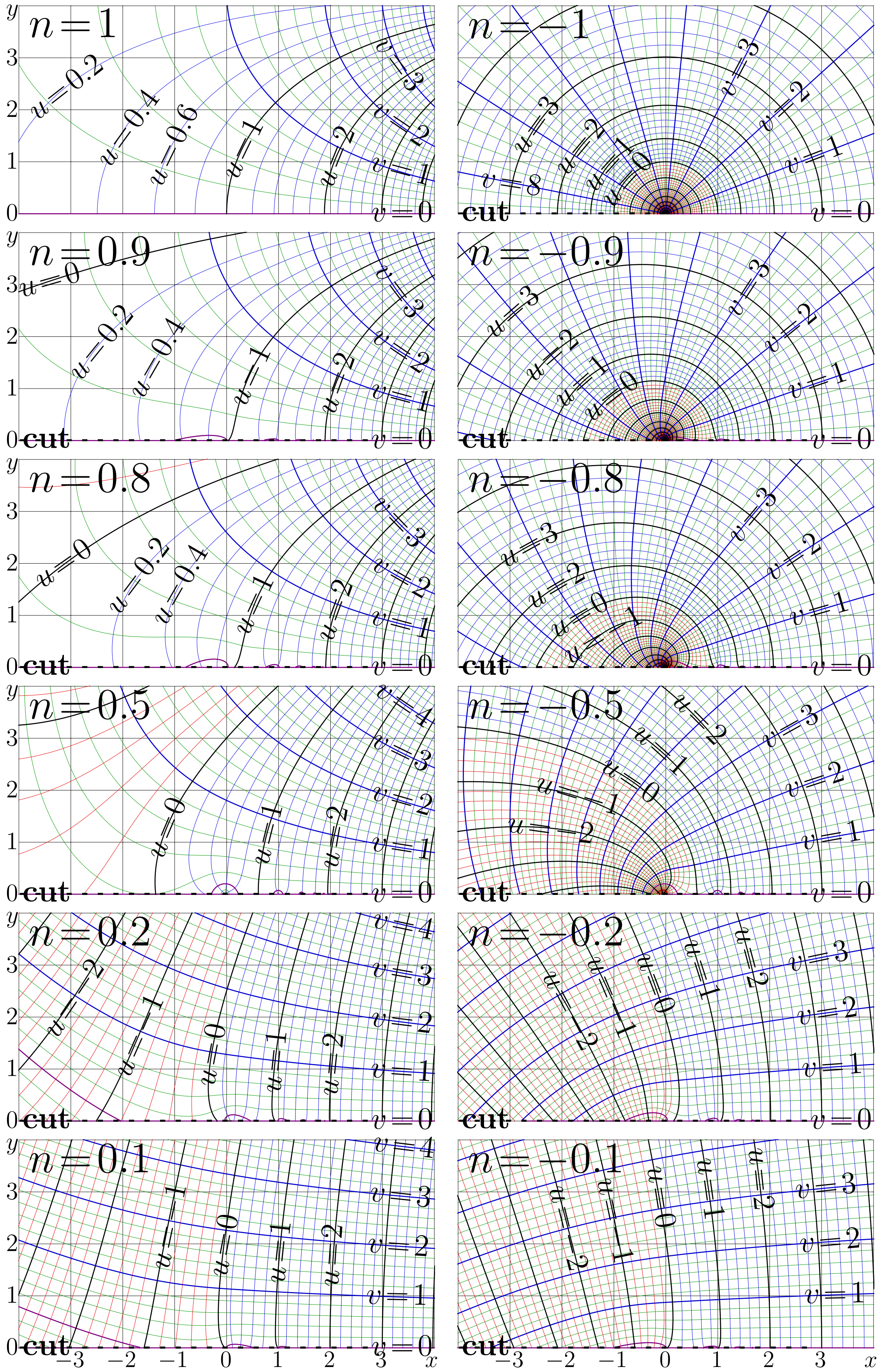
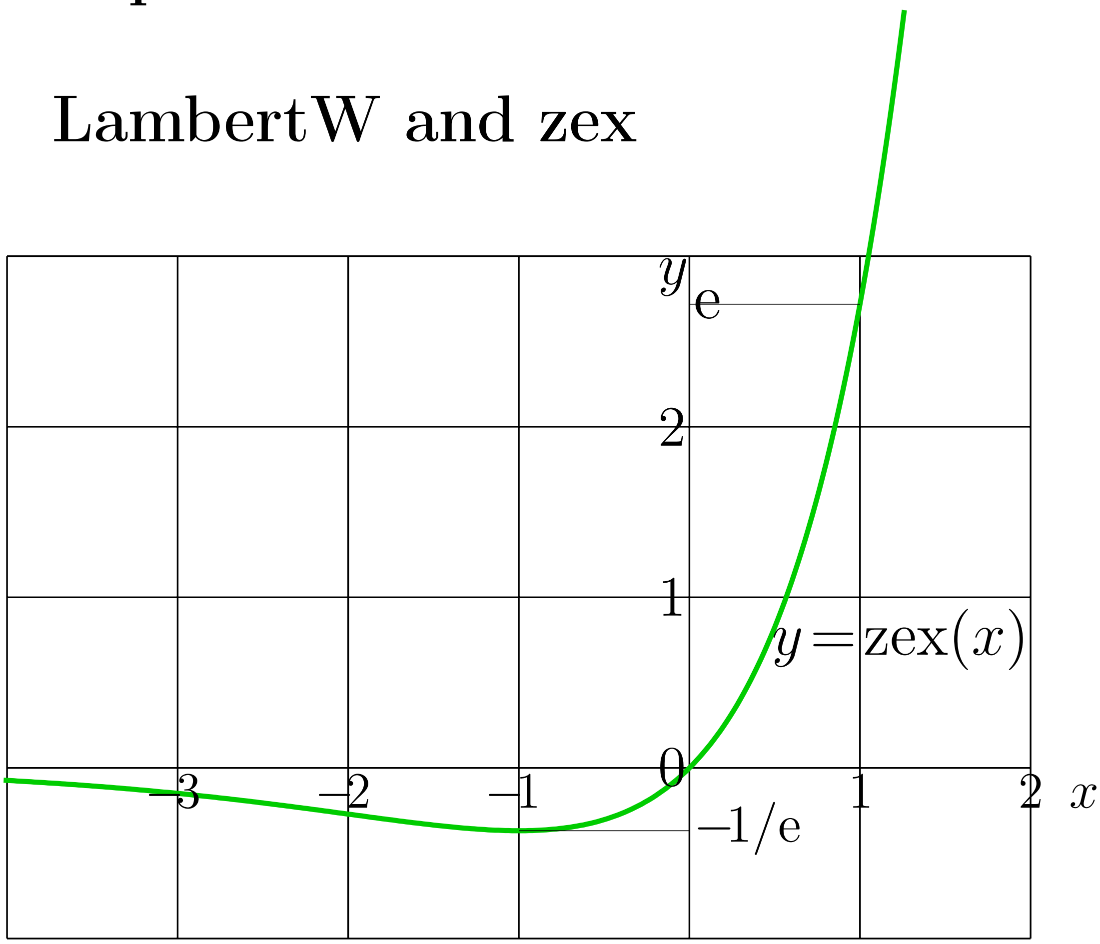


Figure 10.10: $u + \mathrm{i}v = \exp_{\eta,\mathrm{u}}^{n}(x + \mathrm{i}y)$, complex maps of iterates of exponent to base η by equation (10.53) for various n [e1e000map]

Chapter 11

LambertW and zex



http://mizugadro.mydns.jp/t/index.php/File:ZexPlot.png

Figure 11.1: $y = \mathrm{zex}(x) = x\ \exp(x)$ [zexplot]

This chapter deals with function ArcLambertW,

$$\mathrm{ArcLambertW}(z) = \mathrm{ArcProductLog}(z) = \mathrm{zex}(z) = z \exp(z) \quad (11.1)$$

shown in figure 11.1. The long name ArcLambertW is equivalent to even longer name ArcProductLog. This name looks so long, as “Logistic map” or “dihidrogena monoxide” shown in figure 11.2. Formally, the name is correct; the inverse function of ProductLog can be called ArcProductLog in analogy with arcsin or ArcTetration. But name zex is shorter. Complex map of function zex is shown in figure 11.3.

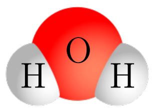


Figure 11.2: Dihidrogena monoxide

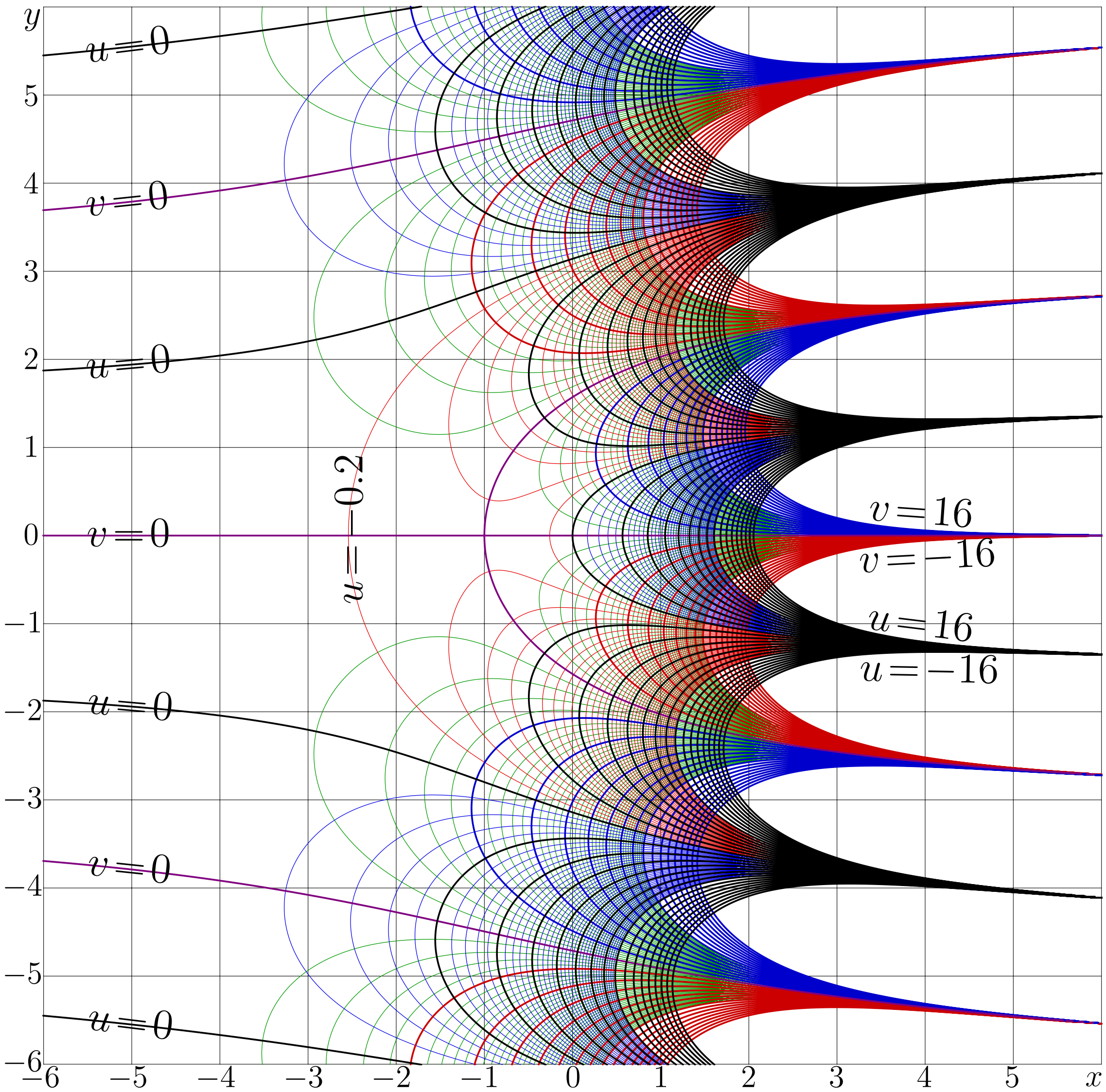


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Figure 11.3: $u+\mathrm{i}v=\mathrm{zex}(x+\mathrm{i}y)$ by (11.1) [zexmap]

Short name zex is created from the first three characters in the right hand side of equation (11.1). Hope, this long explanation helps to avoid confusions. In the chapter name I keep the indication to LambertW, as it is used in many other publications and algorithmic languages. As the negative iterates are allowed, if one can iterate zex, one can iterate also LambertW. Function zex is simpler than LambertW, so, I consider zex first.

Consideration in this chapter is very similar to that of the previous one; the similar exotic iterates are constructed. I hope, the reader can perform the calculus for iterates of zex in the same way, as the deduction is performed in the previous chapter for the exponent to base $\eta=\mathrm{e}^{1/\mathrm{e}}$. On the other hand, I remember the old rule:

If at your presentation, at the first desk you see some Hideki Yukawa and Sofia Kovalevskaya, and at the last desk you see some Bart Simpson and Shoko Okudaira, then you should address to Bart and Shoko. In this case, you may hope, that Hideki and Sofia will understand at least the main idea of your talk.

Adopting that rule to the Book, here I repeat the deduction of the previous section for function zex. However, I would like the Readers at least to try to make this deduction by themselves, using the Book only to verify the results; then the Readers will be able to do the same also for other functions (including exotic ones).

1 Holomorphic zex

For real argument, function zex by (11.1) is shown in figure 11.1. At negative values of the argument, zex is negative, and it is positive for positive argument. At large negative values of the argument, zex decreases, reaches its local minimum $-1/\mathrm{e}$ at -1, then grows up. The graphic passes through point (0,0); zero is fixed point. At zero, its derivative is unity; so the iterates of zex are qualified as exotic; the non-integer iterates are not regular at the fixed point. Here, it may be a good moment to go a little bit back, to the regular iteration, in order to remember, what happens, for example, at the right hand side of equation (6.10), when the derivative T' of the transfer function at the fixed point becomes zero.

Complex map of function zex is shown in figure 11.3. This function is entire, it has no singularities. Map of the inverse function

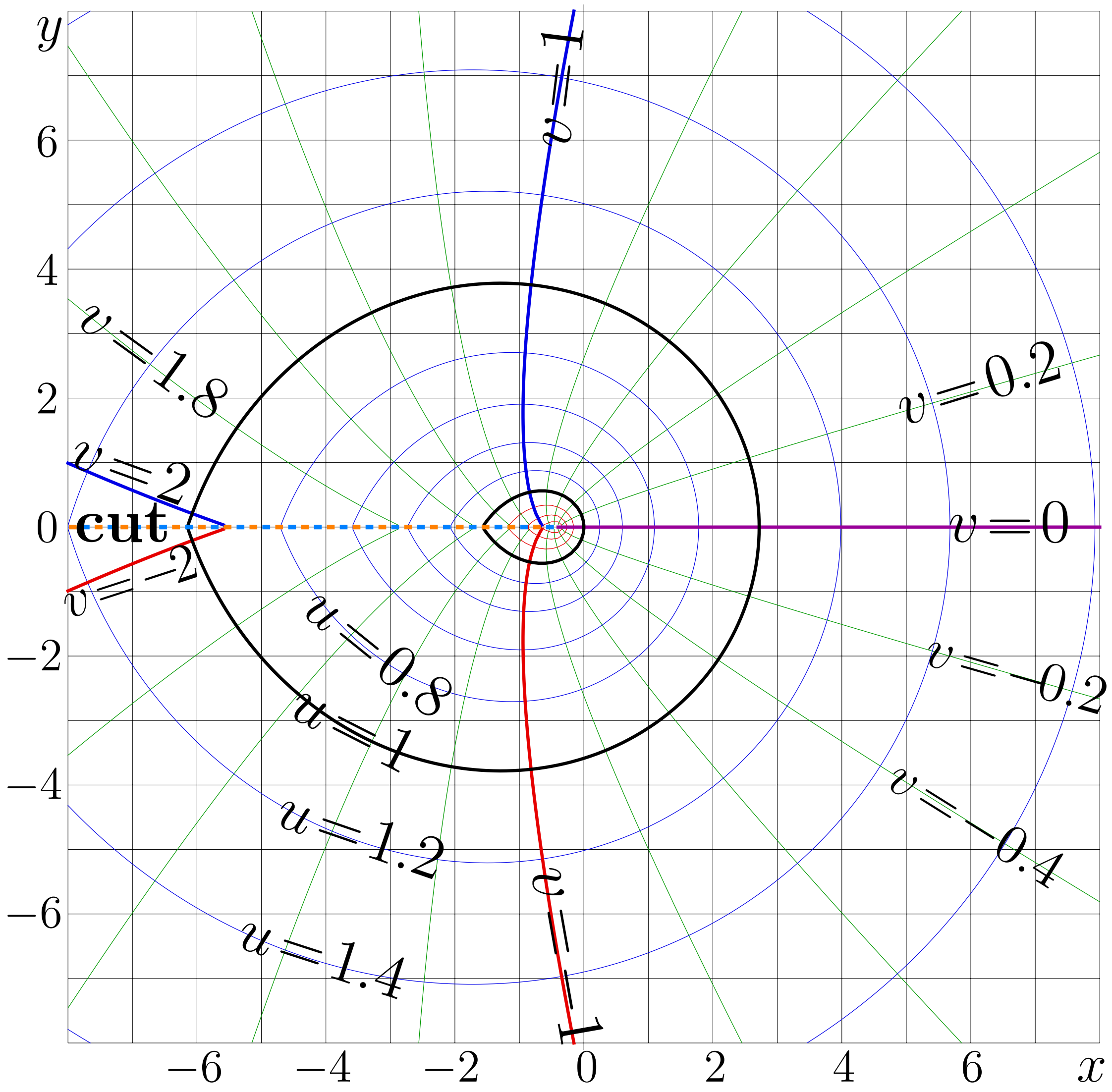
$$\mathrm{zex}^{-1} = \mathrm{ArcZex} = \mathrm{zex}^{-1} = \mathrm{LambertW} \tag{11.2}$$

is shown in figure 11.4. This function has cut, along the negative part of the real axis. This cut is marched with dashed line.

Function LambertW can be defined as solution F of the differential equation

$$F'(z) = \frac{F(z)}{(1+F(z))\; z} \quad \text{[LambertWdifur]} \tag{11.3}$$

with additional condition $F(0)\!=\!0$, where the contour of integration goes along the imaginary axis from zero to the imaginary part of argument z,



http://mizugadro.mydns.jp/t/index.php/File:LambertWmap150.png

Figure 11.4: $u+\mathrm{i}v = \mathrm{ArcZex}(x+\mathrm{i}y) = \mathrm{LambertW}(x+\mathrm{i}y)$

and then, parallel to the real axis, to the value z. Solving the equation (11.3), the Reader may verify, that the inverse function of the solution is zex. Note, that the similar contour of integration is used in chapter 5 to define function Tania by equation (5.3). Indeed, function LambertW is related to the Tania function: (5.3):

$$\mathrm{LambertW}(z) = \mathrm{Tania}\Big(\ln(z) - 1\Big) \qquad [\mathrm{taniaLambertW}] \quad (11.4)$$

As the imaginary part of the logarithm is limited to the $-\pi, \pi$ range, in this expression, the argument of the Tania function is always inside the strip along the real part, seen in the complex map of the Tania function in figure (11.4). Function LambertW is in some sense simpler, than function Tania: it has only one cut line (and one branch point $-1/\mathrm{e}$), while the Tania functions has two.

I could not find good implementation complex double for function LambertW in C++ (in which I have good plotters of complex maps). For this reason, I use the expansions listed below.

For small values of argument, LambertW can be expanded as follows:

$$\begin{aligned}\mathrm{LambertW}(z) = &\ z\sum_{n=0}^{\infty}\frac{(n+1)^{n-1}}{n!}(-z)^n\\ &= z-z^2+\frac{3z^3}{2}-\frac{8z^4}{3}+\frac{125z^5}{24}-\frac{54z^6}{5}+\frac{16807z^7}{720}+..\end{aligned} \tag{11.5}$$

The series converges at $|z|<1/\mathrm{e}\approx 0.367879$. With 48 terms, at $|z|\le 0.2$, the truncated sum provides at least 16 correct decimal digits.

The expansion at of the branch point can be written as follows:

$$\begin{aligned}\mathrm{LambertW}\Big(\frac{-1}{\mathrm{e}}+\frac{t^2}{2\mathrm{e}}\Big) = -1+t-\frac{t^2}{3}+\frac{11t^3}{72}-\frac{43t^4}{540}+\frac{769t^5}{17280}-\frac{221t^6}{8505}\\ +\frac{680863t^7}{43545600}-\frac{1963t^8}{204120}+\frac{226287557t^9}{37623398400}-\frac{5776369t^{10}}{1515591000}+..\end{aligned} \tag{11.6}$$

The expansion can be used for approximation of LambertW while $|t|<1$, id est, while the argument of LambertW is close to $-\exp(-1)\approx -0.367879$.

For large values of $|z|$, using notations $L=\ln(z)$ and $M=\ln^2(z)$ the expansion of $\mathrm{LambertW}(z)$ can be written as follows:

$$\begin{aligned}\mathrm{LambertW}(z) \ = \ & L-M+\frac{M}{L}+\frac{M(-2+M)}{2L^2}\\ &+\frac{M(6-9M+2M^2)}{6L^3}\\ &+\frac{M(-12+36M-22M^2+3M^3)}{12L^4}\\ &+\frac{M(60-300M+350M^2-125M^3+12M^4)}{60L^5}\\ &+\frac{M(-120+900M-1700M^2+1125M^3-274M^4+20M^5)}{120L^6}\\ &+O\Big(\frac{M}{L}\Big)^7\end{aligned} \tag{11.7}$$

where the effective parameter of expansion happens to be $\varepsilon = \ln^2(z)/\ln(z)$; at $|\varepsilon| \ll 1$, the asymptotics (11.7) can be used for the evaluation of LambertW.Here, as usually, the upper superscript after the function indicates the number of iterations, and the upper superscript after the argument and the closing parenthesis indicates the power that is assumed to be evaluated after the evaluation of the function. However, $\ln^{-1}(z) = \exp(z)$ should not be confused with $1/\ln(z)$, nor $\ln^2(z) = \ln(\ln(z))$ should be confused with $\ln(z)^2$, and so on.

The asymptotics above allow to implement the efficient and precise algorithm for evaluation of LambertW. The C++ implementation is loaded as `http://mizugadro.mydns.jp/t/index.php/LambertW.cin`.

With functions zex and $\mathrm{LambertW} = \mathrm{zex}^{-1}$, the superfunction of zex and its abelfunction can be implemented. The superfunction is considered in the next section.

2 SuZex

Superfunction for the transfer function zex is solution F of the transfer equation

$$F(z+1) = \mathrm{zex}\Big(F(z)\Big) \qquad \text{[suzexFeq]} \tag{11.8}$$

For some integer $M > 1$, search for the asymptotic expansion of the solution in the following form:

$$F(z) = F_M(z) + O\left(\frac{\ln(\pm z)^{M+1}}{z^{M+2}}\right) \qquad \text{[SuZexFMa]} \tag{11.9}$$

where

$$F_M(z) = -\frac{1}{z} + \frac{\frac{1}{2}\ln(\pm z)}{z^2} + \frac{1}{z}\sum_{m=2}^{M} \frac{P_m\Big(\ln(\pm z)\Big)}{z^m} \qquad \text{[SuZexFM]} \tag{11.10}$$

$$P_m(z) = \sum_{n=0}^{m} c_{m,n}(-z)^n \qquad \text{[SuZexP]} \tag{11.11}$$

The coefficients c can be found by substitution of expansion (11.9) into the transfer equation (11.8). This can be done with the Mathematica code below:

```
zex[z_] = z Exp[z];
Foo[z_] = -1/z + a Log[z]/z^2
Soo = Series[Foo[z+1]-zex[Foo[z]], {z,Infinity,3}]
Eoo = Coefficient[Soo,1/z^3]
Ao = Extract[Solve[Eoo==0, a], 1]
F2o[z_] = ReplaceAll[Foo[z], Ao]
F20[z_] = F2o[z] + (a Log[z]^2 + b Log[z] + c)/z^3
S2o = Series[F20[z+1] - zex[F20[z]], {z,Infinity,4}]
S20 = ReplaceAll[S2o, Log[1/z] -> -L]
E2o = Coefficient[S20, 1/z^4]
E22 = Coefficient[E2o, L^2]
A1 = Extract[Extract[Solve[E22==0, a], 1], 1]
E2A = ReplaceAll[E2o, A1]
E21 = Coefficient[E2A, L]
B1 = Extract[Extract[Solve[E21==0, b], 1], 1]
E2B = ReplaceAll[E2A, B1]
C1 = Extract[Extract[Solve[E2B==0, c], 1], 1]
F3o[z_] = ReplaceAll[F20[z], {A1, B1, C1}]
F30[z_] = F3o[z]+(a Log[z]^3+b Log[z]^2+c Log[z]+d)/z^4
S3o = Series[F30[z+1] - zex[F30[z]], {z, Infinity, 5}]
S30 = ReplaceAll[S3o, Log[1/z] -> -L]
E3o = Coefficient[S30, 1/z^5]
E33 = Coefficient[E3o, L^3]
A3 = Extract[Extract[Solve[E33==0, a], 1], 1]
E3a = ReplaceAll[E3o, A3]
E32 = Coefficient[E3a, L^2]
B3 = Extract[Extract[Solve[E32==0, b], 1], 1]
E3b = ReplaceAll[E3a, B3]
E31 = Coefficient[E3b, L]
C3 = Extract[Extract[Solve[E31==0, c], 1], 1]
E3c = ReplaceAll[E3b, C3]
D3 = Extract[Extract[Solve[E3c == 0, d], 1], 1]
F4o[z_] = ReplaceAll[F30[z], {A3, B3, C3, D3}]
F40[z_] = F4o[z] +
      (a Log[z]^4+b Log[z]^3+c Log[z]^2+d Log[z]+e)/z^5
S4o = Series[F40[z+1] - zex[F40[z]], {z, Infinity, 6}]
```

Such a calculus leads to the following asymptotics:

$$
\begin{aligned}
F(z) = & -\frac{1}{z} + \frac{\frac{1}{2}\ell}{z^2} + \frac{\frac{-1}{4}\ell^2 + \frac{1}{4}\ell - \frac{1}{6}}{z^3} + \frac{\frac{1}{8}\ell^3 + \frac{-5}{16}\ell^2 + \frac{3}{8}\ell + \frac{-7}{48}}{z^4} \\
& + \frac{\frac{-1}{16}\ell^4 + \frac{13}{48}\ell^3 + \frac{-17}{32}\ell^2 + \frac{23}{48}\ell + \frac{-707}{4320}}{z^5} \\
& + \frac{\frac{1}{32}\ell^5 + \frac{-77}{384}\ell^4 + \frac{37}{64}\ell^3 + \frac{-83}{96}\ell^2 + \frac{1121}{1728}\ell + \frac{-1637}{8640}}{z^6} \\
& + \frac{\frac{-1}{64}\ell^6 + \frac{87}{640}\ell^5 + \frac{-205}{384}\ell^4 + \frac{443}{384}\ell^3 + \frac{-1619}{1152}\ell^2 + \frac{15427}{17280}\ell + \frac{-274133}{1209600}}{z^6} \\
& + O\left(\frac{\ell^7}{z^8}\right)
\end{aligned}
\tag{11.12}
$$

where $\ell = \ln(-z)$.

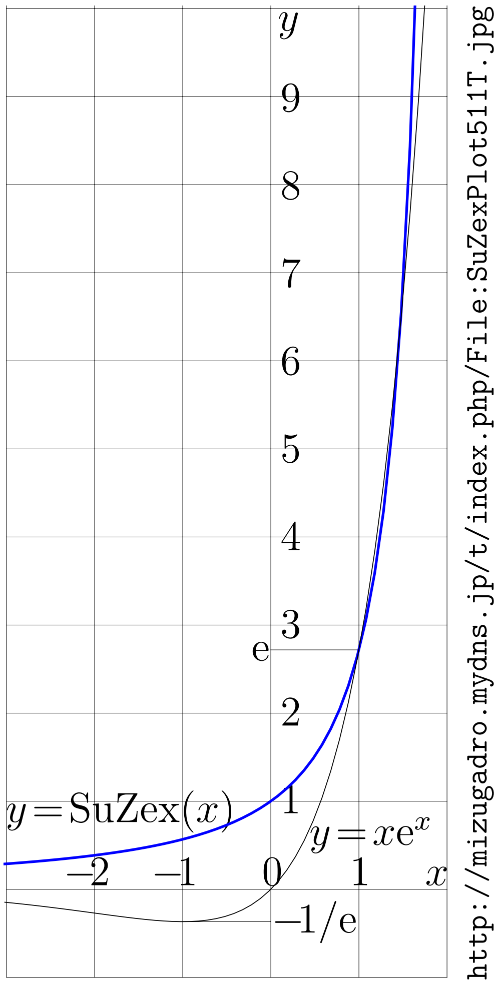


Figure 11.5: $y = \mathrm{SuZex}(x)$

for some fixed integer M, the superfunction F can be expressed through its Mth asymptotic as sollows:

$$F(z) = \lim_{n\to\infty} \mathrm{zex}^n\Big(F_M(z-n)\Big) \tag{11.13}$$

The resulting F does not depend on the number M of terms taken into account in the primary approximation. However, the rate of convergence of the limit for larger M is higher.

In order to simplify the comparison of different representations of the superfunction, it is convenient to define the misplaced function

$$\mathrm{SuZex}(z) = F(z_1+z) \tag{11.14}$$

where $z_1 \approx -1.1259817765745026$ is solution of equation $F(z_1) = 1$. This definition gives a way to evaluate the superfunction of zex. The complex double implementation in C++ is loaded as `http://mizugadro.mydns.jp/t/index.php/SuZex.cin`

For real values of argument, the explicit plot of function SuZex is shown in figure 11.5 with thick line. For comparison, the thin line shows the

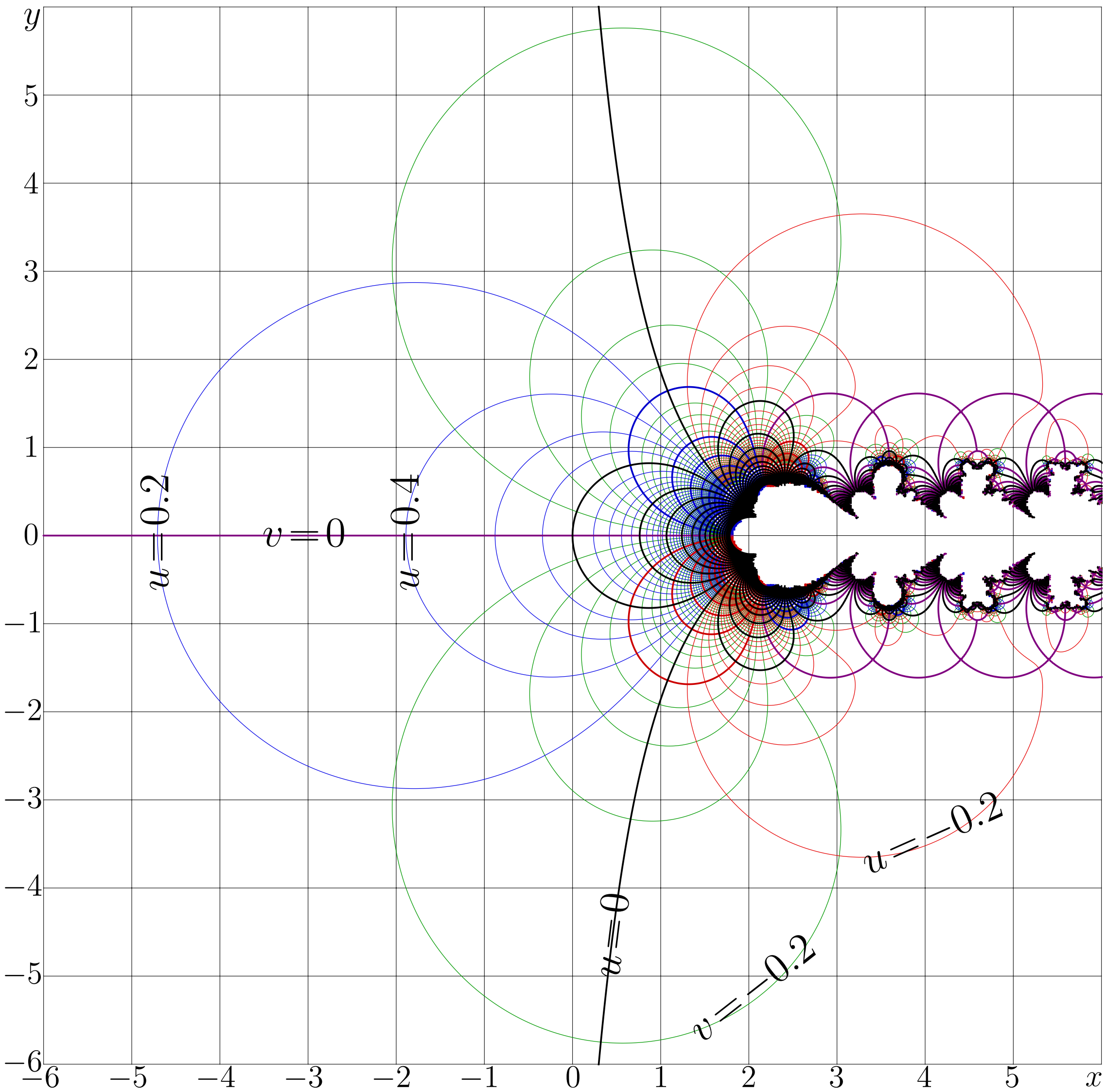


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Figure 11.6: $u+\mathrm{i}v = \mathrm{SuZex}(x+\mathrm{i}y)$

zex function, id est, $y = z\exp(z)$. These curves cross at point $(1, \mathrm{e})$ and in vicinity of point $(1.4, 6.2)$.

For real values of argument, SuZexp is positive monotonously growing function. At $-\infty$, it approaches zero, as the asymptotic representation (11.10) prescribes.
Then, the curve passes through point $(0, 1)$, and then - through point $(1, \mathrm{e})$. At this point it grows a little bit slower, than function zex, but soon overdoes the zex, showing very fast growth; this growth is faster, than growth of any exponential.

The same behaviour can be seen also at figure 11.6, that represents the complex map of function SuZex. zex is entire function, id est, holomorphic in the whole complex plane. The inverse function, shown in figure 11.6, has cut. This inverse function is considered in the next section.

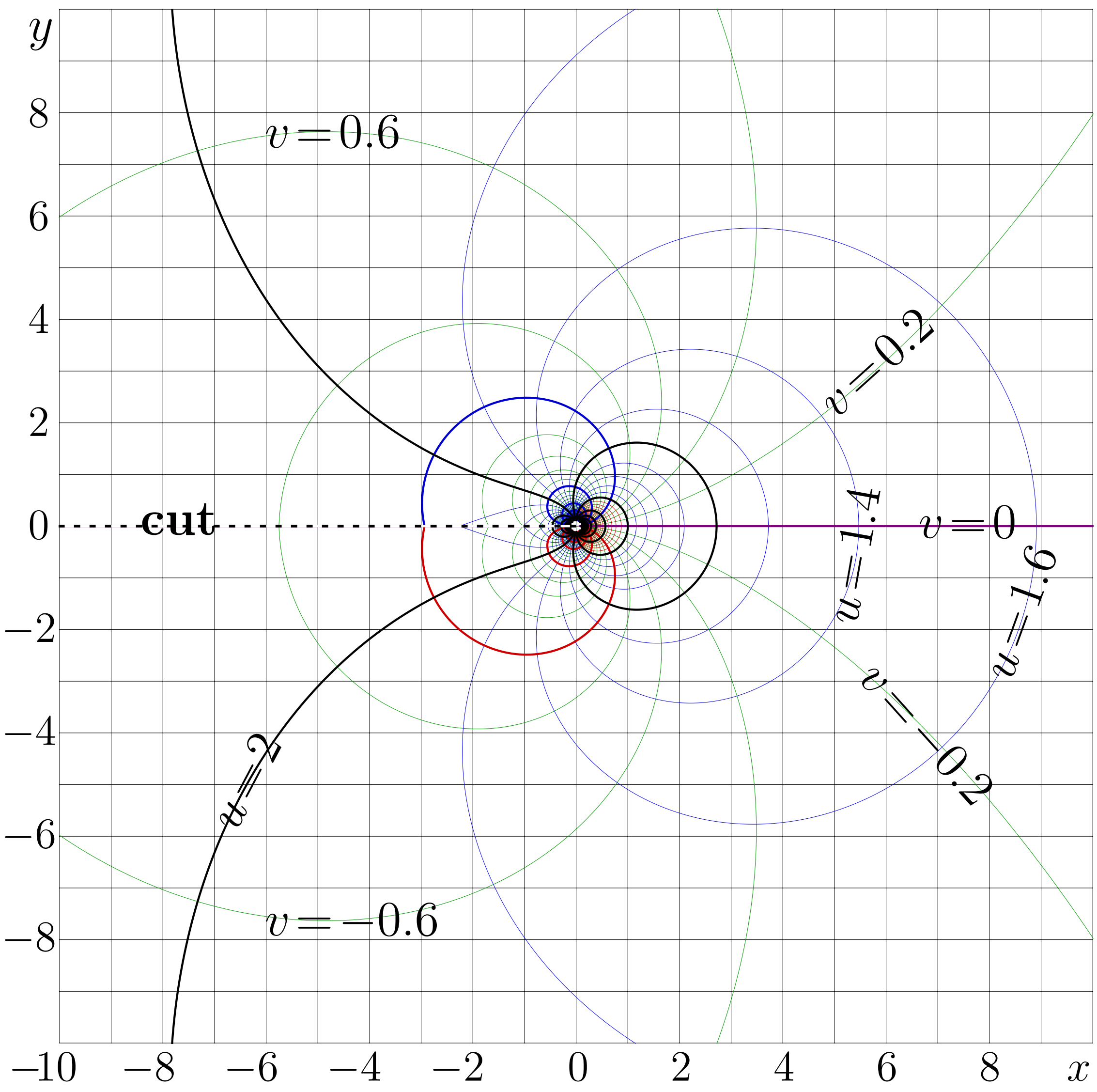

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Figure 11.7: $u+\mathrm{i}v = \mathrm{AuZex}(x+\mathrm{i}y)$ [auzexmap]

3 AuZex

Complex map of function AuZex = SuZex^{-1}, id est, abelfunction for the transferfunction zex, is shown in the right hand side of figure 11.7. This section describes properties of function AuZex.

The asymptotic expansion for the abelfunction for the transfer function AuZex can be obtained inverting the expansion of function SuZex, described in the previous section. However, one may consider as well the Abel equation for the abelfunction G :

$$G(\mathrm{zex}(z)) = G(z) + 1 \quad \text{[AuZexGeq]} \tag{11.15}$$

The asymptotics of solution $G = \mathrm{AuZex}$ can be expressed with

$$G(z) \approx \frac{-1}{z} + \frac{1}{2}\ln(z) + \sum_{n=0}^{N} b_n z^n + .. \tag{11.16}$$

Coefficients b_n for $n > 1$ can be found substituting this expansion into the Abel equation (11.15). This asymptotics provides the precise approximation for small values of z; at large values, the inverse of the Abel equation should be applied. For some fixed M, let

$$G_M(z) \approx \frac{-1}{z} + \frac{1}{2}\ln(z) + \sum_{m=0}^{M} b_n z^n \quad \text{[zexGas]} \tag{11.17}$$

and let

$$\begin{aligned} \mathrm{AuZex}(z) &= \lim_{n\to\infty} F_M\Big(\mathrm{zex}^{-n}(z)\Big) + n \\ &= \lim_{n\to\infty} F_M\Big(\mathrm{LambertW}^{\,n}(z)\Big) + n \end{aligned} \tag{11.18}$$

Coefficient C_0 is chosen in such a way, that $\mathrm{AuZex}(1) = 0$; then, the relation

$$\mathrm{SuZex}\Big(\mathrm{AuZex}(z)\Big) = z \quad \text{[SuAuZexz]} \tag{11.19}$$

holds in wide range of values of z, except the negative part of the real axis. With Mathematica software, the coefficients b of the asymptotic expansion (11.17) can be calculated by the code below.

```
zex[z_] = z Exp[z];
S[k_, L_] = Sum[a[k, m] L^m, {m, 0, k}]
F[K_, z_, L_] = Sum[S[k, L]/z^(k + 1), {k, 0, K}]
Series[zex[F[4,z,L]] - F[4, z+1,L+Log[1+1/z]], {z,Infinity,3}]
a[0,0] = -1;
Series[zex[F[4, z, L]] - F[4, z+1, L+Log[1+1/z]], {z,Infinity,3}]
a[1, 1] = 1/2; a[1, 0] = 0;
Simplify[Series[zex[F[5,z,L]] - F[5,z+1, L+Log[1+1/z]], {z,Infinity,4}]]

n = 2;
s[n]=Series[zex[F[n+3,z,L]]-F[n+3,z+1,L+Log[1+1/z]],{z,Infinity,n+2}];
For[k = 0, k<=n,k++,m=n-k;
a[n,m] = ReplaceAll[a[n, m],Sol[Coefficient[s[n] L,L*L^m] == 0, a[n,m]]];
Print[n, Space, k, Space, m, Space, a[n, m] ] ]

n = 3;
s[n]=Series[zex[F[n+3,z,L]]-F[n+3,z+1,L+Log[1+1/z]],{z,Infinity,n+2}];
For[k = 0, k<=n,k++,m=n-k;
a[n,m] = ReplaceAll[a[n, m],Sol[Coefficient[s[n] L,L*L^m] == 0, a[n,m]]];
Print[n, Space, k, Space, m, Space, a[n, m] ] ]
```

and so on for higher n. The first 9 coefficients are copypasted below:

n	b_n		approximation of b_n
1	$-1/6$	$\approx$	-0.1666666666666666667
2	$1/16$	$\approx$	0.0625
3	$-19/540$	$\approx$	-0.0351851851851851852
4	$1/48$	$\approx$	0.0208333333333333333
5	$-41/4200$	$\approx$	-0.0097619047619047619
6	$37/103680$	$\approx$	0.00035686728395061728
7	$18349/3175200$	$\approx$	0.005778848576467624
8	$-443/80640$	$\approx$	-0.005493551587301587
9	$55721/21555072$	$\approx$	-0.002585052835824441

(11.20)

The readers are invited to plot maps of ranges of validity of realtions

$$\mathrm{SuZex}(\mathrm{AuZex}(z)) = z \qquad (11.21)$$

$$\mathrm{AuZex}(\mathrm{SuZex}(z)) = z \qquad (11.22)$$

These maps can be considered as confirmation, verification of the deduction above.

4 Iterates of zex

With functions SuZex and AuZex, described in the previous sections, the iterates of function zex can be expressed as follows:

$$\mathrm{zex}^n(z) = \mathrm{SuZex}\Big(n + \mathrm{AuZex}(z)\Big) \quad \text{[zexn]} \qquad (11.23)$$

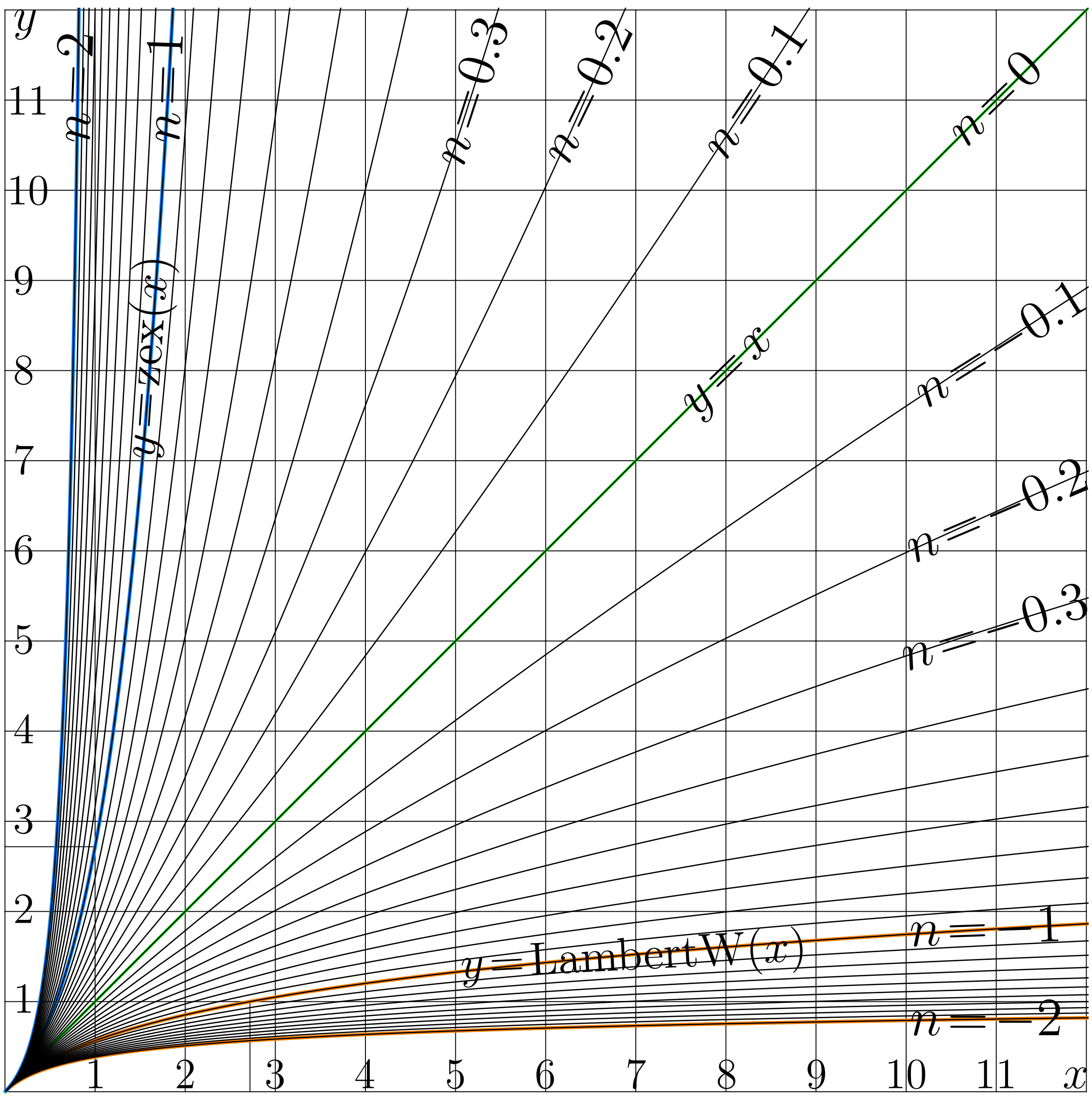
As usually, the number n of iterate has no need to be integer; it can be real or even complex.

Iterates $y = \mathrm{zex}^n(x)$ versus x for different n are shown in figure 11.8. The integer iterates are shown with thick lines.

The readers are invited to check the relation

$$\mathrm{zex}^n\Big(\mathrm{zex}^m(z)\Big) = \mathrm{zex}^{m+n}(z) \quad \text{[zexiteran]} \qquad (11.24)$$

and describe the range of validty in simple terms. At least, this relation should hold in some vicinity of the positive part of the real axis for values of the parameters.



`http://mizugadro.mydns.jp/t/index.php/File:ZexIteT.jpg`

Figure 11.8: $y\!=\!\mathrm{zex}^n(x)$ by (11.23) versus x for various n [zexiteplo]

Iterates of function zex are shown in figure 11.8. They look similar to iterates of other fastly growing functions with real fixed point. The iteration keeps the unity derivative in this point, so, all the curves in figure 11.8 approach the fixed point (id est, to the origin of coordinates) with unity derivative. Id est, with angle 45 degrees to the abscisa axis. This property takes place also for other transfer functions with unity derivative at the fixed point. One of such functions is considered in the next chapter.

I hope, the Readers can plot by themselves the complex maps of non-integer iterates of function zex. The codes supplied to the figures above have the complex double implementation of SuZex and AuZex.

Chapter 12

Sin, super sin and iterates of sin

Not all exotic iterations can be constructed with formulas of the previous sections. In the previous two sections, the transfer functions T with fixed points L are considered such that $T(L)=L$, $T'(L)=1$, $T''(L)>0$. That deduction fails, if $T''(L)=0$. One of examples of such a transfer function is considered here.

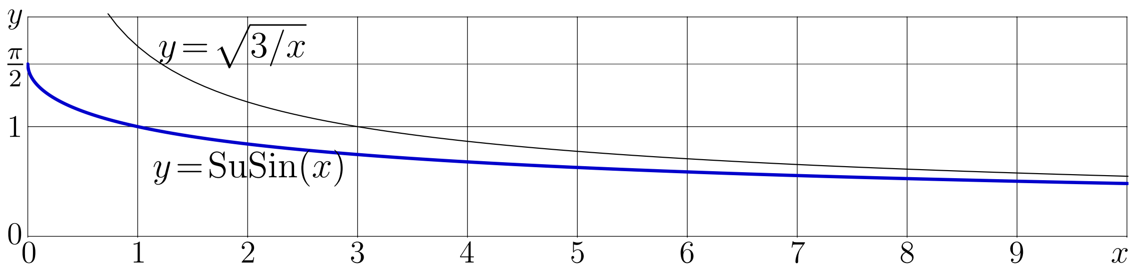
This chapter deals with transfer function $T=\sin$. This function is often used in various applications, so, I think, it deserves a special chapter.

Iterates of sin had been considered since century 19, but the rough approximations had been suggested only in 2011 [76, 77]. Then in 2014, the efficient approximation had been reported in the Far East Journal of Mathematical Science [91]. Below, I retell the key ideas of that publication.

I hope, the Reader can plot the complex maps of sin and arcsin. I recommend that Reader does this as an exercise. And also, the explicit plots of these functions. After to watch the pictures mentioned, one can understand the sense of superfunction of sin, I call it SuSin. SuSun is solution of the transfer equation, I repeat is once again,

$$\sin\big(\mathrm{SuSin}(z)\big)=\mathrm{SuSin}(z+1)$$

Explicit plot of function SuSin of real argument is shown in figure 12.1. In the next section, I describe the construction of this function.



http://mizugadro.mydns.jp/t/index.php/File:Susinplot.jpg

Figure 12.1: $y=\mathrm{SuSin}(x)$ by (12.8) and its asymptotics (12.3) [susinplot]

1 Super sin

Superfunction of sin, called SuSin, is shown in figure 12.1. It is solution of the transfer equation with sin as transfer function; I repeat it once again:

$$F(z+1) = \sin\Big(F(z)\Big) \quad \text{[Fz1sinFz]} \tag{12.1}$$

In this section, the special solution $F = \mathrm{SuSin}$ is constructed with the following properties:

$$\mathrm{SuSun}(0) = \pi/2 \quad \text{[susinp2]} \tag{12.2}$$

$$\mathrm{SuSun}(z) = \sqrt{\frac{3}{z}}\left(1 + O\left(\frac{\ln(z)}{z}\right)\right) \quad \text{[susin1o]} \tag{12.3}$$

The leasing term of the right hand side of (12.3) can be guessed replacing $F(z+1)$ to $F(z) + F'(z)$ in the left hand side of equation (12.1) and solving the resulting differential equation. However, there is certain residual at the substitution of such an approximation into (12.1); and this residual indicates the form of the the next term in expansion (12.3). Expression $\dfrac{\ln(z)}{z}$ appears as the effective small parameter of the expansion. The residual at the substitution of representation (12.3) into the transfer equation (12.1) helps to guess a form of the higher term of the expansion, and so on. After few such steps, I guess and verify the following form of the approximation for the superfunction F of sin:

$$F(z) = F_M(z) + O\left(\frac{\ln(z)^{M+1}}{z^{M+3/2}}\right) \quad \text{[susinasymf]} \tag{12.4}$$

where

$$F_M(z) = \sqrt{\frac{3}{z}}\left(1 - \frac{3}{10}\frac{\ln(z)}{z} + \sum_{m=2}^{M} P_m(\ln(z))\, z^{-m}\right) \quad \text{[susinFM]} \tag{12.5}$$

$$P_M(z) = \sum_{n=0}^{m} a_{m,n}(-z)^m \quad \text{[susina]} \tag{12.6}$$

and coefficients a are constants. These constants can be calculated with the Mathematica code below:

```
P[m_, L_] := Sum[a[m, n] L^n, {n, 0, m}]
F[z_] = Sqrt[3/z] ( 1 + Sum[P[m, Log[z]]/z^m, {m, 1, M}])
M = 9; a[1, 0] = 0;
F1x = F[1 + 1/x];
Ftx = Sin[F[1/x]];
s[1] = Series[(F1x - Ftx)/Sqrt[x], {x, 0, 2}]
t[1] = Extract[Solve [Coefficient[s[1], x^2] == 0, {a[1, 1]}], 1]
A[1, 1] = ReplaceAll[a[1, 1], t[1]]
su[1] = t[1]

m=2;
s[m]=Simplify[ReplaceAll[Series[(F1x-Ftx)/Sqrt[3 x],{x,0,m+1}], su[m-1]]]
t[m] = Simplify[Coefficient[ReplaceAll[s[m], Log[x] -> L], x^(m+1)]]
u[m] = Simplify[Collect[t[m], L]]
v[m] = Table[Coefficient[u[m] L, L^(n+1)] == 0, {n, 0, m}]
w[m] = Table[a[m, n], {n, 0, m}]
ad[m] = Extract[Solve[v[m], w[m]], 1]
su[m] = Join[su[m - 1], ad[m]]

m=3;
s[m]=Simplify[ReplaceAll[Series[(F1x-Ftx)/Sqrt[3 x],{x,0,m+1}],su[m-1]]]
t[m] = Simplify[Coefficient[ReplaceAll[s[m], Log[x] -> L], x^(m+1)]]
u[m] = Simplify[Collect[t[m], L]]
v[m] = Table[Coefficient[u[m] L, L^(n+1)] == 0, {n, 0, m}]
w[m] = Table[a[m, n], {n, 0, m}]
ad[m] = Extract[Solve[v[m], w[m]], 1]
su[m] = Join[su[m - 1], ad[m]]
. . .
```

and so on. The resulting coefficients are shown in table 12.1.

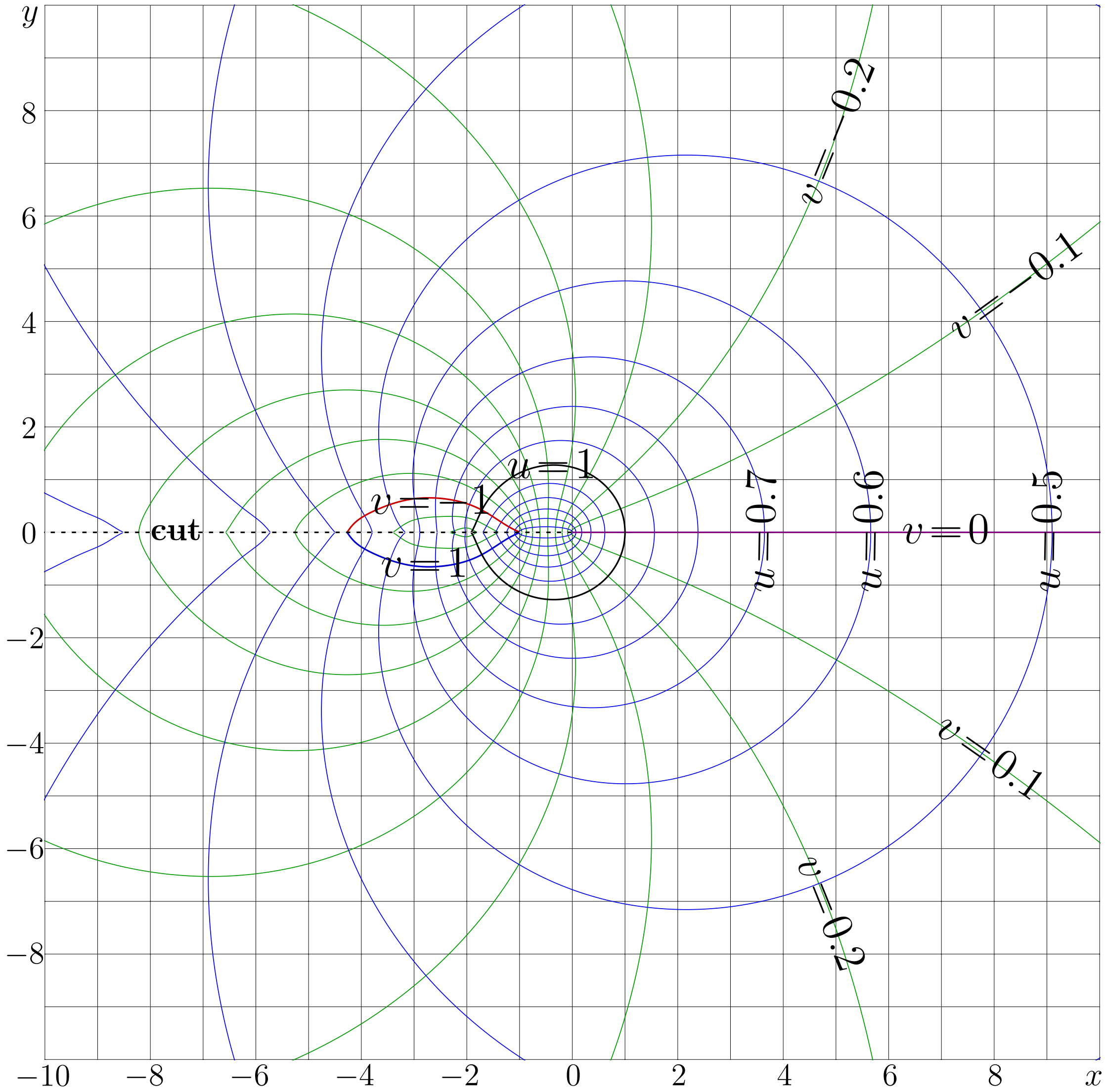
For some positive integer M, define function F with

$$F = \lim_{k\to\infty} \arcsin^k(F_M(z+k)) \quad \text{[SuSinF]} \tag{12.7}$$

The resulting F does not depend on the number M of terms taken into account in the expansion (12.5). However, the larger, M, the faster

Table 12.1: Table of coefficients a in equation (12.6)

$a_{1,0}$	$\frac{3}{10}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	$a_{1,5}$	$a_{1,6}$	$a_{1,7}$
$\frac{79}{700}$	$\frac{9}{50}$	$\frac{27}{200}$	$a_{2,3}$	$a_{2,4}$	$a_{2,5}$	$a_{2,6}$	$a_{2,7}$
$\frac{411}{3500}$	$\frac{1941}{7000}$	$\frac{27}{125}$	$\frac{27}{400}$	$a_{3,4}$	$a_{3,5}$	$a_{3,6}$	$a_{3,7}$
$\frac{1606257}{10780000}$	$\frac{7227}{17500}$	$\frac{1683}{4000}$	$\frac{1917}{10000}$	$\frac{567}{16000}$	$a_{4,5}$	$a_{4,6}$	$a_{4,7}$
$\frac{140345627}{700700000}$	$\frac{70079931}{107800000}$	$\frac{566973}{700000}$	$\frac{98739}{200000}$	$\frac{7533}{50000}$	$\frac{15309}{800000}$	$a_{5,6}$	$a_{5,7}$
$\frac{137678711441}{490490000000}$	$\frac{7364523}{7007000}$	$\frac{305491257}{196000000}$	$\frac{4155111}{3500000}$	$\frac{796311}{1600000}$	$\frac{2218347}{20000000}$	$\frac{168399}{16000000}$	$a_{6,7}$
$\frac{25317035192599}{62537475000000}$	$\frac{8462569406199}{4904900000000}$	$\frac{32174780481}{10780000000}$	$\frac{5367503637}{1960000000}$	$\frac{407711313}{280000000}$	$\frac{181900809}{400000000}$	$\frac{1960281}{25000000}$	$\frac{938223}{160000000}$



http://mizugadro.mydns.jp/t/index.php/File:Susinmap.jpg

Figure 12.2: $u+\mathrm{i}v = \mathrm{SuSin}(x+\mathrm{i}y)$ by (12.8)

is convergence of the limit in (12.7). This gives efficient algorithm for evaluation of superfunction F for the transfer function sin.

For superfunction, declared in the beginning of the section, id est, satisfying equation (12.2), define

$$\mathrm{SuSin}(z) = F(z+x_1) \tag{12.8}$$

where $x_1 \approx 1.4304553465288$ is solution of equation

$$F(1+x_1) = 1 \tag{12.9}$$

Note that the required value at zero is achieved, because $\mathrm{SuSin}(0) = \arcsin(\mathrm{SuSin}(1)) = \arcsin(1) = \pi/2$.

Function SuSin is shown as explicit plot and as complex map in figures 12.1 and 12.2. Function SuSin has sqrt-type singularity as zero, and it has cut line along the negative part of the real axis. In the rest of the complex plane, SuSin is holomorphic. At infinity, SuSin slowly decays to zero, according to its asymptotic (12.3). This asymptotic is shown in figure 12.1 with thin line.

In this Book, SuSin is first example of superfunction, for which the transfer function shows growth slower than linear. In this case, for positive values of the argument (id est, larger than the fixed point), the superfunction is monotonically decreasing, as it is seen in Figure 12.1. One could expect the inverse function (Abel sin) to decrease, at least for some moderate positive values of the argument. This inverse function is considered in the next section.

2 AuSin

Here, the inverse function of super sin is called AuSin; its explicit plot is shown in figure (12.3). In this chapter, I describe evaluation of this function.

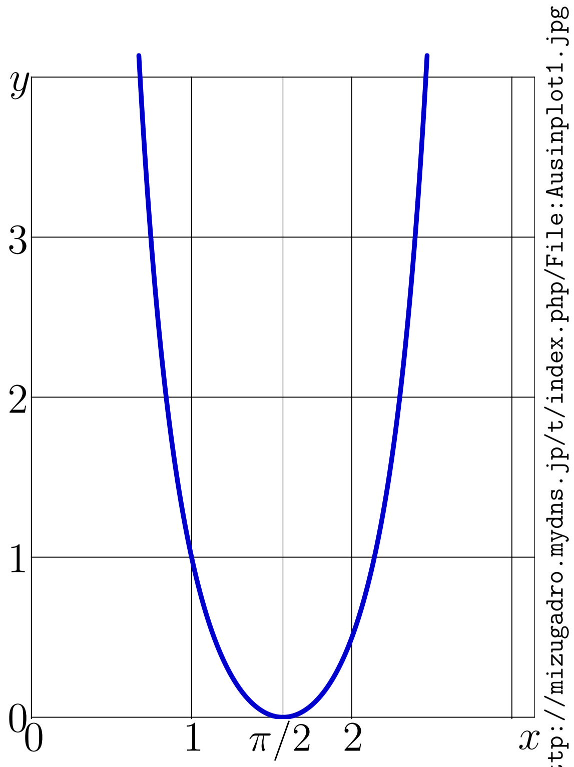


Figure 12.3: $y = \mathrm{AuSin}(x)$

For the sinusoidal transfer function, for superfunction F, the Abel function $G = F^{-1}$ satisfies the Abel equation

$$G(\sin(x)) = G(z)+1 \quad (12.10)$$

From the properties of SuSin, the properties of the inverse function can be guessed. In particular, for large values of the argument, the asymptotic below should hold:

$$G(z) = \frac{2}{z^2} + O(\ln(z)) \quad \text{[AbelsinGas]} \qquad (12.11)$$

Abel sin can be constructed in analogy with super sin. The solution G of the Abel equation (12.10) with the asymptotic representation (12.11) can be constructed, inverting function F by (12.7), id est, $G = F^{-1}$. Then, the constant x_1 should be added,

$$\mathrm{AuSin}(z) = G(z) + x_1 \qquad (12.12)$$

in order to satisfy relation

$$\mathrm{AuSin}(\pi/2) = 0 \quad \text{[ausinp2]} \tag{12.13}$$

First, construct the approximation of function G; let

$$G_M(z)=\frac{3}{z^2}+\frac{5}{6}\ln(z)+\sum_{m=1}^{M} c_m z^{2m} \tag{12.14}$$

Subsitution of $g(z) = G_M + O(z^{2M+2})$ into the Abel equation (12.10) gives the coefficients c. In particular,

$$c_1 = \frac{79}{1050} \tag{12.15}$$

$$c_2 = \frac{29}{2625} \tag{12.16}$$

$$c_3 = \frac{91543}{36382500} \tag{12.17}$$

$$c_4 = \frac{18222899}{28378350000} \tag{12.18}$$

Then, for some fixed M, function G can be expressed as limit

$$G(z)=\lim_{k\to\infty} G_M(\sin^k(z)) - k \quad \text{[singlim]} \tag{12.19}$$

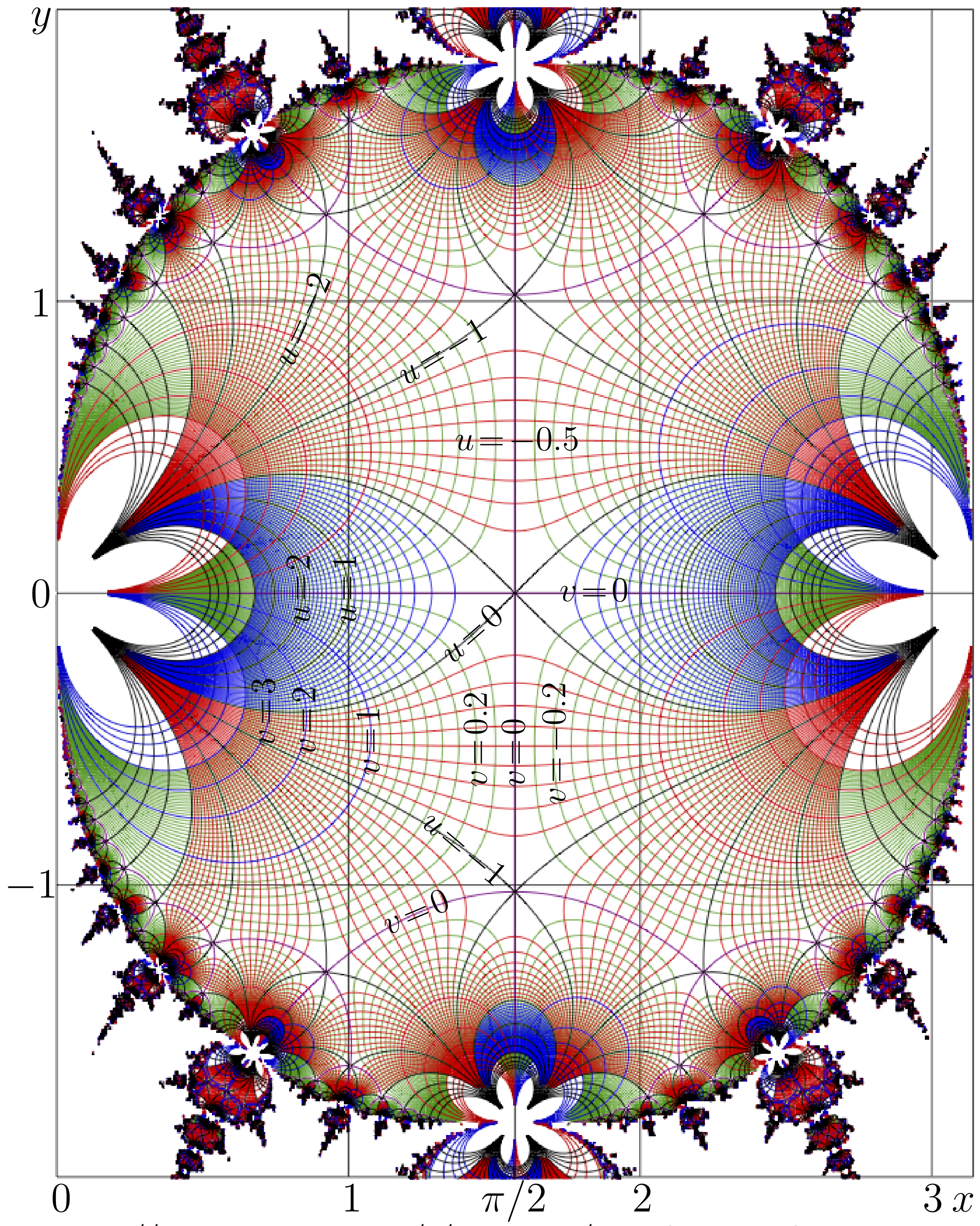
and $\mathrm{AuSin}=\mathrm{SuSin}^{-1}$ appears as

$$\mathrm{AuSin}(z) = G(z) - G(\pi/2) \quad \text{[AuSinDe]} \tag{12.20}$$

Term $G(\pi/2)\approx 2.089622719729524$ in equation (12.20) provides condition (12.13).

Complex map of abelsinus AuSin by (12.20) is shown in fgure 12.4. As for other real-holomorphic functions, this map is symmetric with respect to reflection from the real axis, id est, with respect to flipping upside-down. In addition, the map is symmetric with repeat to reflections from the axis $x=\pi/2$; the first evaluation of function sin in formula (12.19) provides this symmetry.

For the central part of figure 12.4, the limit (12.19) converges and defines the holomorphic function in a pretty regular way. At lateral parts of figure 12.4, the lines of level of the real part and those of the imaginary part form the fractal-like structures. There, the AuSin cannot be considered as inverse function of SuSin. Actually, the range of validity of the inverse function is even narrower.



http://mizugadro.mydns.jp/t/index.php/File:Ausinmap52r5.jpg

Figure 12.4: $u+\mathrm{i}v = \mathrm{AuSin}(x+\mathrm{i}y)$ by formula (12.20) [ausinmap]

Range of validity of relation

$$\mathrm{SuSin}\Big(\mathrm{AuSin}(z)\Big) = z \quad \text{[SuSinA]} \tag{12.21}$$

is shown in figure 12.5, this range is shaded with rectangular grid.

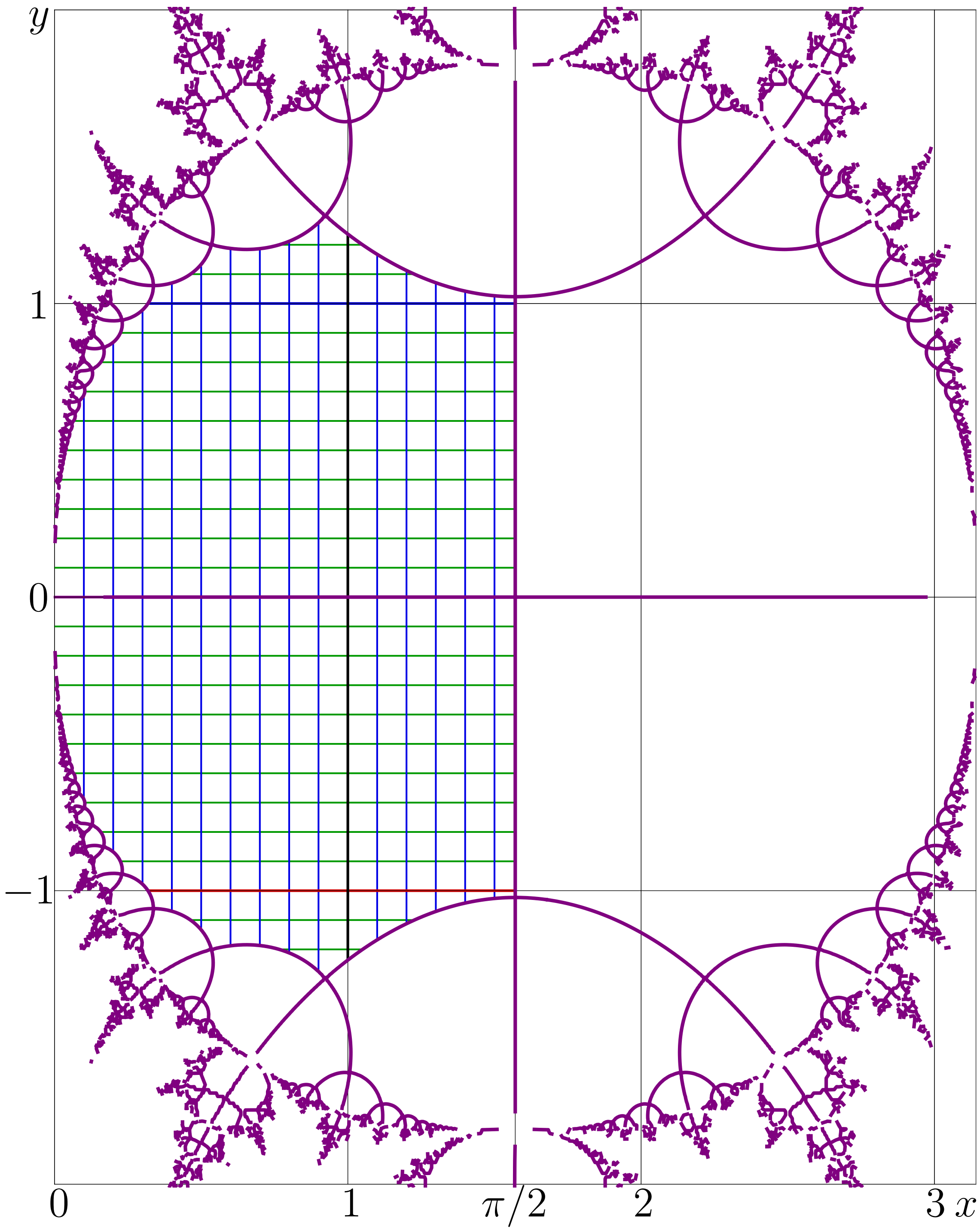


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Figure 12.5: $u+\mathrm{i}v=h(x+\mathrm{i}y)$ and lines $\Im(\mathrm{AuSin}(x+\mathrm{i}y)=0$

Technically, the shading in figure 12.5 is built up as the complex map of function

$$h(z)=\mathrm{SuSin}\Big(\mathrm{AuSin}(z)\Big) \tag{12.22}$$

In addition, in figure 12.5, lines of level $\Im(\mathrm{AuSin}(x+\mathrm{i}y))$ are shown; they are borrowed from figure 12.4. These lines bound the range of validity of relation $h(z)=z$.

The readers are cordially invited to download the generator of the figure and modify it, to plot the map of the agreement function

$$A(z) = -\lg\left(\frac{|h(z)-z|}{|h(z)|+|z|}\right) \qquad \text{[aausinsu]} \qquad (12.23)$$

and to verify, that in the shaded part, relation $\mathrm{AuSin} = \mathrm{SuSin}^{-1}$ holds with at least 14 decimal digits. At the right hans side of the figure 12.5, relation $\mathrm{AuSin} = \mathrm{SuSin}^{-1}$ is not valid; $\mathrm{AuSin}(x+\mathrm{i}y)$ is symmetric with respect to line $x=\pi/2$; so, it cannot distinguish values for the right hand side in figure 12.3 from those at the left hand side.

For the efficient (fast and precise) numerical implementation of AuSin, various expansions can be used. The Taylor expansion at $\pi/2$ can be written as follows:

$$\mathrm{AuSin}\left(\frac{\pi}{2}+t\right) = \sum_{n=1}^{\infty} b_n t^{2n} \qquad (12.24)$$

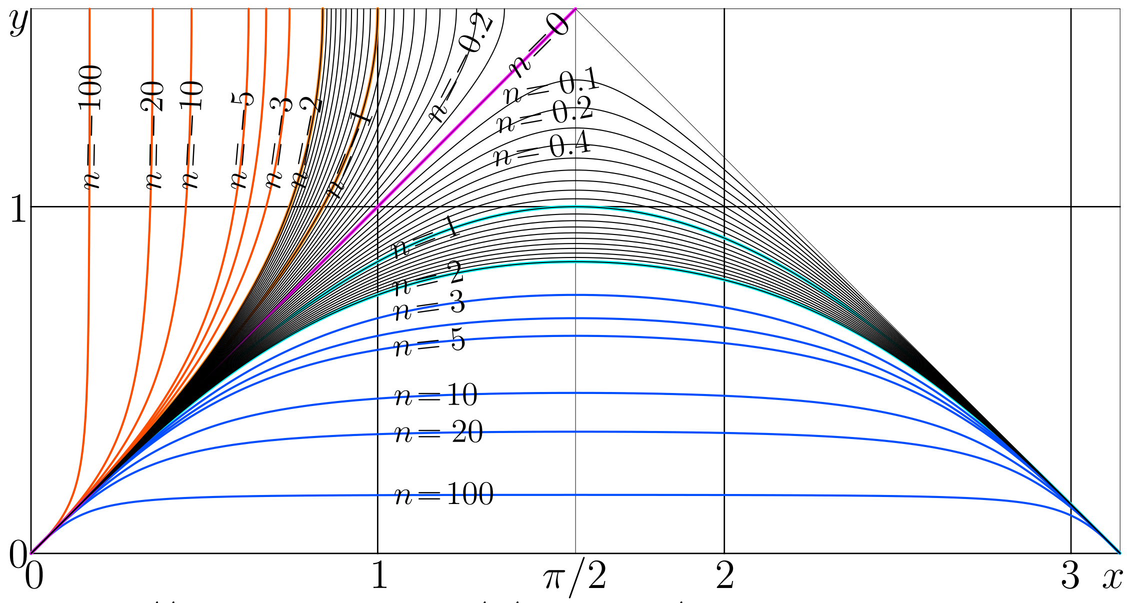
Coefficients b in the expansion (12.24) are evaluated with the Cauchy integral, using the representation of AuSin through function G by (12.14), (12.19), (12.20). Approximations of first nine of these coefficients are shown in Table 12.2.

Table 12.2:

n	b_n
1	2.29163807440958
2	1.96043852439688
3	1.07862851256147
4	0.59622997993395
5	0.28333997139829
6	0.14193261194548
7	0.06423734271234
8	0.03026687705508
9	0.01351721250427

Series in the expansion (12.24) converges at $|t|<\pi/2$. It is sufficient to take few terms of this expansion in order to reproduce the most of map in figure 12.4. Readers are invited to plot this map (or, at least, to look for it in the Appendix, figure 22.2 at page 300).

Taking into account some tens of coefficients in series (12.24), the numerical approximations of AuSin provides the precision at least not worse, than the precision of the original representation through the asymptotic formulas (12.14), (12.19), (12.20). However the original representation is still necessary to calculate the coefficients of the secondary expansions, and, in particular, those of the Taylor expansion (12.24). Optimisation of such representations may have sense before to include them to some software as built-in functions, while the each microsecond at the evaluation is important for the resulting efficiency at the multiple evaluations. However, even with the primitive approximations described above, the functions are evaluated with approximately 14 decimal digits and allow to plot complex maps in real time. This indicates the good efficiency of the representations suggested.



http://mizugadro.mydns.jp/t/index.php/File:Sinplo2t100.jpg

Figure 12.6: $y = \sin^n(x)$ by (12.25) versus x for various n [sinite]

3 Iterated sin

With functions SuSin and AuSin $= \mathrm{SuSin}^{-1}$, defined above, the iterates of sin can be expressed as follows:

$$\sin^n(z) = \mathrm{SuSin}\Big(n + \mathrm{AuSin}(z)\Big) \quad \text{[sinn]} \qquad (12.25)$$

This formula looks pretty similar to representations of iterates of any other function with determined superfunction and the corresponding Abel function. In this representation, number n of iterates has no need to be integer; it can be real and even complex. For real values of number n of iterates, $\sin^n(x)$ is plotted versus x in figure 12.6.

For positive number n of iterate, graphics of $y = \sin^n(x)$ are symmetric with respect to line $x = \pi/2$. The larger is n, the closer the curve approaches the abscissa axis.

For negative n, the graphic reaches the branch point at $y = \pi/2$ and cannont be continued be continued above, as the iterates get complex values. A usually, the 0th iterate corresponds to the identity function, $\sin^0(x) = x$, and this relation holds while $x < \pi/2$.

Through the iterates (12.25) of sin, the SuSin can be expressed as follows:

$$\mathrm{SuSin}(z) = \sin^z(\pi/2) \quad \text{[susinzsin]} \qquad (12.26)$$

From the point of view of computation, representation (12.26) does not have much sense. Anyway, for the evaluation of the right hand side of (12.26), the approximations of SuSin should be used. However, such a representation may have sense at the building-in of superfunctions (and non-integer iterates) into the programming languages.

4 Application

In this section, I discuss applications of multiple iterates of sin. However, I assume, that the Readers can freely use SuSin and AuSin, described above, to plot new graphics. I load the complex double implementations of these functions as
`http://mizugadro.mydns.jp/t/index.php/Susin.cin` and
`http://mizugadro.mydns.jp/t/index.php/Ausin.cin` Perhaps, many beautiful figures can be plotted playing with SuSin and AuSin.

For the physical applications, the real number of iterate is simpler to interpret, than the complex iterate. Keeping in mind the application, I suggest the example of parameterisation of the shape of the sled runner with high iterate of sin. This example is shown in figure 12.7, that shows

$$y = \sin^n(\pi/2) - \sin^n(x) \tag{12.27}$$

plotted over the photo of the sled (with a boy in it) is taken from Wikimedia Common [9].
The curve, that overlaps wight he sled runner, is the 100th iterate of sin; the number of iterate $n = 100$ is the only adjusted parameter used for the fitting. However, the photo is shifted, scaled and rotated, in order to have the tip at $x = 0$ and the last support of the sled runner at $x = \pi/2$, $y = 0$; in this point the sled runner is horizontal, this is provided by the slight rotation of the photo.

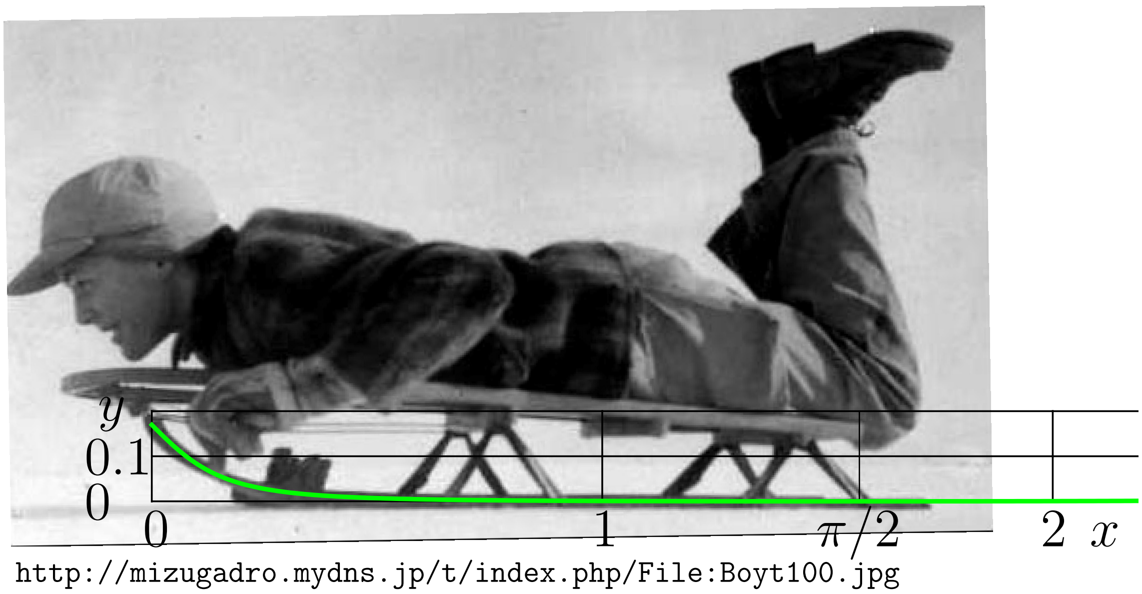


Figure 12.7: Boy at the sledge and $y = \sin^n(\pi/2) - \sin^n(x)$ for $n = 100$

Chapter 13

Nemtsov function

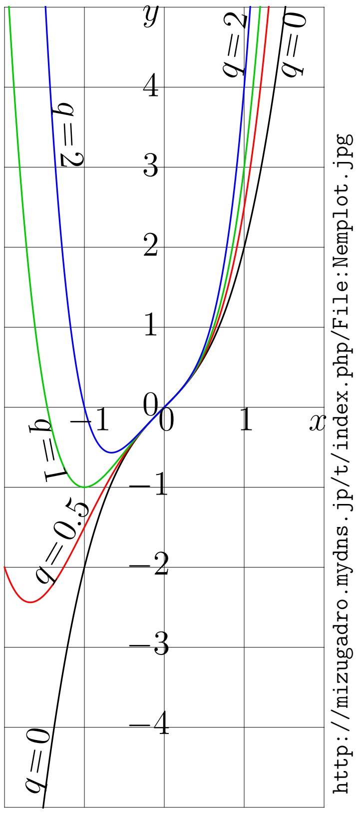


Figure 13.1: $y=\mathrm{Nem}_q(x)$ for various q

This chapter represents the last (for this book) example of exotic iteration, and deal with the specific polynomial

$$T(z) = \mathrm{Nem}_q(z) = z + z^3 + qz^4 \qquad (13.1)$$

I assume, that q is positive real parameter. For various values of q, the explicit plot of this function is shown in figure 13.1. This is simple, but still non-trivial case of the transfer function T, for which

$$T(L)=L~, \quad T'(L)=1~, \quad T''(L)=0~, \quad T'''(L)>0~, \quad T''''(L)>0 \qquad (13.2)$$

Note, that for $T=\sin$, considered in the previous section, $L=0$ and $T''''(L)=0$; function sin is antisymmetric $T(-z)=-T(z)$, and in this sense, simpler, than $T=\mathrm{Nem}_q$.

Historically, consideration of the Nemtsov function is one of the last attempts to write an elementary function, for which I would not be able construct superfunction, abelfunction and the non-integer iterates. I found, that I need the special name for function by (13.1). That happened 2015.01.28, in the same day, as the murder of Boris Nemtsov had been reported [1]; so I pick up the first three letters of his last name to create the name the function.

In this Book, I do not follow the historical timeline of events, so, I put chapter about the Nemtsov function here, in order to have the exotic iterations in one bunch. For real argument, graphics of the Nemtsov function for various values of parameter q are shown in figure 13.1. Complex maps of function Nem_q and its inverse function $\mathrm{ArqNem}_q=\mathrm{Nem}_q^{-1}$ are shown in figure 13.2 for $q=0$, $q=1$ and $q=2$. Function ArqNem is described in the following section.

[1] http://mizugadro.mydns.jp/t/index.php/Putin_killed_Nemtsov

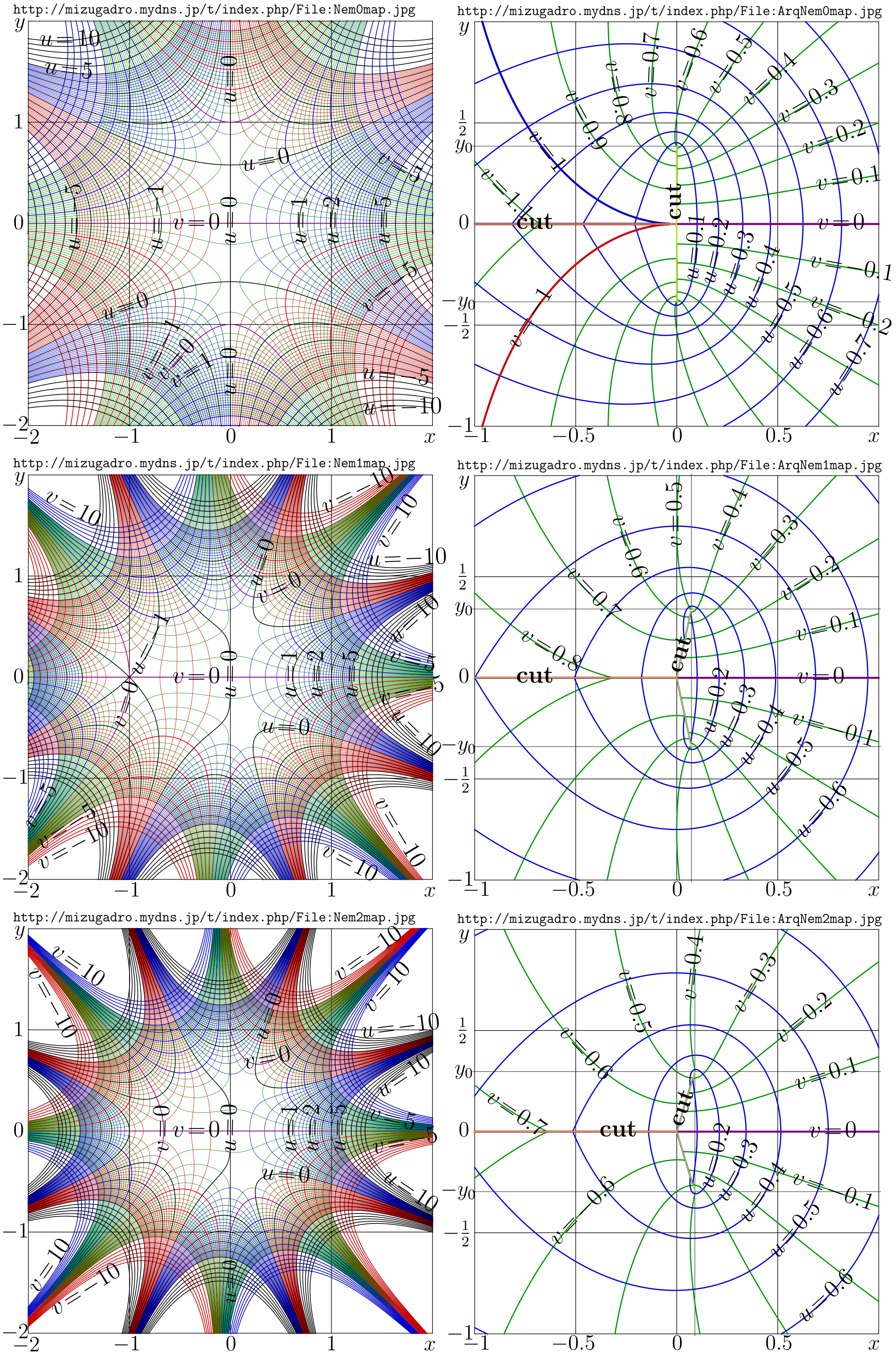


Figure 13.2: Maps $u+\mathrm{i}v = \mathrm{Nem}_q(x+\mathrm{i}y)$ for $q=0,1,2$, left column, and maps $u+\mathrm{i}v = \mathrm{ArqNem}_q(x+\mathrm{i}y)$, right column, for the same q

http://mizugadro.mydns.jp/t/index.php/File:Arqnemplot.jpg

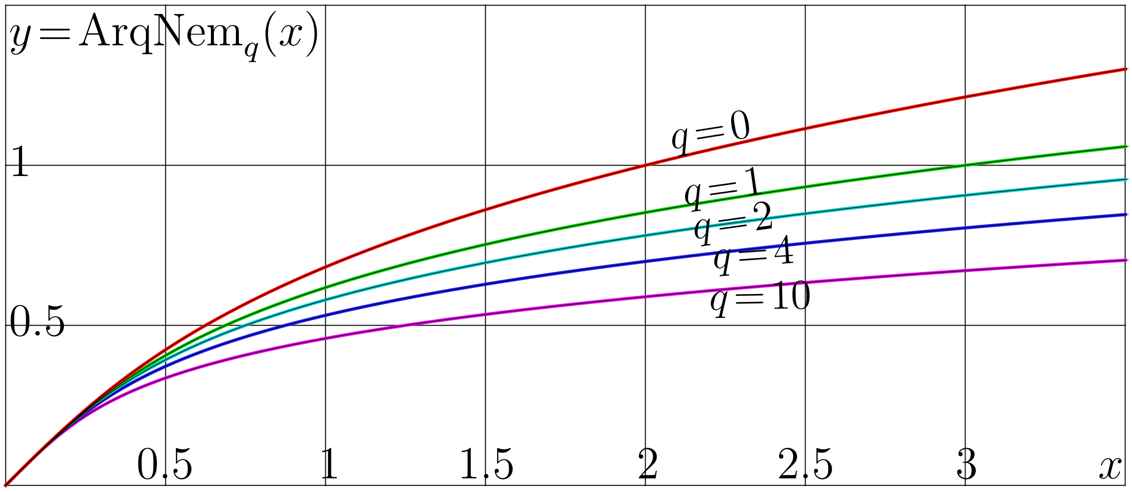


Figure 13.3: $y = \mathrm{ArqNem}_q(x)$ for $q = 0, 1, 2$ [arqnemplo]

1 ArqNem

For the efficient evaluation of iterates of a function, we need both, its supedfunction and the abelfunction. For implementations of these functions, we need both the transfer function $T = \mathrm{Nem}_q$ and its inverse $\mathrm{ArqNem}_q = T^{-1}$. For real values of the argument, plot $y = \mathrm{ArqNem}_q(x)$ for $q = 0$, $q = 1$, $q = 2$, $q = 4$, $q = 10$ is shown in figure 13.3. Inversion of the Nemtsov function happens to be not trivial, and its description deserves the special section.

The Nemtsov function Nem_q by (13.1) is the 4th order polynomial; for $q > 0$, equation

$$\mathrm{Nem}_q(x) = z \quad \text{[Nemqxz]} \tag{13.3}$$

at given z has four solutions x. Any of original solutions (suggested, for example, by Mathematica routine **Solve**) happen to be ugly at the complex map, and even worse at evaluation of abelfunction for the Nemtsov function. The reader can plot the complex maps for the four roots of equation (13.3) and see the root of my discontent with them. But these solutions can be used to construct the inverse function shown the right hand column of figure 13.2. I have implemented several combinations of the "primary" solutions, giving special name to each resulting inverse function: ArcNem_q, ArkNem_q, ArqNem_q. The third of them, ArqNem_q, happened to be satisfactory; so, I use it as Nem^{-1}. The cut lines of function ArqNem_q go from $-\infty$ to zero, and then to each of the branch points, seen at the maps in the right column of Figure 13.2.

For the positive values of the argument, there is no need to make any difference between three functions mentioned above; for $x > 0$, relations

$\mathrm{ArcNem}_q(x) = \mathrm{ArkNem}_q(x) = \mathrm{ArqNem}_q(x)$ hold. Graphics of these functions are shown in figure 13.3.

For handling of cuts of function ArqNem_q, the branch points should be calculated. These branch points are shown in figure 13.4. It refers to function NemBran, that expresses the upper branch point of function ArqNem_q as function of q. Below I describe, how to evaluate this function.

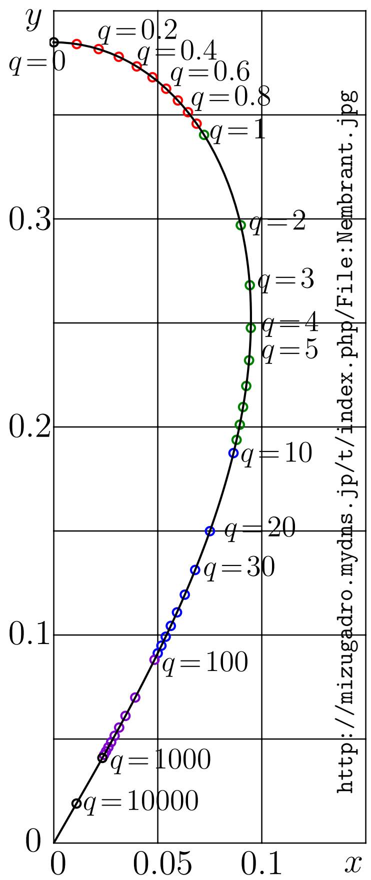


Figure 13.4: Parametric plot: $x+\mathrm{i}y = \mathrm{NemBran}(q)$

Assume, some q is given. We need to find solutions of equation $\mathrm{Nem}_q{}'(z) = 0$. This solution can be expressed in Mathematica language with code

```
T[z_]=z+z^3+q z^4
s = Solve[T'[z]==0,z]
ReplaceAll[z,Extract[s,1]]
```

The output indicates, how to program function NemBran, shown in figure 13.4:

```
z_type nembra(DB q){ z_type Q,v,V;
  Q=q*q;
  v=-1.-8.*Q+4.*sqrt(Q+4.*Q*Q);
  V=pow(v,1./3.);
   return (.25/q)*(-1.+1./V+V); }

z_type NemBran(DB q){ z_type z,zz=z*z;
  z=nembra(q); return z*(1.+zz*(1.+q*z)); }
```

Here DB denotes <double> and z_type denotes complex<double> ; NemBran appears as combination of functions Nem and nembra.

Assume, some $q>0$ is given; let $x_0+\mathrm{i}y_0 = \mathrm{NemBran}(q)$. For function ArqNem_q, I draw the cut lines form $-\infty$ to zero, and from zero to point (x_0, y_0) and to point $(x_0, -y_0)$. As the cut lines are specified, it is easy to program evaluation of ArqNem_q, picking up the corresponding branch of the solution. The C++ code is shown in Table 13.1. One can extract the code also from the URL marked in figure 13.4. The readers are invited to check numerically relations

$$\mathrm{Nem}_q\big(\mathrm{ArqNem}_q(z)\big) = z \tag{13.4}$$
$$\mathrm{ArcNem}_q\big(\mathrm{Nem}_q(z)\big) = z \tag{13.5}$$

and investigate the range of validity of each of these equation.

Table 13.1: C++ implementation of function ArqNem_q. Values of q and corresponding x_0, y_0 should be already calculated and stored in global variables Q, tr, ti

```
z_type arnemU(z_type z){ DB q=Q; DB q2=q*q; DB q3=q2*q;
z_type a=q-z; z_type b=1.+4.*q*z; z_type r=81.*(a*a)+12.*(b*b*b);
z_type R=-I*sqrt(-r);
z_type s=27.*a + 3.*R; z_type S=pow(s,1./3.);
z_type B=(0.26456684199469993*S)/q - (1.2599210498948732*b)/(q*S);
z_type h=0.25/q2 + B;
z_type H=I*sqrt(-h);
z_type g=0.5/q2  - B + (.25+2.*q2)/(q3*H);
z_type G=I*sqrt(-g);
return - 0.25/q - 0.5*H + 0.5*G ;}

z_type arnemD(z_type z){ DB q=Q; DB q2=q*q; DB q3=q2*q;
z_type a=q-z; z_type b=1.+4.*q*z; z_type r=81.*(a*a)+12.*(b*b*b);
z_type R=I*sqrt(-r);
z_type s=27.*a + 3.*R;  z_type S=pow(s,1./3.);
z_type B=(0.26456684199469993*S)/q - (1.2599210498948732*b)/(q*S);
z_type h=0.25/q2 + B;
z_type H=-I*sqrt(-h);
z_type g=0.5/q2  - B + (.25+2.*q2)/(q3*H);
z_type G=-I*sqrt(-g);
return - 0.25/q - 0.5*H + 0.5*G ;}

z_type arnemR(z_type z){ DB q=Q; DB q2=q*q; DB q3=q2*q;
z_type a=q-z; z_type b=1.+4.*q*z; z_type r=81.*(a*a)+12.*(b*b*b);
z_type R=sqrt(r); z_type s=27.*a + 3.*R;
z_type S=pow(s,1./3.);
z_type B=(0.26456684199469993*S)/q - (1.2599210498948732*b)/(q*S);
z_type h=0.25/q2 + B;
z_type H=sqrt(h);
z_type g=0.5/q2  - B + (.25+2.*q2)/(q3*H);
z_type G=sqrt(g);
return - 0.25/q - 0.5*H + 0.5*G ;}

z_type arnemL(z_type z){ DB q=Q; DB q2=q*q; DB q3=q2*q;
z_type a=q-z; z_type b=1.+4.*q*z; z_type r=81.*(a*a)+12.*(b*b*b);
z_type R=-sqrt(r);
z_type s=27.*a + 3.*R; z_type S=pow(s,1./3.);
z_type B=(0.26456684199469993*S)/q - (1.2599210498948732*b)/(q*S);
z_type h=0.25/q2 + B;
z_type H=sqrt(h);
z_type g=0.5/q2  - B + (.25+2.*q2)/(q3*H);
z_type G=sqrt(g);
return - 0.25/q - 0.5*H + 0.5*G ;}

z_type arqnem(z_type z){ DB x,y; x=Re(z);y=Im(z);
if( y>ti || (x<0 && y>=0)) return arnemU(z);
//if(y<0) return conj(arnemU(conj(z)));
if(y<-ti || (x<0 && y<=0)) return arnemD(z);
if(x*ti>fabs(y)*tr) return arnemR(z);
return arnemL(z);}
```

2 SuNem

Functions Nem_q and $\mathrm{ArqNem}_q = \mathrm{Nem}_q^{-1}$ are defined, implemented, tested and described above; they can be used for construction of superfunction SuNem_q, abelfunction $\mathrm{AuNem}_q\!=\!\mathrm{SuNem}_q^{-1}$ and corresponding non-integer iterates of the Nemtsov function. I begin with function SuNem_q. It is shown in figure 13.5 dor vvarious values of q. In this section, I describe, how is it constructed.

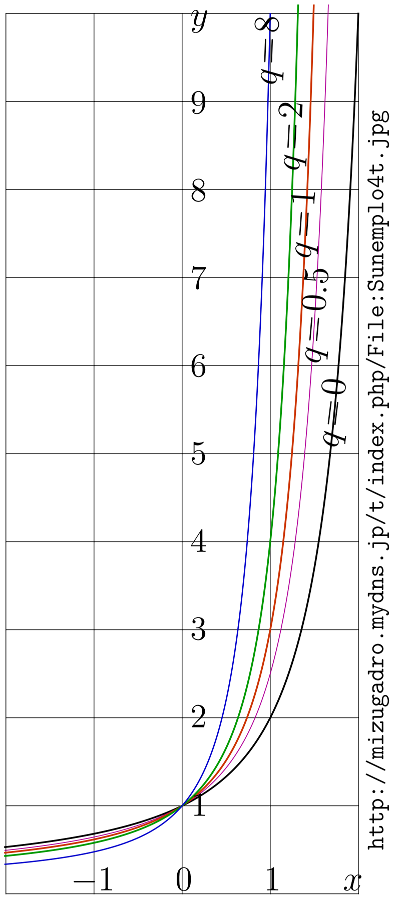


Figure 13.5: $y\!=\!\mathrm{SuNem}_q(x)$

For the Nemtsov function Nem_q, the superfunction is solution F of the transfer equation

$$F(z\!+\!1) = \mathrm{Nem}_q(F(z)) \qquad [\mathrm{Feq}] \quad (13.6)$$

In analogy with approach of the previous chapter, I look for solution F with the certain asymptotic behaviour,

$$F_q(z)\!=\!\frac{1}{\sqrt{-2z}}\left(1\!-\!\frac{q}{\sqrt{-2z}}\!+\!O\Big(\frac{\ln(-z)}{z}\Big)\right) \qquad [\mathrm{F}] \qquad (13.7)$$

In order to construct the computationally-efficient asymptotic approximation of superfunction F_q, define set of polynomials

$$P_m(z) = \sum_{n=0}^{\mathrm{IntegerPart}[m/2]} a_{m,n} z^n \qquad [\mathrm{P}] \qquad (13.8)$$

where a are constant coefficients. Then, I set

$$F_{q,M}(z) = \varepsilon \left(1 - q\varepsilon + \sum_{m=2}^{M} P(\ln(-z))\varepsilon^m \right) \qquad [\mathrm{FqM}] \quad (13.9)$$

where

$$\varepsilon = \frac{1}{\sqrt{-2z}} \qquad [\mathrm{epsilon}] \qquad (13.10)$$

I substitute the approximation (13.9) into the transfer equation (13.6). The asymptotic analysis of the residual (id est, its asymptotic minimisation) determines coefficients a. The asymptotic representation (13.8),(13.9),(13.10) should approximate superfunction F at least for large negative values of the argument.

Table 13.2: Computation of coefficients a in equation (13.8) [T2]

```
T[z_] = z + z^3 + q z^4
P[m_, L_] := Sum[a[m, n] L^n, {n, 0, IntegerPart[m/2]}]
A[1, 0] = -q; A[1, 1] = 0;
a[2, 0] = 0; A[2, 0] = 0;
F[m_,z_]:=1/(-2z)^(1/2)(1-q/(-2z)^(1/2)+
Sum[P[n,Log[-z]]/(-2z)^(n/2),{n,2,m}])

m = 2;
s[m] =Numerator[Normal[Series[
     (T[F[m,-1/x^2]]-F[m,-1/x^2+1])2^((m+1)/2)/x^(m+2),{x,0,1}]]]
t[m] = Numerator[Coefficient[Normal[s[m]], x] ]
sub[m] = Extract[Solve[t[m] == 0, a[m, 1]], 1]
SUB[m] = Simplify[sub[m]]
f[m, z_] = ReplaceAll[F[m, z], SUB[m]]

m = 3
s[m] =Simplify[ReplaceAll[Series[
(T[F[m,-1/x^2]]-F[m,-1/x^2+1])2^((m+3)/2)/x^(m+3),{x,0,0}],SUB[m-1]]];
t[m] = ReplaceAll[Normal[s[m]], Log[x] -> L];
u[m] = Table[Coefficient[t[m] L, L^n]==0, {n,1,1+IntegerPart[m/2]}];
tab[m] = Table[a[m, n], {n, 0, IntegerPart[m/2]}];
sub[m] = Extract[Solve[u[m], tab[m]], 1]
SUB[m] = Join[SUB[m-1], sub[m]];

(* and so on for  m=4,  m=5, etc. *)
```

The original (and non-trivial) part of this research is guessing of representation (13.8),(13.9),(13.10). This representation is a little bit more complicated, that the similar representation (12.4), (12.5), (12.6) for function SuSin, described in the precious section.

Once representation (13.8),(13.9),(13.10) is written out, the following construction is straightforward. This analysis can be performed with the Mathematica code shown in Table 13.2.

Coefficients a are chosen in such a way, that

$$F_{q,M}(z) = F_q(z) + O\big(\varepsilon^{M+1}\big) \tag{13.11}$$

Tens of coefficients a in equation (13.8) can be computed in such a way. The first coefficients are:

$a_{2,0} = 0$, $\quad a_{2,1} = \frac{1}{4}(3 + 2q^2)$

$a_{3,0} = q + 3q^3$, $\quad a_{3,1} = -(3q)/2 - q^3$

$a_{4,0}=\frac{1}{8}(5-4q^2-44q^4)$, $a_{4,1}=\frac{1}{8}(-9-12q^2-4q^4)$, $a_{4,2}=\frac{3}{32}(9+12q^2+4q^4)$

$a_{5,0}=\frac{1}{12}(-39q-104q^3-4q^5)$, $a_{5,1}=\frac{7}{4}(3q+8q^3+4q^5)$, $a_{5,2}=-\frac{9q}{4}-3q^3-q^5$

Coefficients a happen to be polynomials with respect to parameter q. For the following numerical implementation of superfunction, they are expressed through the Horner rule. The C++ implementation is loaded together with generators of figures of this chapter.

Assume given number M of terms of sum in the asymptotic expansion (13.9). Then, the superfunction F_q can be defined as limit

$$F_q(z) = \lim_{n\to\infty} \mathrm{Nem}_q^n\Big(F_{q,M}(z-n)\Big) \quad \text{[Flim]} \tag{13.12}$$

I remind, the upper index after the name of the function indicates the number of its iterate. The limit does not depend on the number M. However, for large M, the limit converges faster. For q of order of unity, and argument z of order of unity, with $M\!=\!30$, it is sufficient to make $n\!=\!20$ iterates in order to approximate limit in equation (13.12) with 14 decimal digits. That greatly exceeds the precision, required to plot the camera-ready copies of all the figures presented. However, the extra digits help to reveal faults of the representation of function, if any mistake takes place.

The transfer equation has translational invariance. If some $z \to F(z)$ is the solution, then, for a constant C, function $z \to F(z\!+\!C)$ is also solution, id est, also superfiunction of the same transfer function. In order to make figures more beautiful it is convenient, that at zero, the superfunction has value unity. For this reason, I define superfunction SuNem_q as superfunction F with displaced argument:

$$\mathrm{SuNem}_q(z) = F_q(x_1\!+\!z) \quad \text{[SuNem]} \tag{13.13}$$

where $x_1 = x_1(q)$ is real solution of equation

$$F_q(x_1) = 1 \tag{13.14}$$

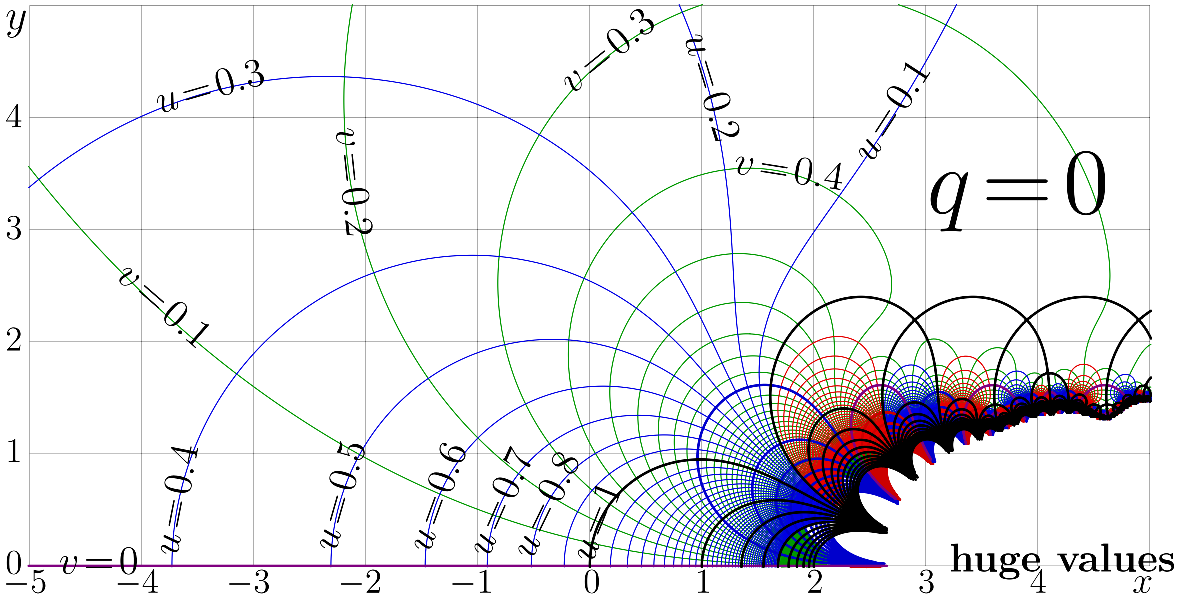
This provides condition

$$\mathrm{SuNem}_q(0) = 1 \tag{13.15}$$

This condition is not so important to evaluate iterates of the Nemtsov function, but it helps to compare different superfunctions. Many superfunctions, described in this Book, have value unity at zero.

Definition (13.13) of function SuNem is used to generate the explicit plot in figure 13.5 and also the complex maps of SuNem_q for $q\!=\!0$, $q\!=\!1$ and $q\!=\!2$ in figure 13.6. These maps are symmetric with respect to reflection from the real axis, so, the only upper half of the complex plane is shown.

`http://mizugadro.mydns.jp/t/index.php/File:Sunem0ht.jpg`



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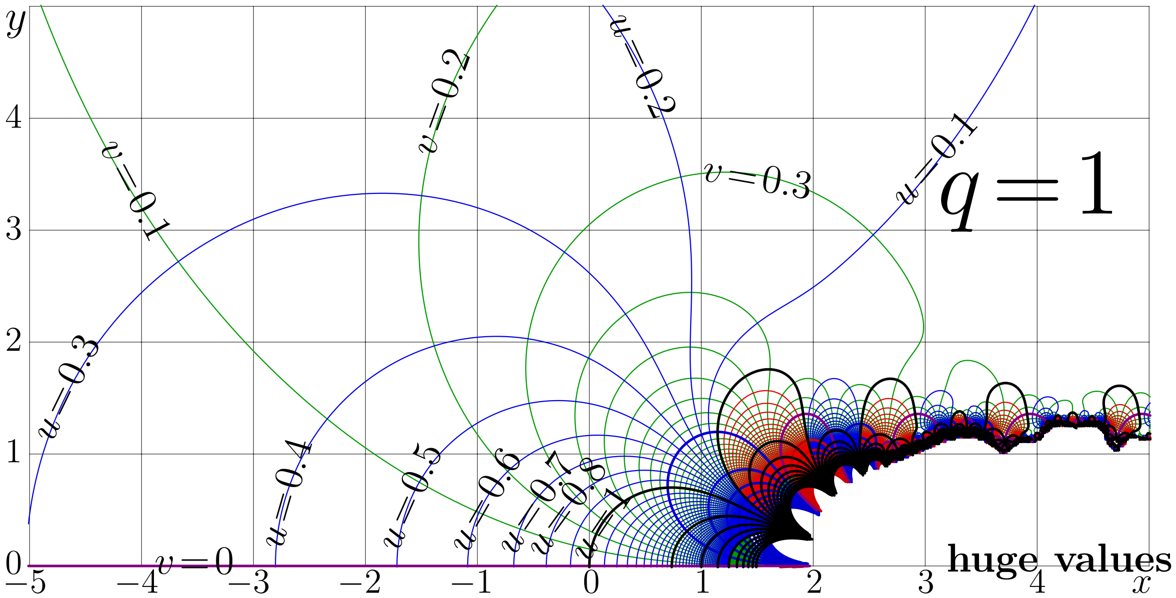


`http://mizugadro.mydns.jp/t/index.php/File:Sunem2ht.jpg`

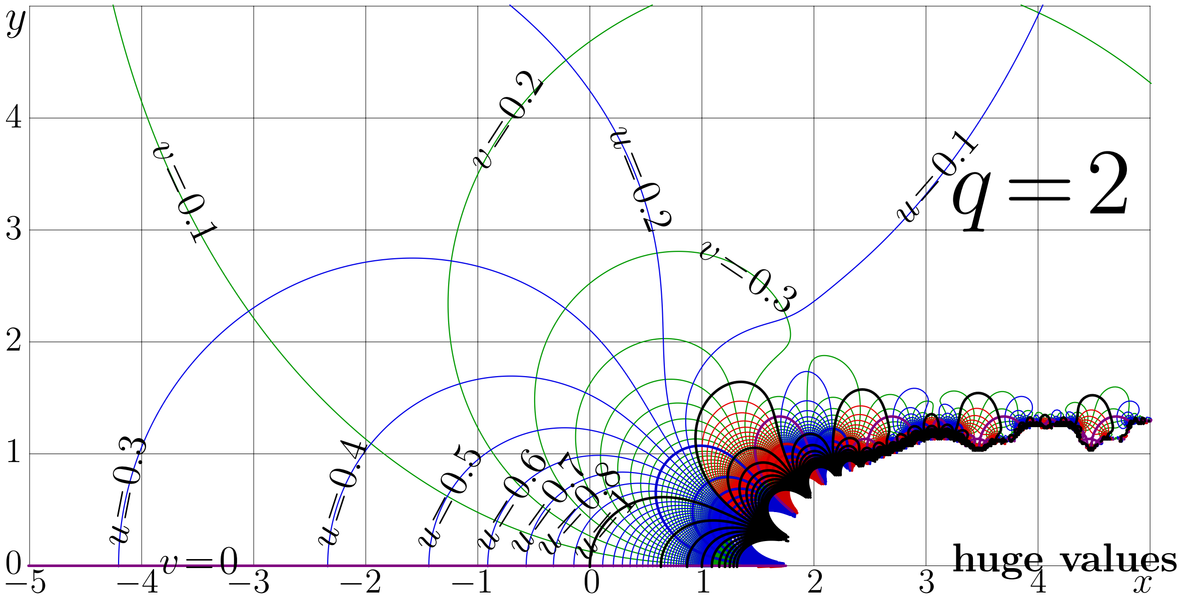


Figure 13.6: $u+\mathrm{i}v = \mathrm{SuNem}_q(x+\mathrm{i}y)$, for $q=0,1,2$ [suma]

Maps in figure 13.6 demonstrate the asymptotic behaviour of function SuNem. In the most of the complex plane, the function slowly decays to zero. This decay is determined by the leading term of the asymptotic expansion (13.9). However, this asymptotic expansion is not valid in vicinity of the origin of coordinate, nor in vicinity of the the positive part of the real axis. In the half-strips $x > 2$, $1 < |y| < 2$, the maps show complicated, oscillating behaviour of the function. For positive values of the argument, function SuNem_q shows fast growth. This growth is seen both, in the explicit plot at the right hand side picture of figure 13.5 and in the complex maps in figure 13.6.

For computation of iterates of the Nemtsov function Nem_q, the Abel function $\mathrm{AuNem}_q = \mathrm{SuNem}_q^{-1}$ is also required. Function AuNem_q is described in the next section.

3 AuNem

In this section, I construct abelfunction for the Nemtsov function Nem_q by (13.1). This abeldunction is inverse of the superfunction, id est, $\mathrm{AuNem}_q = \mathrm{SuNem}_q^{-1}$.

First, consider inverse function of the superfunction F by (13.7); let $G_q = G = F^{-1}$. It can be expanded as follows:

$$\begin{aligned} \mathrm{G}_{q,M}(z) &= -\frac{1}{2z^2} + \frac{q}{z} + \frac{1}{2}\left(2q^2+3\right)\log(z) + \frac{q^2}{2} + \frac{1}{4}\left(2q^2+3\right)\log(2) \\ &\quad + \sum_{n=1}^{M} c_n z^n \quad \text{[auneqm]} \qquad (13.16) \\ G_q(z) &= \mathrm{G}_{q,M}(z) + O(z^{M+1}) \quad \text{[G]} \qquad (13.17) \end{aligned}$$

This form can be obtained, inverting expansions (13.7), (13.9) for superfunction F_q.

Coefficients c depend on q; these coefficients can be computed either with asymptotic analysis of equation

$$G_{q,M}(F_{q,M}(x)) = z \qquad (13.18)$$

or from the Abel equation

$$G_q(\mathrm{Nem}_q(z)) = G_q(z) + 1 \qquad (13.19)$$

Coefficients c in equation (13.16) can be calculated in Mathematica with the code shown in Table 13.3.

Table 13.3: Mathematica code for calculation of coefficients c in equation (13.16)

```
T[z_] = z + z^3 + q z^4
P[m_,L_]:= Sum[a[m, n] L^n, {n, 0, IntegerPart[m/2]}]
F[m_,z_]:=1/(-2z)^(1/2)(1-q/(-2z)^(1/2)+
                          Sum[P[n,Log[-z]]/(-2z)^(n/2),{n,2,m}])
G[m_,x_]:=-1/(2x^2)+q/x+q^2/2+1/4(3+2q^2)Log[2]+1/2 (3+2q^2)Log[x]+
                                                Sum[c[n]x^n,{n,1,m}]
Series[ReplaceAll[F[3,h+G[3, z]], a[2,1]-> 1/4 (3+2 q^2)], {z,0,4}]
(*The line above is just test *)
m=1;
sg[m]=Coefficient[Series[G[m+3,T[z]]-G[m+3,z]-1,{z,0,3}], z^(m+2)]
st[m]=Solve[sg[m] == 0, c[m]]
su[m]=Extract[st[m], 1]
SU[m]= su[m];

m= 2;
sf[m]=Series[ ReplaceAll[G[m+3,T[z]]-G[m+3,z]-1,SU[m-1]],{z,0,m+2}]
sg[m]=Simplify[Coefficient[sf[m] 2^m, z^(m+2)]]
st[m]= Solve[sg[m] == 0, c[m]]
SU[m]= Join[SU[m - 1], su[m]]

m = 3;
sf[m]=Series[ ReplaceAll[G[m+3,T[z]]-G[m+3,z]-1,SU[m-1]],{z,0,m+2}]
sg[m]=Simplify[Coefficient[sf[m] 2^m, z^(m+2)]]
st[m]= Solve[sg[m] == 0, c[m]]
su[m]= Extract[st[m], 1]
SU[m]= Join[SU[m-1], su[m]]

(*and so on for m=4, m-5, etc... *)
```

For an integer $M>0$, the abelfunction G_q can be evaluated through the asymptotic approximation $G_{q,M}$ as the limit

$$G_q(z) = \lim_{n\to\infty} \Big(\mathrm{G}_{q,M}\big(\mathrm{ArqNem}_q{}^n(z)\big) + n \Big) \quad [\text{AuNemLim}]\,(13.20)$$

Then, the function AuNem is expressed through function G with addition of the constant,

$$\mathrm{AuNem}_q(z) = G_q(z) - G_q(1) \qquad (13.21)$$

in such a way, that $\mathrm{AuNem}(1) = 0$. The readers are invited to check numerically the ranges of validity of relation

$$\mathrm{SuNem}_q\big(\mathrm{AuNem}_q(z)\big) = z \qquad (13.22)$$
$$\mathrm{AuNem}_q\big(\mathrm{SuNem}_q(z)\big) = z \qquad (13.23)$$

and also estimate the residuals at the substitution of the numerical implementations into these relations.

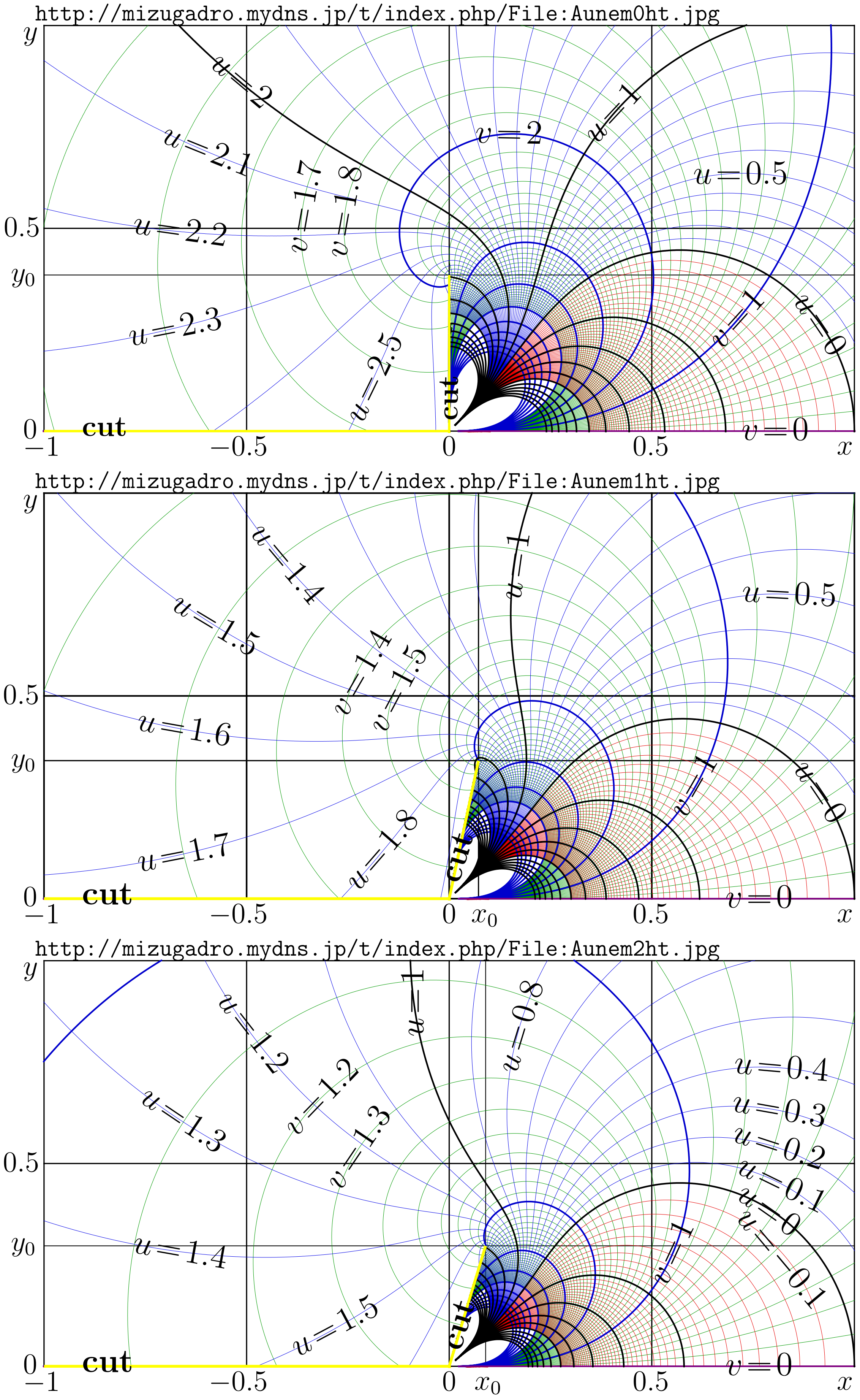


Figure 13.7: $u+\mathrm{i}v = \mathrm{AuNem}_q(x+\mathrm{i}y)$ for $q=0$, $q=1$, $q=2$ [aunema]

Complex maps of $u+\mathrm{i}v = \mathrm{AuNem}_q(x+\mathrm{i}y)$ for $q=0$, $q=1$ and $q=2$ are shown in the x, y plane in figure 13.7.

The Readed is invited to compare the cut lines in figure 13.7 with cuts in maps at the right hand side column of figure suma. The cut lines of function AuNem_q are the same, as those of function ArqNem_q; because it is the first function to evaluate while computing through the asymptotic representation $G_{q,M}$ by equation (13.20). The branch points are determined by function NemBran(q) shown in figure 13.4.

Lines $v=\mathrm{const}$ in maps of figure 13.7 approach the cut lines in pretty specific way, that corresponds to decrease of u, which represents the real part of the function. This means, that, if one goes along any line $v = \mathrm{const}$, increasing u, one approaches to the origin of coordinates, where the asymptotic representation is accurate, without to cross the cut lines. This happens due to the specific choice of the inverse function ArqNem_q, with this goal, the cut lines of function ArqNem_q are chosen in the function ArqNem. (The default choice of the inverse function, provided by the routine “Solve” in Mathematica, does not provide this.)

In such a way, the Abel function AuNem for the Nemtsov function is constructed and implemented. With functions SuNem_q and AuNem_q, the iterates of function Nem_q can be calculated. They are shown in figures 13.8, 13.9 and described in the next section.

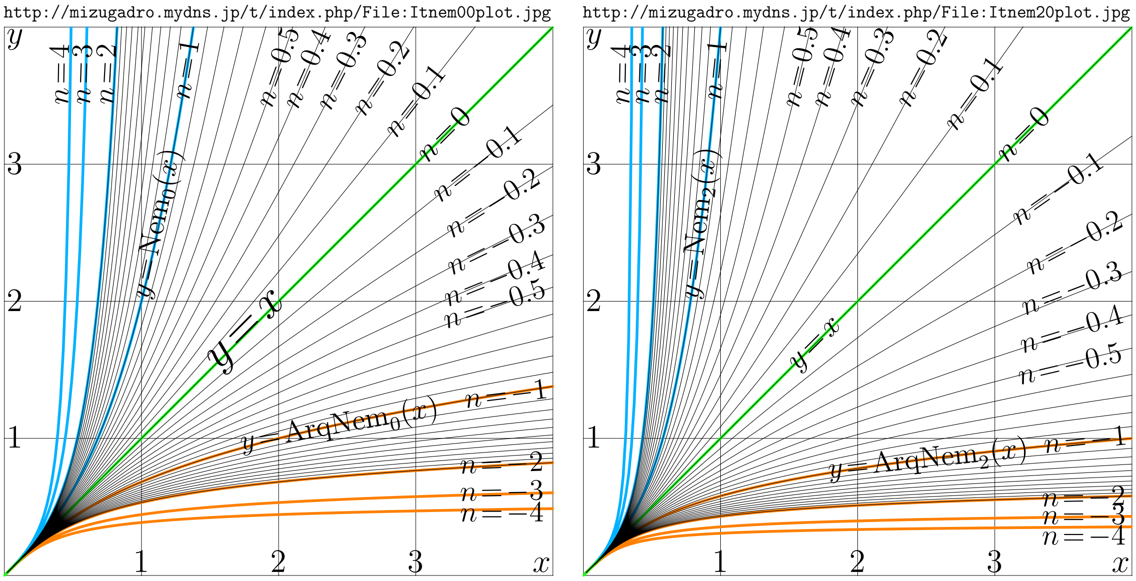


Figure 13.8: $y=\mathrm{Nem}_q^n(x)$ at $q=0$, left, and $q=2$, right, for various n by (13.24)

http://mizugadro.mydns.jp/t/index.php/File:Nem1it.jpg

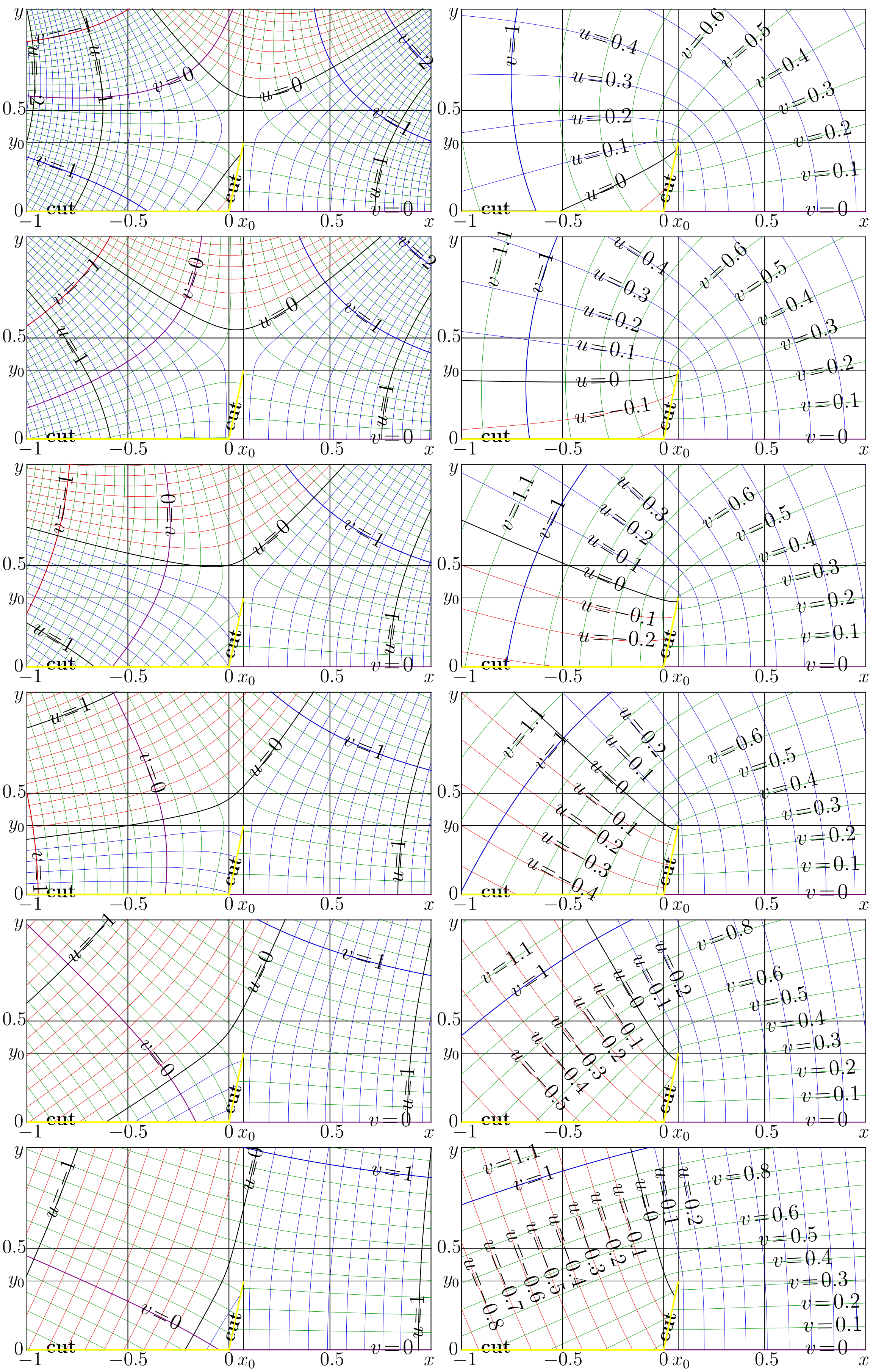


Figure 13.9: $u+\mathrm{i}v = \mathrm{Nem}_1^n(x+\mathrm{i}y)$ by (13.24) for $n = 0.6$ .. 0.1, left, and for $n = -0.6$ .. -0.1, right [iteramap]

4 Iterates of the Nemtsov function

With functions Nem_q and SuNem_q from the previous sections, the iterates of the Nemtsov function can be defined,

$$\mathrm{Nem}_q^n(z) = \mathrm{SuNem}_q\Big(n + \mathrm{AuNem}_q(z)\Big) \quad [\mathrm{Nqn}] \quad (13.24)$$

Figure 13.8 shows the explicit plot $y = \mathrm{Nem}_q(x)$ for $q \to 0$ in the left hand side picture and for $q = 2$ in the right hand side picture, for various values of the number n of iterate. The integer values of n correspond to the thick lines.

For $q = 1$, complex maps of iterates Nem_1^n are shown in figure 13.9. The figure shows, how, at the increase of number n of iterate from -1 to 1, function ArqNem_1 gradually changes to the identity function (with rectangular grid as the complex map) and then to the Nemtsov function Nem_1.

Iterates in figure 13.8 look similar to iterates of other quickly growing holomorphic functions [54, 64, 61, 65, 88]. In particular, at $n \approx 0$, the iterate Nem_q^n looks similar to identical function; at $n = 1$, it is just Nemtsov function Nem_q, and at $n = -1$, it is the inverse function, id est, ArqNem_q.
Iterates of a growing real-holomorphic function are also real-holomorphic; the complex maps are symmetric with respect to reflection from the real axis, so, the only upper half of the complex plane is shown shown in figure 13.9. The left column shows maps for the positive iterates; the number n varies from 0.6 at the top map with step -0.1 to -0.1 at the bottom map. In the similar way, the right hand side column represents maps for n from -0.6 at the top to -0.1 at the bottom. Only case with $q = 1$ is presented, but one can download the generator of the figure and plot similar maps for other values of q, and, of course, other values of number n of iterate; this number can be even complex.

Iterates by 13.24 shown in figures 13.8, 13.9 provide the smooth (holomorphis) transition from the Nemtsov function Nem_q to the identity function and then to the inverse function ArqNem. Iterates have the group property,

$$T^{m+n}(z) = T^m(T^n(z)) \quad [\mathrm{Tmn}] \quad (13.25)$$

This ratio holds only for certain range of values of parameters, that includes the positive part of the real axis for z. In order to keep the Book of reasonable thickness, I skip out this analysis and suggest the Readers to do it as an excersise.

5 End of exotic iteration

Figure 13.10: B.Nemtsov

The 4th order polynomial of special kind (13.1) is considered in this section. I call it Nemtsov function after Boris Nemtsov, see fig. 13.10; the need to denote this function coincided with the tragic event. This polynomial is treated as transfer function: the inverse function ArqNem, the superfunction SuNem and the abelfunction AuNem are constructed.

Construction of function ArqNem happend to be non-trivial; so, its map is shown in the right hand side column of figure 13.2 for various values of parameter. Choice of the cut lines of this function is important. The readers are invited to try to construct abelfunction with other choice of the cut lines and plot the complex maps of the result. And compare the resulting complex maps to those in figure 13.7. And the same for the iterates in figure 13.9.

Function Nem is my last attempt to construct a difficult-to-iterate growing real-holomomorphic function with real fixed point. The fixed point is chosen at zero, because the update to the more general case is straightforward, it is specified in the las row of table 3.1.

For the real-holomorphic growing transfer function with real fixed point, the suprfunction can be constructed with regular iteration considered in chapter 6, if the derivative at fixed point is not unity, or with exotic iteration, if this derivative is unity. Then, one can guess the heuristic "solution" with correct asymptotic behaviour of the superfunction, using analogy with the differential equation, discussed in the previous chapter; the same analogy works for the Nemtov function too. This leads to iterates that I call "exotic". The exotic iterates lead to pretty regular and real-holomorphic superfunctions, abelfunctions and correspondingly regular non-integer iterates of the transfer function.

In the next chapter, I consider, in some sense, even more exotic case, when the transfer function has no real fixed point. The example of such a function without real fixed point is just natural exponent, $T = \exp$. Historically, namely this transfer function had been considered and iterated (it can be iterated any real or even complex number of times) [54].

Chapter 14

Natural tetration tet

Here, I consider the exponential transfer function, $T = \exp$. For this transfer function, the transfer equation can be written as follows:

$$f(z+1) = \exp\big(f(z)\big) \tag{14.1}$$

In order to narrow the set of solutions, the additional condition is assumed:

$$f(0) = 1 \tag{14.2}$$

In order to provide the uniqueness, in addition, I require, that the solution $f(z)$ is holomorphic at the whole complex plane except $z \le -2$, and also limited at least in the strip $\Re(z) \le 1$. I refer the solution f as "tetration" (or "natural tetration") and denote it with symbol tet. In this chapter I tell, how this function is constructed; I use the main formulas and pictures from publications [54, 64].

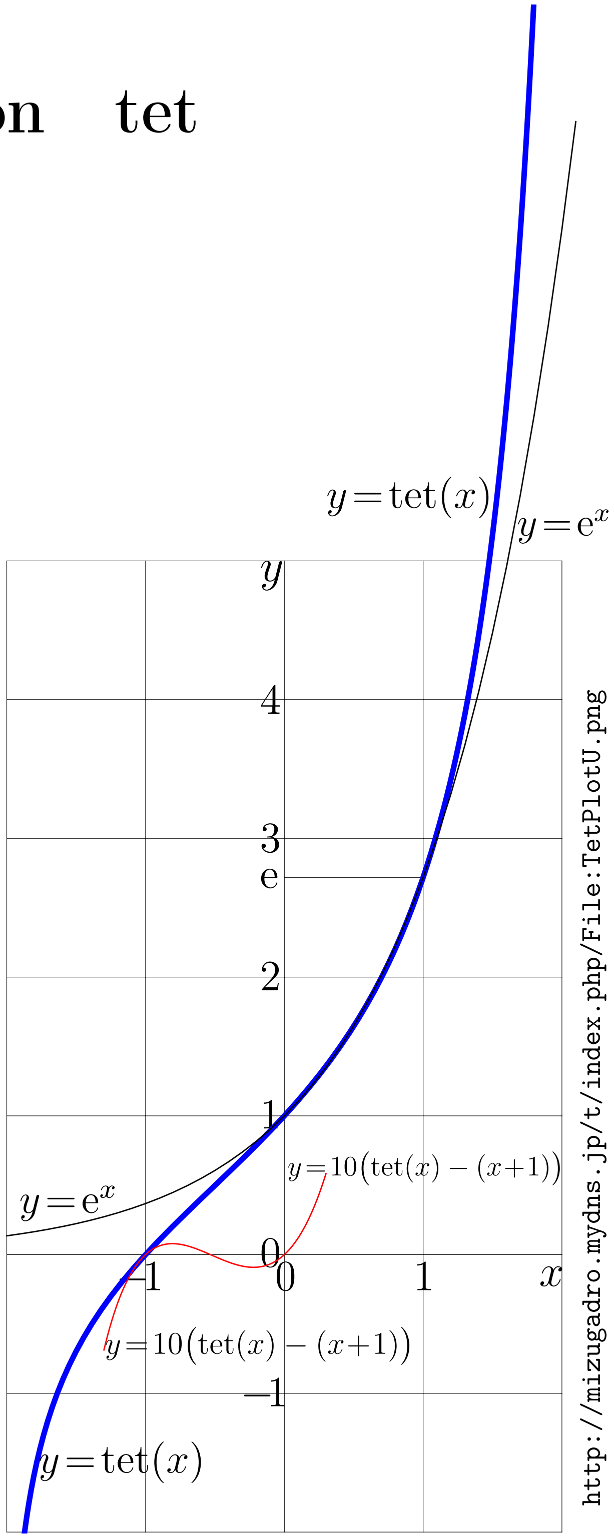


Figure 14.1: $y = \mathrm{tet}(x)$, $y = \exp(x)$, and $y = 10\big(\mathrm{tet}(x) - (x+1)\big)$ [TetPlot]

Colleagues often ask questions not only about superfunctions and way of the evaluation, but also about guessing of properties of these functions. In particular, these questions refer to the properties of the natural tetration: “How did you guess?”

The questions about guessing are important not only for the history, but also for the colleagues, who want to use the similar way for other transfer functions, superfunctions and, perhaps, even to some more complicated objects. So, I consider these questions seriously. Especially, this apply to the natural exponent and the natural tetration, as the natural tetration is first non-trivial superfunction, for which the efficient algorithms of evaluation had been suggested and described [54, 64].

In this chapter, I provide not only the formulas and pictures for the natural exponental and the natural tetration, but also explanations, why namely this tetration should be considered and recognised as the most “true” and the “most natural” among various possible superfunctions; and why any researcher, following the same idea, should come to the same tetration.

1 Exponent

Before to deal with solution f of equations (14.1), (14.2), it worth to remind properties of exponent. For the real argument, graphic of exponent is shown in figure 14.1 with thin line. Complex map of the transfer function $T = \exp$ is shown in figure 14.2.

I hope to be not condemned for drawing so elementary functions as exponent. (I did not do it in the previous section about sin; but for some elementary functions I provide the maps. The Book should allow the understanding, even if it happens to be in hand of a pure experimentalist. With the detailed descriptions of the elementary things, I hope, that even Aleksander Kaminskii, or Akira Shirakawa, or Yulya Kuznetsova can understand at least the main idea of the Book. The Book must allow the reading even by the least-educated academician [16] [1].

[1] `http://royallib.com/read/Strugatsky_Arkady/Tale_of_the_Troika.html` Tale of the Troika by Arkady and Boris Strugatsky. PROLOG. *"..Our slogan is ‘elevators for everyone.’ No matter who. The elevator must be able to withstand the entrance of the least-educated academician."*

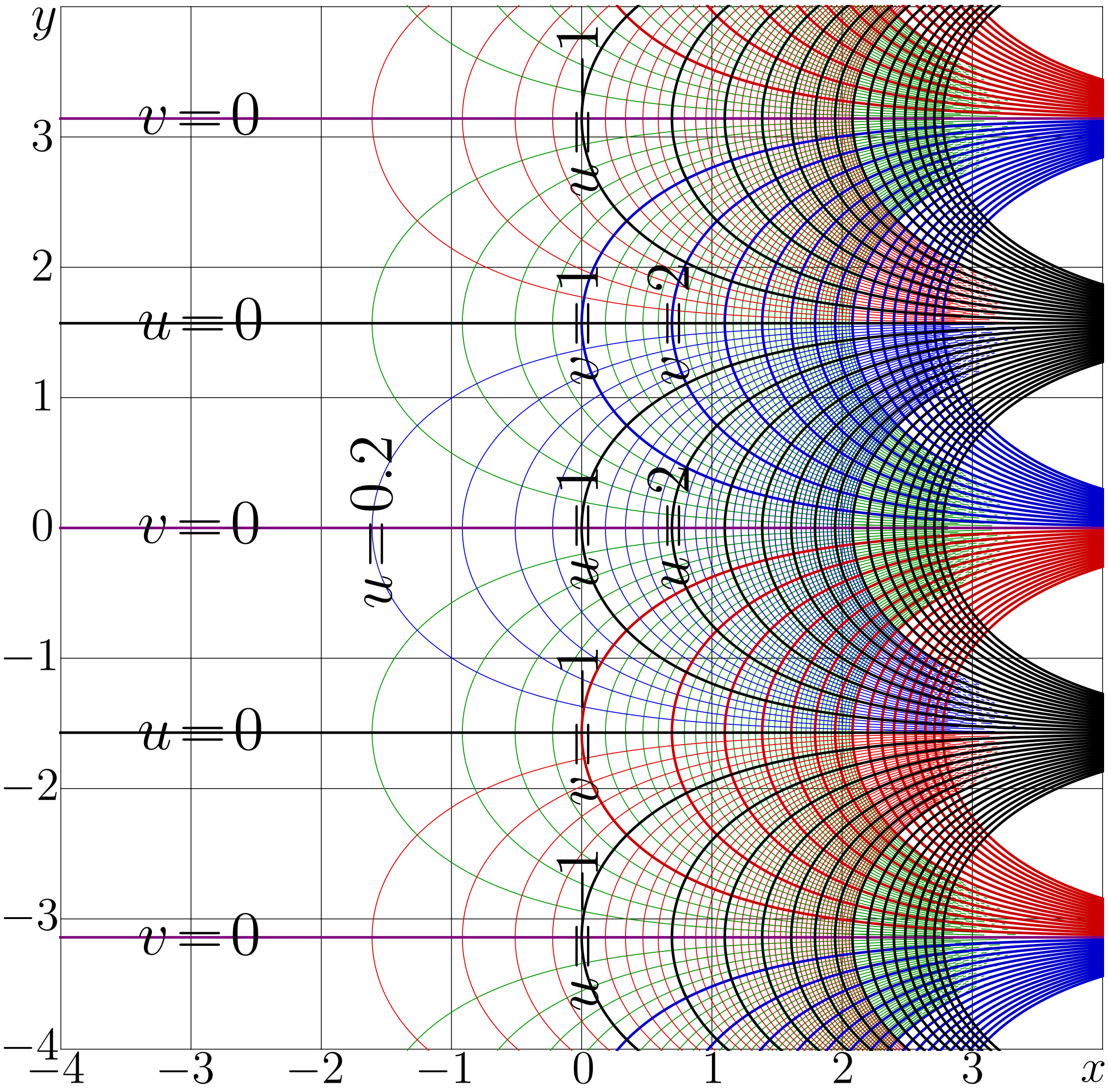


Figure 14.2: $u+\mathrm{i}v = \exp(x+\mathrm{i}y)$ [expMap]

The exponent can be considered as superfunction of function "multiplication by constant number e"; $\mathrm{e} = \exp(1) \approx 2.71828182846$. In such a way, exp is superfunction for the transfer function T by

$$T(z) = \mathrm{e}\, z \quad \text{[Tez]} \tag{14.3}$$

I repeat the formula from the school course of algebra:

$$f(z+1) = \mathrm{e}\, f(z) \quad \text{[expz1ez]} \tag{14.4}$$

The solution of this equation can be constructed with regular iteration, in vicinity of the fixed point $L=0$ of the transfer function T by (14.3). The Reader is invited to make this exercise and check, that the primary expansion stops a the first term, giving the exact solution $f=\exp$ at the first iteration.

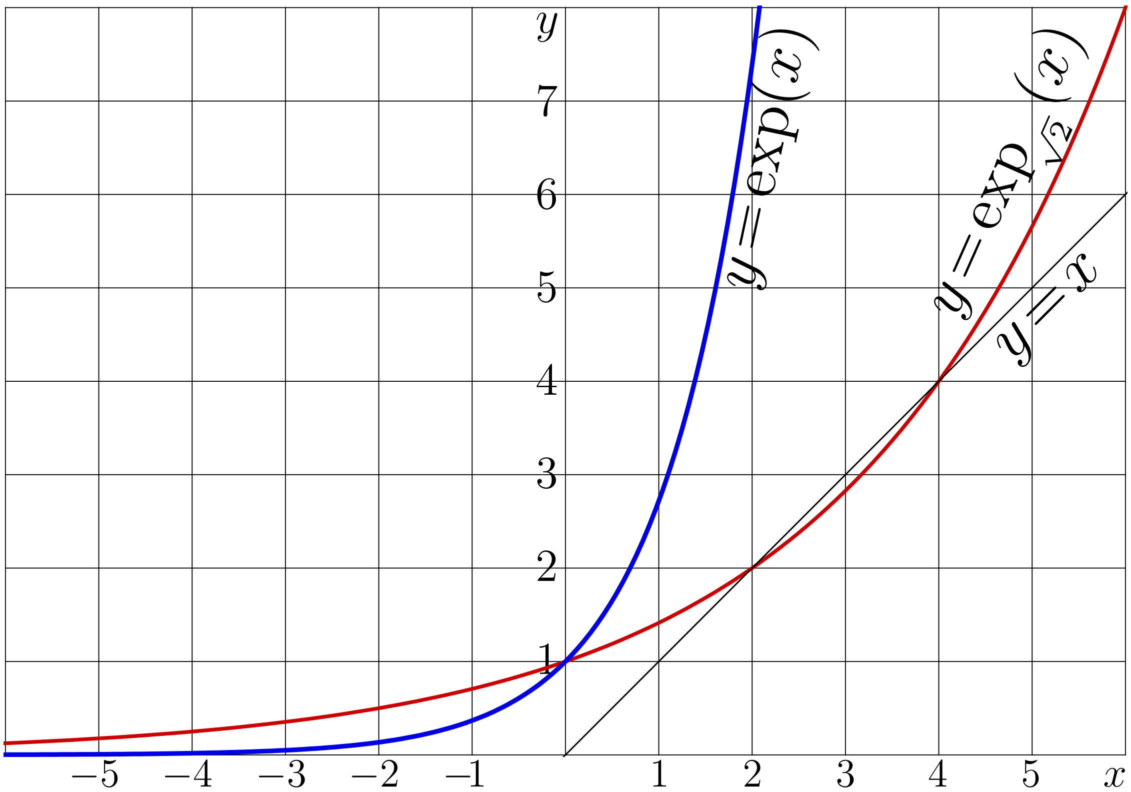


Figure 14.3: $y\!=\!\exp(x)$ and $y\!=\!\exp_{\sqrt{2}}(x)$ [ExpQ2ePlot]

For the transfer function $z \mapsto \mathrm{e}\, z$, I write also the Abel equation:

$$g(\mathrm{e}\, z) = 1 + g(z) \quad \text{[abelog]} \tag{14.5}$$

Equation (14.5) is considered by Henryk Trappmann [86]. The Reader can guess, that Henryk got the natural logarithm as the solution, id est, $g\!=\!\ln$. Readers are invited to think, what additional requirements should be associated with equation (14.5), in addition to equation

$$g(1) = 0 \tag{14.6}$$

in order to provide the uniqueness of the solution $g\!=\!\ln$.

Explicit plot of exp is shown in figure 14.3. (The same dependence is shown with thin curve in figure 14.1.) It worths to compare the graphic of the natural exponent to that of the exponent to base $b\!=\!\sqrt{2}$, which crosses the straight line $y = x$. This line is also shown in figure 14.3. The graphic of the natural exponent $y\!=\!\exp(x)$ does not cross the line $y\!=\!x$. The natural exponent has no real fixed point.

Fixed point L of exponential and logarithm to base $b\!=\!\exp(a)$ is solution of equation $L = \log_b(L)$. This solution can be expressed through the Tania function, considered in chapter 4:

$$L = \mathrm{filog}(a) = \frac{\mathrm{Tania}(\ln(a) - 1 - \pi \mathrm{i})}{-a} = \frac{\mathrm{WrightOmega}(\ln(a) - \pi \mathrm{i})}{-a} \quad (14.7)$$

The second equality in formula (14.7) can be considered as definition of the new function filog. This function is described also in TORI, `http://mizugadro.mydns.jp/t/index.php/Filog`.
Function filog is considered in details below, in chapter 18. Here, we need this function only for the single value of the argument, namely, $L = \mathrm{filog}(1)$. As function Tania is already described, it should be considered as special function. In this sense, quantity L should is exact:

$$\begin{aligned} L = & -\mathrm{Tania}(-1 - \pi \mathrm{i}) \\ \approx & \ 0.3181315052047641353 + 1.3372357014306894089\,\mathrm{i} \end{aligned} \quad (14.8)$$

The rough approximation (with two significant figures) for L by (14.8) can be found even from figure 5.2, counting isolines with finger.

Function Tania is used in this Book already twice: first, in chapter 2, as realistic superfunction for the transfer function Doya (that describes increase of the intensity of light in a laser amplifier with simple model of continuously pumped of active medium. and, second time, here, as exact representation for the fixed point of exponent. This is the same function. Recycling, reusing of the results is base of any science, and the physical mathematics (see the Introduction) is not exception.

Exponent is real-holomorphic function, $\exp(z^*) = \exp(z)^*$; hence $L^* \approx 0.1 - 1.3\,\mathrm{i}$ is also the fixed point. In principle, each of these fixed points can be used for the regular iteration, considered in chapter 6. However, such iteration does not lead to the real-holomorphic superfunction. I wanted to suggest a way of evaluation of the real-holomorphic tetration, that could be used as the definition. In order to specify it, I assume, that there exist some special superfunction of the natural exponent, and this superfunction is characterised with specific behaviour. As the real part of the argument goes to $-\infty$, the superfunction approaches L in the upper part of the complex half plane, and L^* in the loser part. At the beginning, this is nothing more, but just guess. In the next section, this guess is used to construct both, the definition and the way of evaluation.

2 Heuristic tetration

Since publication of the first article about real-holomorphic tetration, the colleague ask me similar questions:
“Why did you interested in holomorphic tetration?"
“How did you guess the asymptotic behaviour to tetration at $\pm\mathrm{i}\infty$?”
“How did you guess the initial approximation fit_3?” [87].
In order not to be like Rip van Winkle [2], revealing new and new details, here I mention the only one of motivations, that is related to physics.

Initially, I wanted to use some fast-growing function in order to represent the factorial of the number of particles in the Bose-Einstein condensate, this factorial appears at the attempt to write-out the first approximation for the normalised multi-particle wave function. The fastly-growing function is described in the article by Hooshmand [49], but it happened to be not suitable for the asymptotic analysis, because it is not holomorphic. The corresponding extension

$$f(z) = \mathrm{uxp}(z) = \begin{cases} \ln\big(\mathrm{uxp}(z{+}1)\big) & \text{at} \qquad \Re(z) \le -1 \\ z+1 & \text{at} \quad -1 < \Re(z) \le \ 0 \\ \exp\big(\mathrm{uxp}(z{-}1)\big) & \text{at} \quad \ 0 \ < \Re(z) \end{cases} \tag{14.9}$$

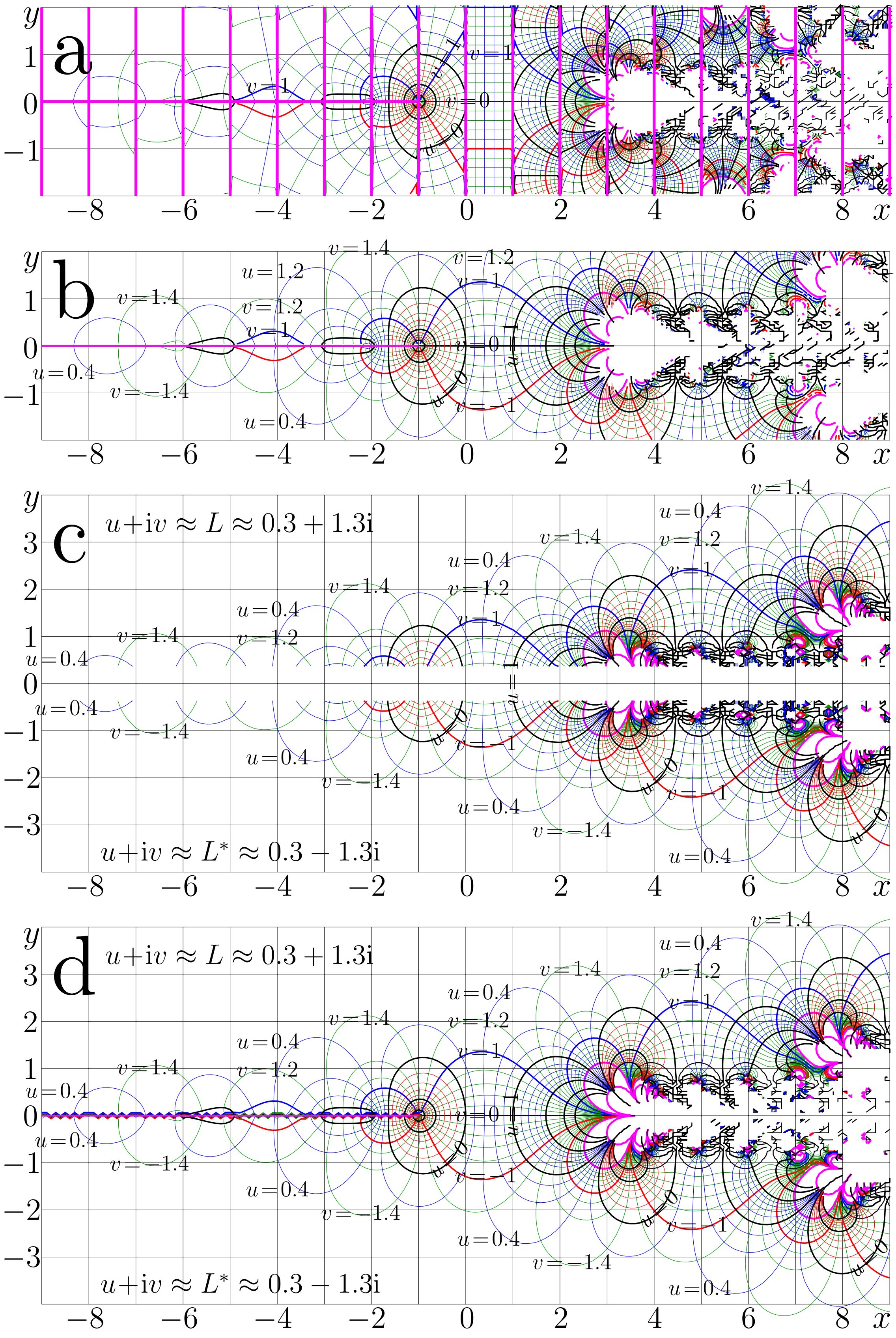
has many cut lines, they divide the complex plane to almost separated strips. Complex map of function uxp by (14.9) is shown in the upper picture of figure 14.4 with lines of constant log amplitude u and lines of constant phase v,

$$\exp(u{+}\mathrm{i}v) = f(x{+}\mathrm{i}y) \tag{14.10}$$

This representation is different from that, usef for the most of complex maps in this Book; usually, the lines of constant real part and those of constant imaginary part are drawn. While I explain, how did I get the holomorphic tetration, I represent maps in the same form, as they appear in the original paper [54].

As I already mention above, the vertical cuts of the range of holomorphism of function uxp divide the complex plane to almost independent strips seen in the top map in figure 14.4. These strips raise the question: Is it possible to suggest a “more holomorphic” (id est, with less cuts) solution of the transfer equation (14.1)? Or no solution f of the equation (14.1) may have wide range of holomorphism?

[2] `http://classiclit.about.com/library/bl-etexts/wirving/bl-wirving-rip.htm` Rip Van Winkle by Washington Irving. (1783-1859)



http://mizugadro.mydns.jp/t/index.php/File:Analuxp01u400.jpg

Figure 14.4: $\exp(u\!+\!\mathrm{i}v) = f(x\!+\!\mathrm{i}y)$ for the following functions: $f\!=\!\mathrm{uxp}$, (a); $f\!=\!\mathrm{Fit}_3$, (b); $f\!=\!\mathrm{Fit}_6$, (c); $f\!=\!\mathrm{tet}$, (d) [analuxpmap]

At the first look into results by Hooshand [49], construction of the real-holomorphic solution seemed to be impossible. On the other hand, the initial assumption used there, about monotonous derivative (for the real argument) of the superfunction of exp, looks doubtful. I tried to construct some alternative proof, that the cuts are unavoidable, without to use the strange assumption. I begun to investigate the case, assuming existence to the holomorphic solution. I expected to get some contradiction, and to use the contradiction for the proof. The holomorphic solution had been constructed, and no contradiction had been detected [54]; so, I had to accept the existence.

Historically, the construction of this solution begun with approximations. I had considered several real-holomorphic elementary functions, that have logarithmic singularity at -2 and take the same values, as tetration, at few integer values of the argument. One of them (that happened to be better than some others) is

$$\begin{aligned}\mathrm{fit}_2(z) =& \ln(2+z)\\ &+(1+z)\left(1+\tfrac{z}{2}\exp\Big((z-1)s_2(z)\Big)\left(\mathrm{e}-2+\ln\tfrac{4}{3}\right)-\ln 2\right)\end{aligned} \qquad (14.11)$$

where

$$s_2(z) = \exp\Big(\exp(z-2.51)\Big) - 0.6 + 0.08(z+1) \quad \text{[fit2s]} \quad (14.12)$$

Constants in the expression (14.12) are chosen in order to minimise the residual at the substitution $f = \mathrm{fit}_2$ into the transfer equation (14.1). This approximation could be improved, comparing (14.12) with the precise approximation through the Cauchy integral, considered below. However, at the heuristic search for the rough approximations, the representation through the Cauchy Integral had not yet been written; so, choosing the approximation, I had to use the residual as the criterion.

After construction of function fit_2, it happened, that the linear combination of functions $z \mapsto \mathrm{fit}_2(z)$ and $z \mapsto \ln(\mathrm{fit}_2(z+1))$ gives the residual even smaller; in such a way, the approximation fit_3 appeared:

$$\mathrm{fit}_3(z) = 0.6\,\mathrm{fit}_2(z) + 0.4\,\ln\big(\mathrm{fit}_2(z+1)\big) \qquad (14.13)$$

The range of approximation of tetration can be extended. Let

$$\mathrm{Fit}_3(z) = \begin{cases} \ln\big(\mathrm{Fit}_3(z+1)\big) & \text{at} \quad \Re(z) \le -1 \\ \mathrm{fit}_3(z) & \text{at} \quad -1 < \Re(z) \le 0 \\ \exp\big(\mathrm{Fit}_3(z-1)\big) & \text{at} \quad 0 < \Re(z) \end{cases} \quad \text{[Fit3]} \, (14.14)$$

Logampliture and phase of this function are show in figure 14.4b.

For comparison, two more maps are shown in figure 14.4, they are numbered as c and d. There represent the asymptotic approximation, $f = \mathrm{Fit}_6$ and tetration $f = \mathrm{tet}$, described below in section 4; namely this tetration is goal of this chapter.

The asymptotic approximation

$$\mathrm{Fit}_6(z) = \left\{ \begin{array}{l} L + \exp(k(z{+}r)) \ , \ \Re(z) < -8 \\ \exp\Big(\mathrm{fit}_6(z{-}1)\Big) \ , \ \Re(z) \ge -8 \end{array} \right. \qquad [\mathrm{Fit6}] \ (14.15)$$

is good at large values of imaginary part of the argument. For natural tetration, the increment $k{=}L$. This looks as just coincidence. However, everyone can check it with the asymptotic analysis, substituting the primary expression of fit6 into the transfer equation (14.1). Value of constant $r \approx 1.075820830781 - 0.9466419207254\,\mathrm{i}$ appears as adjusting parameter of this approximation. On the other hand, it seems to be approximation of the important mathematical constant. I would call this r with name “The Kneser constant”. This is one of constants, required for the expansion of iterates of exponent discussed in [10] and used for the approximation of tetration [64].

Function $\mathrm{Fit}_6(z)$ approximates tetration $\mathrm{tet}(z)$ at $\Im(z) > 0.4$; function $\mathrm{Fit}_6(z^*)^*$ approximates $\mathrm{tet}(z)$ at $\Im(z) < -0.4$; combination of these functions is shown in map “c” in figure 14.4. In vicinity of the real axis, roughly, in the strip $|\Im(z)| < 0.4$, both these functions $\mathrm{Fit}_6(z)$ and $\mathrm{Fit}_6(z^*)^*$ look ugly, and this strip in the map is left white.

Approximations $f = \mathrm{Fit}_3$ and $f = \mathrm{Fit}_6$ by (14.14) (14.15) are already sufficient to plot the complex maps and explicit plots of tertration; together, they provide of order of 3 decimal digits in the range of maps shown in figure 14.4. The last map "d" in figure 14.4 visually looks as superposition of the maps "b" and "c" above; this gave the general view of tetration that had to be constructed.

The approximations above (even Fit_3) allow to guess the asymptotic behaviour of tetration. It should approach the fixed points L or L^* of logarithm, while the imaginary part of the argument approaches the plus or minus infinity. These values are indicated in maps “c” and “d” of figure 14.4.

In such a way, this section explains, how did I guess, which the asymptotic behaviour should the tetration have. Postulating this behaviour, one can construct the algorithm for evaluation of tetration with any required precision. The postulated properties of tetration are collected in the next section.

3 Properties of tetration

Following recommendations by colleagues, friends and relatives, this content of this Book gradually goes from the simple examples to the more general formulas. In order to follow this way, here I define only the natural tetration. This section continues the following article [54] of year 2009. .

Since the two upper maps in figure 14.4 were plotted, the main properties of this function are clear. I postulate them below.

Solution F of the transfer equation (14.1) with additional condition $F(0)=0$ is called natural tetration, or simply tetration tet, if the following conditions are satisfied:

T1. Function $F(z)$ is real-holomorphic in the whole complex plane except the halfline $z \le -2$. Id est, $F(z^*)=F(x)^*$. At $z=-2$, function $F(z)$ has logarithmic singularity, id est, the branch point.

T2. Function $F(z)$ is bounded in the strip $|\Re(z)| \le 1$.

T3. Function $F(x)$ asymptotically approaches the fixed point L in the upper half plane: for any real x, the relation below holds:

$$\lim_{y\to+\infty} f(z+\mathrm{i}y) = L \qquad \text{[T715]} \qquad (14.16)$$

In addition, for positive y, the relation below holds:

$$\lim_{x\to-\infty} f(z+\mathrm{i}y) = L \qquad \text{[T716]} \qquad (14.17)$$

T4. In the strip $-1 \le \Re(z) \le 2$, the following condition holds:

$$\arg(F(z)) < 2 \quad \text{[argF]} \qquad (14.18)$$

Conditions T1-T4 above are a little bit redundant. The following development of the formalism of superfunctions is expected to indicate, which of these properties should be kept as definition of tetration, and which should appear as theorems, following from the shortened definition.

From postulates (14.16), (14.17) and real holomorphism $f(z^*) = f(z)^*$, it follows, that

$$\lim_{y\to-\infty} f(z+\mathrm{i}y) = L^* \qquad \text{[T717]} \qquad (14.19)$$

and for negative y, the relation below holds

$$\lim_{x\to-\infty} f(z+\mathrm{i}y) = L^* \qquad \text{[T718]} \qquad (14.20)$$

These conditions are used in the next section for construction and evaluation of tetration through the Cauchy integral.

4 Cauchy integral

For holomorphic function F, the Cauchy formula takes place [103]:

$$F(z) = \frac{1}{2\pi \mathrm{i}} \oint_\Omega \frac{F(t)}{t-z}\, \mathrm{d}t \quad \text{[Cauchy]} \tag{14.21}$$

where contour Ω belongs to the simply connected range of holomorphism of function F and only once passes around point z in the counter clock wise direction. With equation (14.21), the transfer equation (14.1) leads to the integral equation for the values of superfunction along the imaginary axis [54]. The description is repeated below.

Engineer, physicist or mathematician, using formula (14.21), has certain freedom in choice of the contour of integration. The researcher acts in a way, similar to that of an Engineer, who makes a project of the loop railroad for the rocky semi-island.

The smart engineer takes into account the locations of ports, cities, farms, industries, in order to help the people to reach the places of their destination. Also, the engineer tries to avoid swamps, steep slops, narrow curvy canyons, to make the railroad fast, cheap and safe.

Vainglorious tyrant, dictator, already famous in sports, war, art, archeology and ornithology, who wants to show himself also as a powerful all-mighty engineer, may draw a rectangle on the map, and promote it as a project of the trace of the railway. Such a " project " will require a lot of bridges, ramps, excavations and tunnels, makes him famous also as vain waster of the state budget and may bring him to the situation "no money" [3].

Sorry, in the choice of the contour of integration, described in [54], I look like as a tyrant, rather than as a smart engineer: I choose the contour of integration in the shape of rectangle. The only excuse is, that the this contour leads to efficient way of evaluation of tetration.

Let F be real-holomorphic of equation (14.1);
Let A be real positive number, so big, that $F(\mathrm{i}A)\approx L$
Let the range of hlomorphizm of function $F(z)$ includes the domain $-1 \le \Re(z) \le 1$ and, in this range, let $|\arg F(z)| < \pi$.

[3] `http://weirdrussia.com/2016/05/28/meme-medvedev-says-we-have-no-money-but-you-hang-in-there/` Medvedev Says "We have no money, but you hang in there" (2016).

`http://www.bbc.com/news/blogs-trending-36482124` Russian PM: 'No money for pensions, but have a good day!' 2016.06.09

`http://www.cnbc.com/2016/06/09/there-is-no-money-left-bye-russian-pm-causes-social-media-storm.html` Holly Ellyatt. 'There is no money left, bye!': Russian PM causes social media storm. .. "no money left" in Russia's budget.. "There just isn't any money now. .."

These conditions allow to convert the contour integral into the "solvable" integral equation; of course, at the appropriate choice of the contour of integration. Let contour Ω consists of the four segments:
A. Segment along line $\Re(t)=1$ from $t=1-\mathrm{i}A$ to $t=1+\mathrm{i}A$.
B. Segment from point $t=1+\mathrm{i}A$ to $t=-1+\mathrm{i}A$, passing above point z.
C. Segment along line $\Re(t)=-1$ from $t=-1+\mathrm{i}A$ to $t=-1-\mathrm{i}A$.
D. Segment from point $t=-1-\mathrm{i}A$ to $t=1-\mathrm{i}A$, passing below point z.

For this contour Ω, the Cauchy integral can be written as follows:

$$\begin{aligned} F(z) \;=\; & \frac{1}{2\pi}\int_{-A}^{A} \frac{F(1+\mathrm{i}p)\,\mathrm{d}p}{1+\mathrm{i}p-z} - \frac{1}{2\pi}\int_{-A}^{A} \frac{F(-1+\mathrm{i}p)\,\mathrm{d}p}{-1+\mathrm{i}p-z} \\ & - \frac{F_{\mathrm{up}}}{2\pi\mathrm{i}}\int_{-1-\mathrm{i}A}^{1-\mathrm{i}A} \frac{\mathrm{d}t}{t-z} + \frac{F_{\mathrm{down}}}{2\pi\mathrm{i}}\int_{-1-\mathrm{i}A}^{-1-\mathrm{i}A} \frac{\mathrm{d}t}{t-z} \end{aligned} \tag{14.22}$$

where F_{up} и F_{down} are some mean values of function F in vicinity of the segments **B** and **D** of the contour Ω.

Taking into account the transfer equation 2.12, and assuming holomorphism of function T^{-1}, equation (14.22) can be rewritten as follows:

$$F(z)=\frac{1}{2\pi}\int_{-A}^{A} \frac{\exp\!\big(F(\mathrm{i}p)\big)\,\mathrm{d}p}{1+\mathrm{i}p-z} - \frac{1}{2\pi}\int_{-A}^{A} \frac{\ln\!\big(F(\mathrm{i}p)\big)\,\mathrm{d}p}{-1+\mathrm{i}p-z} + K(z) \tag{14.23}$$

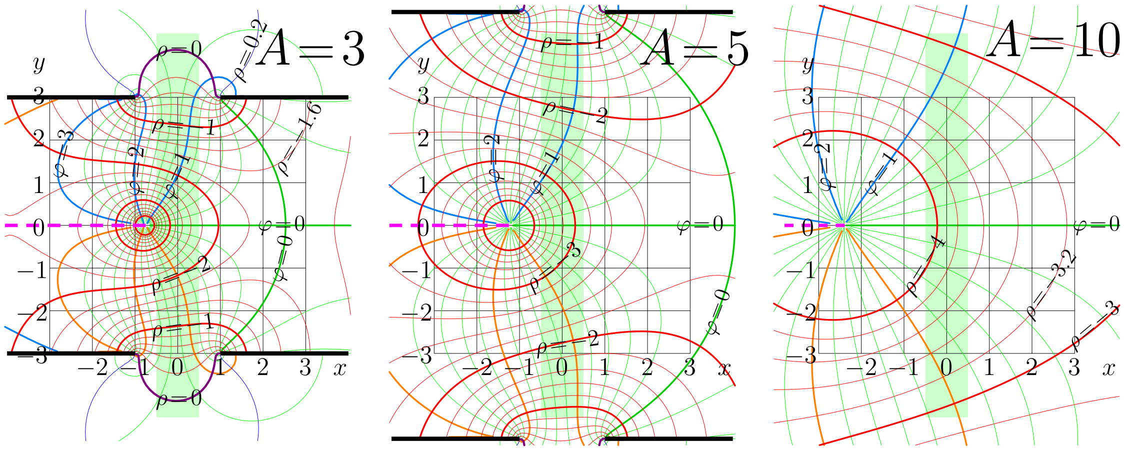
where

$$\begin{aligned} K(z) \;=\; & F_{\mathrm{up}} \cdot \left(\frac{1}{2} - \frac{1}{2\pi\mathrm{i}} \ln\frac{1-\mathrm{i}A+z}{1-\mathrm{i}A-z}\right) \\ + \; & F_{\mathrm{down}} \cdot \left(\frac{1}{2} - \frac{1}{2\pi\mathrm{i}} \ln\frac{1-\mathrm{i}A-z}{1-\mathrm{i}A+z}\right) \qquad [\mathrm{K}] \end{aligned} \tag{14.24}$$

This representation implies that the modulus of phase of function F along the imaginary axis remains less than π, so, the contour of integration does not cross the cut line of the logarithmic function in (14.24).

Equations (14.23),(14.24) are still exact. However, they become approximations, if we replace e $F_{\mathrm{up}} \to L$ and $F_{\mathrm{down}} \to L^*$. This replacement leads to the closed representation for K. We get the "solvable" integral equation for the approximation $F_A(\mathrm{i}y)$ of superfunction F along the imaginary axis:

$$F_A(\mathrm{i}y) = \frac{1}{2\pi}\int_{-A}^{A} \frac{\exp\!\big(F_A(\mathrm{i}p)\big)\,\mathrm{d}p}{1+\mathrm{i}p-\mathrm{i}y} - \frac{1}{2\pi}\int_{-A}^{A} \frac{\ln\!\big(F_A(\mathrm{i}p)\big)\,\mathrm{d}p}{-1+\mathrm{i}p-\mathrm{i}y} + K_A(\mathrm{i}y) \tag{14.25}$$



http://mizugadro.mydns.jp/t/index.php/File:Analuxp02t900.jpg

Figure 14.5: $\exp(\rho+\mathrm{i}\varphi)=K_A(x+\mathrm{i}y)$ by (14.26) for $A=3,5,10$

where

$$K_A(z) = L\cdot\left(\frac{1}{2}-\frac{1}{2\pi\mathrm{i}}\ln\frac{1-\mathrm{i}A+z}{1+\mathrm{i}A-z}\right) + L^*\cdot\left(\frac{1}{2}-\frac{1}{2\pi\mathrm{i}}\ln\frac{1-\mathrm{i}A-z}{1+\mathrm{i}A+z}\right) \tag{14.26}$$

Equations (14.25) и (14.26) include no unknown parameters; neither F_{up} not F_{down} appear there. In such a way, equations (14.25) and (14.26) can be used to calculate F_A.

Representation of the integral by the uppest and lowest pars of the contour Ω through function K_A by (14.26) is not trivial. The first look at the expression causes the seduction to rewrite each logatightm as difference of two logarithms, in order to simplify the expression in the argument. However, in other to get a robust representation for tetration, this is not a good idea. On the representation (14.26), the cuts are directed away from the imaginary axis; they go horizontally, parallel to the abscissa axis. This helps to avoid approaching of the argument of the primary approximation of tetration to the boundaries of holomorphism of function K_A. Complex map of function K_A is shown in figure 14.5 for $A=3$, $A=5$ and $A=10$ with lines of constant logamplitude ρ and constant phase ϕ such that $\exp(\rho+\mathrm{i}\varphi)=K_A(x+\mathrm{i}y)$ in the x, y plane.

Solution F_A of equation (14.25) can be approximated with the iterates, described below. At $A \gg 1$, solution F_A provides the good approximation for the superfunction $F \approx F_A$;

$$F(z) = \lim_{A\to\infty} F_A(z) \quad [\mathrm{tetF}] \tag{14.27}$$

In order to get tetration tet, the additional consition $\mathrm{tet}(0)=1.$ should be satisfied. So, I define

$$\mathrm{tet}(z) = F(x_0+z) \quad [\mathrm{tet}] \tag{14.28}$$

where x_0 is solution of equation $F(x_0)=1$. Value x_0 depends on the initial condition at the iterations, and also on the order of update of the values of the discrete approximation of function F_A. For the initial approximation by (14.15), the resulting x_0 happens to be or order of 0.1; and it is important, that it remains smaller than unity.

Formula (14.28) can be considered as "constructive definition" of tetration tet, with all reverences about the existence and the uniqueness of the limit, at $A\to\infty$ and the limit of inifinite increase of the number of points for the discrete approximation of the integrals. From the point of view of the "pure" mathematics, such a "definition" deserves critics, but I expect, that with time, the "pure mathematicians" present a more efficient and elegant way of constructive definition of tetration. For a while, tetration may remain as solution f of the transfer equation $\mathrm{e}^{f(z)}=f(z{+}1)$ bounded in the strip $|\Re(z)|<1$ with additional condition $f(0)=1$.

For the approximation of limit in equation (14.27), some finite value of A should be chosen. Value of increment $k=L\approx 0.318+1.337\,\mathrm{i}$ in approximation (14.15) indicates, that for the complex double implementation, the reasonable value of constant A should be of order of 20. It happened, that for $A=24$, the residual is a little bit smaller, and this value is used for the primary numerical implementation [4].

For the numerical implementation of equation (14.25), the integrals are replaced to their approximations with the Gauss-Legendre quadrature formula. Then, the resulting equation can be solved with iterations, updating values of the function one by one.

The attempt of the parallel assignment of the new values (that is easy to program with the high-level programming languages) leaded to the diverging algorithm. In order to get the convergence, I update first the odd nodes, and then the even ones. Then, after some teens of iterates, the procedure provides the accurate solution with 14 significant figures; this precision is estimated, evaluating the residual in various tests of the internal self-consistency. [5]

Solution of equation (14.25) approximates values of superfunction F

[4]Using the numerical implementation of the Cauchy integral for the first time, I did not guess the simple estimate through the increment k; so I had to increase value of A until the residual at the substitution of the primary approximation into the transfer equation (14.1) became of order of the rounding errors of the complex double arithmetics

[5]following Axiom 4 (see Introduction), I made certain efforts trying to refute, negate the concepts of existence and uniqueness of tetration.

along the imaginary axis. Then, equation (14.23) extends this approximation to the strip

$$-1 < \Re(z) < 1 \quad \text{[strip1]} \tag{14.29}$$

Accuracy of this primary approximation is poor in vicinity of the edges of the strip. Therefore, for the numerical implementation of tetration, the narrower strip is used,

$$-\frac{1}{2} \le \Re(z) \le \frac{1}{2} \quad \text{[striphalf]} \tag{14.30}$$

applying formula

$$F(z) = T^n\big(F(z-n)\big) \quad \text{[Tn]} \tag{14.31}$$

for some appropriate integer n, positive or negative, dependently on the sign of $\Re(z)$.

At the increasing of value of parameter A in equation (14.22), function F approaches to the solution of the transfer equation (14.1). This solution does not yet satisfy the condition $F(0)=1$, but, and the required tetration appears as the appropriate displacement of the argument with equation (14.28).

In figure 14.5, the strip $|x| \le 0.5$, $|y| \le 4.5$ is shaded; roughy, this is region, where the function K_A is used at the evaluation of superfunction F at the calculation of map in figure 14.4**d**. The complex maps verify, that the phase of reconstructed function does not exceed 2; and therefore, does not exceed π. This justifies use of logarithm in formula (14.23). However, this cannot be considered as a rigorous proof of existence and the uniqueness of the resulting function. I hope, the rigorous proof will be reported soon by the "pure" mathematicians. While I present only the computational evidence of the existence and the uniqueness.

In general case, applying this method to general transfer function T with complex fixed points, $F(z-n)$ could happen at the point of singularity of at the cut line of function T or T^{-1}. This would indicate that the resulting superfunction is also singular. (For tetration, this happens at the real argument, equal to or smaller than -2.) This may limit the range of applicability of the method suggested here.

In the first calculus, the approximation fit_3 had been used as the initial probe function for the iterational solution of equation (14.25). Then it happened, that the iterates with other (more primitive) probe functions lead to the same result, providing the same tetration. With the algorithm above, in 2008, the natural tetration had been constricted [54]. For $A=24$, the Gauss-Legendre quadrature formula with 2048 nodes gives the accurate approximation: at the substitution into equation (2.12), it gives the residual of order of 10^{-14}, while the variables complex(double) are used. Of order of 14 significant figures of the solution can be evaluated in real time. This indicates stability of evaluation of tetration through the Cauchy integral.

The algorithm above had been used to plot first naps and explicit plots of tetration to base $b > \exp(1/\mathrm{e})$ [104, 105]. In particular, this algorithm is used to plot the map at the bottom of figure 14.4. The direct representation through the Cauchy integral is fast enough to plot the maps and the explicit plots of tetration in real time. However, it becomes slow, if the tetration is used for evaluation of other function, for example, its superfunction (pentation) considered below in chapter 19. For the application of tetration, even faster approximations may have sense. One of them is considered in the next section.

5 Taylor expansion at zero

I wanted the check the claim, that solution $f = \mathrm{tet}$ of equations (14.1) and (14.1), that satisfies properties T1-T4, exists and is unique. As it is declared above, the numerical test does not substitute the rigorous proof, in the similar way, as the rigorous proof does not substitute the numerical tests. For the serious tests, it is important, that the function is fast to evaluate. In order to boost the evaluation, the approximations through the elementary functions had been suggested [64]. One of them refers to the Taylor expansion at zero. It is considered in this section.

Derivatives of tetration can be calculated by differentiation of the primary representation by (14.25). More accurate values can be obtained with the Cauchy integral formula with the circular contour of integration (assuming, that the primary representation is already implemented). Radius of this circle should be less than 2, and can be slightly larger than unity; then, the error of the result reduces due to denominator in the fraction under the integral in the right hand sidle of equation (14.21). In such a way, the coefficients in the expansion below are evaluated.

$$\mathrm{naiv}(z) = \sum_{n=0}^{N-1} c_n z^n \qquad \text{[vladinaiv]} \qquad (14.32)$$

$$\mathrm{tet}(z) = \mathrm{naiv}(z) + \mathcal{O}(z^N) \qquad \text{[vladinaiv2]} \qquad (14.33)$$

Table 14.1: Coefficients in expansions (14.32), (14.36) and (14.40)

n	c_n	s_n	$\Re(t_n)$	$\Im(t_n)$
0	1.00000000000000	0.30685281944005	0.37090658903229	1.33682167078891
1	1.09176735125832	0.59176735125832	0.01830048268799	0.06961107694975
2	0.27148321290170	0.39648321290170	−0.04222107960160	0.02429633404907
3	0.21245324817626	0.17078658150959	−0.01585164381085	−0.01478953595879
4	0.06954037613999	0.08516537613999	0.00264738081895	−0.00657558130520
5	0.04429195209047	0.03804195209047	0.00182759574799	−0.00025319516391
6	0.01473674209639	0.01734090876306	0.00036562994770	0.00028246515810
7	0.00866878181723	0.00755271038865	0.00002689538943	0.00014180498091
8	0.00279647939839	0.00328476064839	−0.00003139436775	0.00003583704949
9	0.00161063129058	0.00139361740170	−0.00001376358453	−0.00000183512708
10	0.00048992723148	0.00058758348148	−0.00000180290980	−0.00000314787679
11	0.00028818107115	0.00024379186661	0.00000026398870	−0.00000092613311
12	0.00008009461254	0.00010043966462	0.00000024961828	−0.00000013664223
13	0.00005029114179	0.00004090111776	0.00000007899707	0.00000003171468
14	0.00001218379034	0.00001654344436	0.00000000637479	0.00000002270476
15	0.00000866553367	0.00000663102846	−0.00000000341142	0.00000000512289
16	0.00000168778232	0.00000264145664	−0.00000000162203	0.00000000031619
17	0.00000149325325	0.00000104446533	−0.00000000038743	−0.00000000027282
18	0.00000019876076	0.00000041068839	−0.00000000001201	−0.00000000013440
19	0.00000026086736	0.00000016048059	0.00000000002570	−0.00000000002543
20	0.00000001470995	0.00000006239367	0.00000000000935	0.00000000000045
21	0.00000004683450	0.00000002412797	0.00000000000170	0.00000000000186
22	−0.00000000154924	0.00000000928797	−0.00000000000005	0.00000000000071
23	0.00000000874151	0.00000000355850	−0.00000000000016	0.00000000000012
24	−0.00000000112579	0.00000000135774	−0.00000000000005	−0.00000000000001
25	0.00000000170796	0.00000000051587	−0.00000000000001	−0.00000000000001

Evaluations of coefficients c are shown in the first column of table 14.1.

The Taylor expansion $\mathrm{tet}(z)$ at $z=0$ converges for $|z|<2$. The radius of convergence is determined by the distance from the point of expansion (id est, from zero) to the nearest singularity, which is -2.

For the numerical implementation, the number of terms chosen $N=50$. Complex map of the resulting naive approximation by (14.32) is shown in the left hand side of figure 14.6 with lines of constant real part and content imaginary part, $u+\mathrm{i}v = \mathrm{naiv}(x+\mathrm{i}y)$. The thick lines show levels $u=\Re(L)$ and $v=\pm\Im(L)$.

In order to verify the precision of the approximation $f = \mathrm{naiv}$, the central and the right hand side pictures of figure 14.6 show the maps of

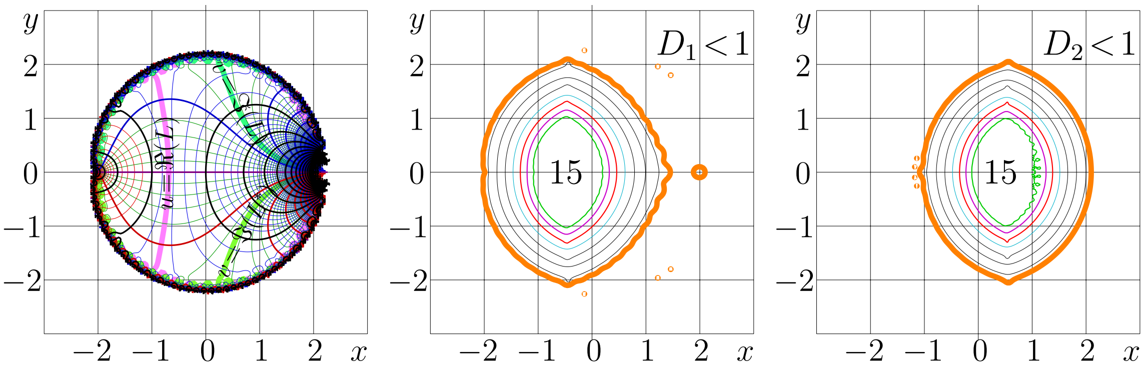


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Figure 14.6: $u+\mathrm{i}v = \mathrm{naiv}(x+\mathrm{i}y)$ by (14.32) for $N = 50$, left; agreements $D_1 = D_{\mathrm{naiv1}}(x+\mathrm{i}y)$ and $D_2 = D_{\mathrm{naiv2}}(x+\mathrm{i}y)$ by (14.34) and (14.35), central and right hand side maps. [vladi04]

agreements

$$D_{\mathrm{naiv1}}(z) = -\lg\left(\frac{|\ln(\mathrm{naiv}(z+1) - \mathrm{naiv}(z)|}{|\ln(\mathrm{naiv}(z+1)| + |\mathrm{naiv}(z)|}\right) \tag{14.34}$$

$$D_{\mathrm{naiv2}}(z) = -\lg\left(\frac{|\exp(\mathrm{naiv}(z-1) - \mathrm{naiv}(z)|}{|\exp(\mathrm{naiv}(z-1)| + |\mathrm{naiv}(z)|}\right) \tag{14.35}$$

Functions of agreement D indicate, how many significant figures of tetration can be expected to appear at evaluation of tetration with approximation "naive" by (14.32). Levels $D=1,2,4,6,8,10,12,14$ are shown. Level $D=1$ is shown with thick lines. Symbol "15" indicates the domain, where the agreement is better than 14. We may expect, at $|z|<1$, the polynomial by (14.32) provides of order of 14 significant figures; this is close to the maximal precision for variables complex double.

Evaluations with 50 terms is considered for verifiertion of the expansion. At the evaluation of tetration, for example, at the implementation for real argument, the number of terms can be significantly reduced without loss of precision.

The polynomial approximation naiv by (14.32) can be used for the precise and fast evaluation of tetration, while the modulus of its argument is smaller or of order of unity. For the efficient implementation, this is good, but it is not sufficient. In the next section, the advanced expansion is considered, that allows to extend the range of the approximation of tetration for moderate values of the argument.

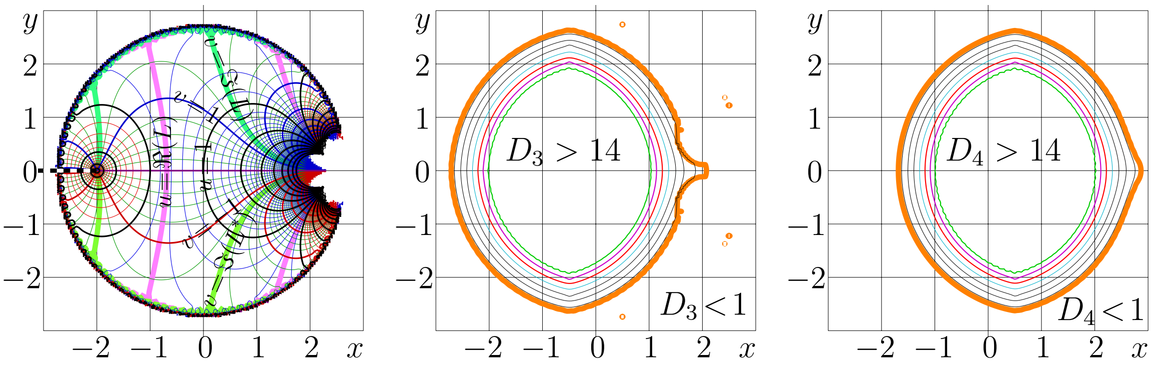

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Figure 14.7: u+iv=maclo(x+iy) by (14.36) at $N\!=\!101$, left; agreements D_3 и D_4 by (14.38) и (14.39), centre and right

6 Improved approximation

The range of the accurate approximation of tetration can be extended, if we take into account the logarithmic singularity of tetration. I "switch-out" the singularity at -2, expanding function $\mathrm{tet}(z) - \log(z\!+\!2)$ instead of $\mathrm{tet}(z)$. This expansion gives the approximation below, I call it "maclo":

$$\mathrm{maclo}(z) = \ln(z\!+\!2) + \sum_{n=0}^{N-1} s_n z^n \ ; \qquad [\mathrm{maclo}] \ (14.36)$$

$$\mathrm{tet}(z) = \mathrm{maclo}(z) + \mathcal{O}(z^N) \ . \qquad [\mathrm{macloN}] \ \ (14.37)$$

For $n = 101$, function maclo is shown in the left map of figure 14.7.

The series, used for approximation (14.36) converges at $|z| < 3$; the function reproduces the logarithmic branch point and even part of the cut at $z\!<\!-2$. Approximate values of first coefficients s are shown in the second column of table 14.1.

The range of approximation of tetration tet with function maclo is wider, than that by the Taylor expansion of tetration at zero; compare figure 14.6 and figure 14.7. The central and right hand side maps of figure 14.7 show agreements

$$D_3(z) = -\lg\left(\frac{\Big|\ln\!\big(\mathrm{maclo}(z\!+\!1)\big) - \mathrm{maclo}(z)\Big|}{\Big|\ln\!\big(\mathrm{maclo}(z\!+\!1)\big)\Big| + \Big|\mathrm{maclo}(z)\Big|}\right) \qquad (14.38)$$

$$D_4(z) = -\lg\left(\frac{\Big|\exp\!\big(\mathrm{maclo}(z\!-\!1)\big) - \mathrm{maclo}(z)\Big|}{\Big|\exp\!\big(\mathrm{maclo}(z\!-\!1)\big)\Big| + \Big|\mathrm{maclo}(z)\Big|}\right) \qquad (14.39)$$

Within the central loop, the residuals at the substitution $f \to \mathrm{maclo}$ into equations (14.1) are of order of 10^{-15}.

While $|z|<2$, the approximation maclo(z) with a hundred terms provides of order to 14 significant figures of tetration tet(z).However, while the module of the argument increases and becomes larger than two, the accuracy of this approximation quickly drops down. In order to even extend the range of the fast approximation, the Taylor expansion at some point at the imaginary axis can be used. In the next section, the Taylor expansion at point $3\,\mathrm{i}$ is described.

7 Expansion of $\mathrm{tet}(z)$ at $z=3\,\mathrm{i}$

For evaluation of tetration, we should cover some strip of unity width along the imaginary axis with good (fast and precise) approximations. The approximation maclo from the previous section does not approximate tetration at point $3\,\mathrm{i}$. For me, this is sufficient reason (or, may be, a pretext) to prepare the Taylor expansion of tetration namely in this point [6]. This expansion is described below.

The truncated Taylor expansion of tet(z) at point $z=3\,\mathrm{i}$ is denoted with name "tai" (TAylor expansion centered at the Imaginary axis):

$$\mathrm{tai}(z) = \sum_{n=0}^{N-1} t_n \,(z-3\,\mathrm{i})^n \qquad \text{[vladitai]} \qquad (14.40)$$

Approximations of the coefficients t are calculated with the Cauchy integral. The real and imaginary parts of the first coefficients are presented in the last two columns of table 14.1. The series converges at $|z-3\mathrm{i}| < \sqrt{2^2+3^2} = \sqrt{13} \approx 3.6$. For the numerical implementation I choose value $N=51$; then, at $|z-3\,\mathrm{i}|<2$, approximation tai by (14.40) provides of order of 14 significant figures. The complex map or this approximation is shown in the left hand side of figure 14.8.
The right hand side map in figure 14.8 shows the ageement

$$D_5(z) \;=\; -\lg\left(\frac{\Big|\ln\!\big(\mathrm{tai}(z\!+\!1)\big) - \mathrm{tai}(z)\Big|}{\Big|\ln\!\big(\mathrm{tai}(z\!+\!1)\big)\Big| + \Big|\mathrm{tai}(z)\Big|}\right) \qquad \text{[vladiD5]} \quad (14.41)$$

[6]Constant 3 appears as minimal integer number for which (with coefficient i) approximation maclo by (14.36) fails.

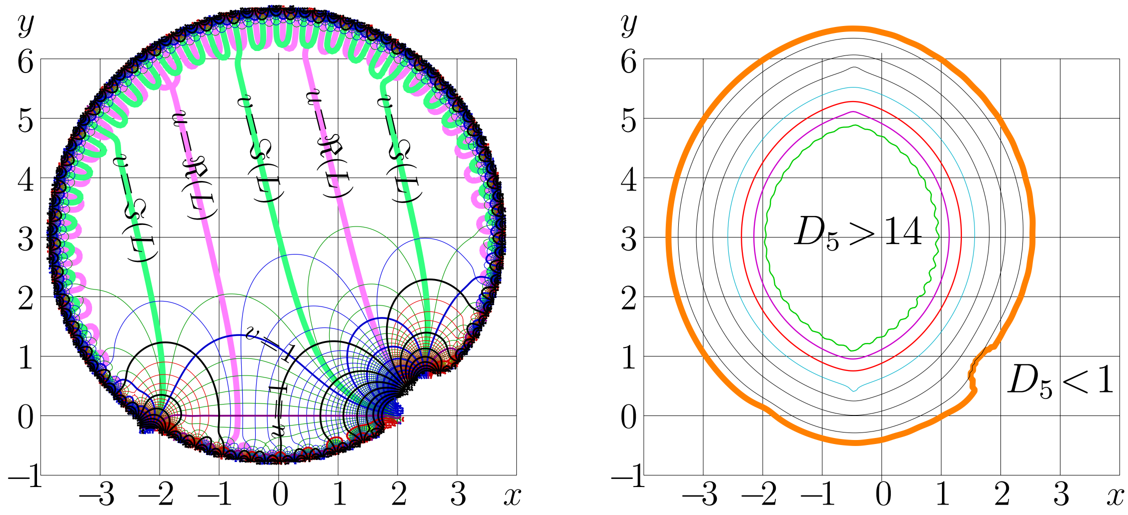

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Figure 14.8: $u+\mathrm{i}v=\mathrm{tai}(x+\mathrm{i}y)$ by (14.40) at $N=51$ and agreement $D_5(x+\mathrm{i}y)$

As in figures 14.6, 14.7, and 14.8, the levels for the agreement are drown with increment 2, beginning with 2; one additional level $D_5=1$ is shown with thick line. Inside the inner loop, the agreement with at least 14 digits takes place.

Approximation tai by (14.40) significantly extends the domain, where the tetration can be precisely evaluated through elementary functions. For positive values of $\Im(z)$, tetration $\mathrm{tet}(z)$ can be approximated with

$$\mathrm{tet}(z) \approx \mathrm{tai}(z) \tag{14.42}$$

For negative $\Im(z)$, tertian can be approximated with

$$\mathrm{tet}(z) \approx \mathrm{tai}(z^*)^* \tag{14.43}$$

These representations are sufficient to plot map in figure 14.4d. I assume, that the transfer equation (14.1) is applied some integer number of times, in order use $\mathrm{tai}(z)$ with the argument from the strip $|\Re(z)| \le 1/2$. However, the expansions above do not provide the accurate approximation of $\mathrm{tet}(z)$ at $|\Im(z)| > 5$.

One could extend the range of approximation, using the truncated Taylor expansions at point $5\,\mathrm{i}$ (or even $6\,\mathrm{i}$), this would significantly extend the range of approximation, and continue such an exercise with new and new points along the imaginary axis. However, there exist more intelligent and elegant way to deal with cases, when the imaginary part of the argument is large. This way is described in the next section.

8 Asymptotic expansion

The approximation of tet at large values of its argument can be build up using the asymptotic representation

$$\mathrm{tet}_{\mathcal{A}}(z) = L + \sum_{n,m} \mathcal{A}_{m,n} \exp\left(Lnz + \alpha mz\right) \qquad \text{[fimao]} \quad (14.44)$$

$L \approx 0.31813150520476413{+}1.3372357014306895\,\mathrm{i}$ is, as before, the fixed point of logarithm, $L{=}\ln(L)$, and $\mathcal{A}$ are constant coefficients.

Substitution $f = \mathrm{tet}_{\mathcal{A}}$ into the transfer equation (14.1) gives the chain of equations for coefficients $\mathcal{A}$. These equations do not determine $\mathcal{A}_{m,0}$; so, the solution still has the countable set of "free" parameters for natural m. Difficulty of determination of these parameters had been discussed in 1950 by Helmuth Kneser [10]. However, even a relatively small amount of terms taken into account in expansion (14.44) can be used for the precise approximation and evaluation of tetration at large values of the imaginary part of the argument.

Looking at the general (and a little bit ugly) expansion (14.44), I suggest the approximate, but more beautiful formula

$$\mathrm{fima}(z) = \sum_{n=0}^{N} a_n \varepsilon^n + \beta\varepsilon \exp(2\pi \mathrm{i} z)\ , \qquad \text{[fima]} \quad (14.45)$$

where the small parameter

$$\varepsilon{=}\exp(Lz{+}Lr) \qquad \text{[fimave]} \quad (14.46)$$

Mnemonics of name fima is following: Functional expansion for large IMAginary part of the argument. Substitution of $f(z) = \mathrm{fima}(z) + O(\varepsilon^{N+1})$ into the transfer equation (14.1) gives the coefficients

$$a_0 = L \approx 0.31813150520 + 1.33723570143\,\mathrm{i} \quad (14.47)$$

$$a_1 = 1 \quad (14.48)$$

$$a_2 = \frac{1/2}{L-1} \approx -0.1513148971 - 0.2967488367\,\mathrm{i} \quad (14.49)$$

$$a_3 = \frac{a_2 + 1/6}{L^2-1} = \frac{2+L}{6(L{-}1)(L^2{-}1)} \approx {-0.036976}{+}0.098730\,\mathrm{i} \quad (14.50)$$

$$a_4 = \frac{6+6L+5L^2+L^3}{24(L{-}1)^3(L{+}1)(L^2{+}L{+}1)} \approx 0.02581{-}0.01738\,\mathrm{i} \quad (14.51)$$

$$a_5 = \frac{24{+}36L{+}46L^2{+}40L^3{+}24L^4{+}9L^5{+}L^6}{120(L{-}1)^4(L{+}1)^2(1{+}L\,{+}2L^2{+}L^3{+}L^4)}$$
$$\approx -0.0079444196{+}0.00057925018\,\mathrm{i} \quad (14.52)$$

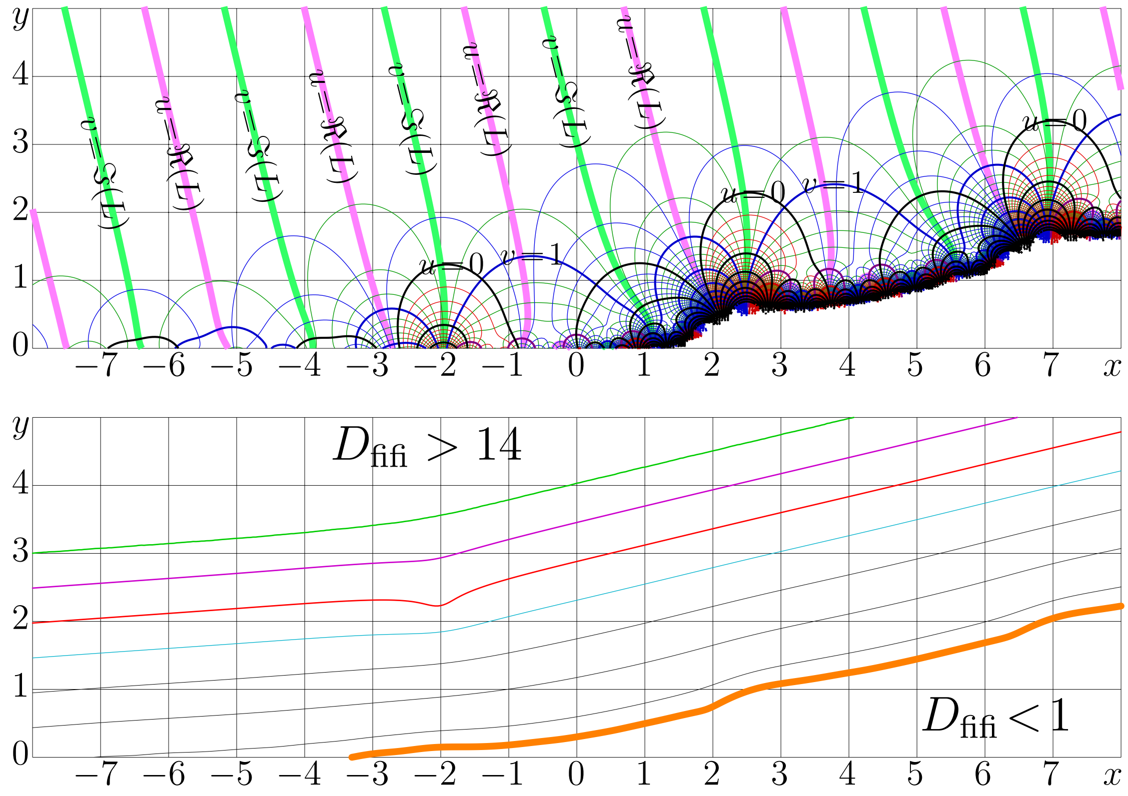

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Figure 14.9: Top: $u+\mathrm{i}v=\mathrm{fima}(x+\mathrm{i}y)$ by (14.45); bottom: map of $D_{\rm fifi} = D_{\rm fifi}(x+\mathrm{i}y)$ by (14.55) [figfima]

It is not difficult to take into account more terms, but $N=5$ already allows to cover the rest of the complex plane (not covered with approximations "maclo" and "tai") with accurate approximations of natural tetration.

Coefficients R and β in the right hand sides of formulas (14.45) and (14.46) remain as "adjusting parameters". Their values are chosen in order to approximate tetration, evaluated with a little bit slower Cauchy integral:

$$r \approx 1.0779614375280 - 0.94654096394782\,\mathrm{i} \qquad \text{[fimaR]} \quad (14.53)$$

$$\beta \approx 0.12233176 - 0.02366108 \quad \mathrm{i} \qquad \text{[fimaB]} \quad (14.54)$$

These values can be interpreted as approximations of the fundamental mathematical constants. I suggest to call them "the Kneser constants", as the expansion with these coefficients had been suggested in 1950 by H.Kneser [10]. Many digits in approximations of these constants can be calculated, in a way, similar to that in centuries 19 and 20 the mathematicians competed in precision of evaluation of number π.

Complex map of function fima is shown in the top picture of figure 14.9; the upper half of the complex plane is shown. This map should be compared to the map of tetration in figure 14.4 (although the levels

$\Re(L)$ and $\Im(L)$ are not drawn in figure 14.4) and to maps of other approximations in figures 14.6, 14.7 and 14.8

There is no fundamental limit on the precision of evaluation of tetration (for example, through the Cauchy integral), so, parameters β and r in should be considered as fundamental mathematical constants. The numerical computations, described in this book, have precision of order of 14 decimal digits (that is close to the best precision achievable with variable complex double), and parameter r is evaluated with the similar precision. Precision of evaluation of parameter β is not so high; perhaps, calculus with variables "long complex double" are necessary to improve the precision of evaluation of β and add more digits in the right hand side of equation (14.54).

In order to show the residual at the substitution $f = \mathrm{fima}$ into the transfer equation (14.1), figure (14.9) shows the agreement

$$D_{\mathrm{fifi}}(z) = -\lg\left(\frac{|\mathrm{fima}(z) - \exp(\mathrm{fima}(z-1))|}{|\mathrm{fima}(z)| + |\exp(\mathrm{fima}(z-1))|}\right) \qquad [\mathrm{fifi}]\ (14.55)$$

This agreement can be considered as an estimate, for how many orders of magnitude the value of the function is larger, than the error of its evaluation with approximation fima. As in the previous maps of agreement, the levels are drown with interval two orders of magnitude; only for level $D_{\mathrm{fifi}} = 1$, the exception is done; this level is shown with thick line. Below this level, the approximation fima does not reproduce even the qualitative behaviour of natural tetration. In the upper region, above the highest level, contrary, the approximation provides at least 14 significant figures, that is close to the maximal precision, achievable with variables complex double.

This section suggests the asymptotic approximation denoted with name "fima" by (14.45). Approximation fima is valid in the most of the upper part of the complex plane. Its conjugation $z \mapsto \mathrm{fima}(z^*)^*$ provides the approximation for the most of the lower part of the complex plane. With the transfer equation, these approximations can be extended also to the larger values of the real part of the argument. Together with approximations "maclo" and "tai", the whole complex plane happens to be covered with overlapping regions, and for each of these region, the efficient approximation based on the series expansion, is described. Now it would be methodically correct to analyse, verify the overlappings, agreement of these approximations. This overlapping is considered in the next section.

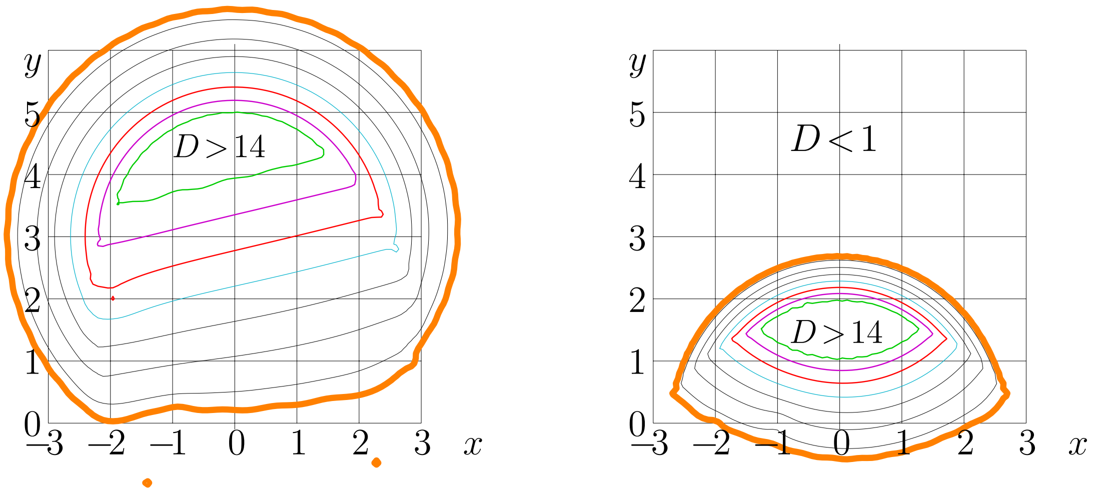

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Figure 14.10: Comparison of approximations tai by (14.40) to fima by (14.45) and to maclo by (14.36): agreements $D\!=\!D_6(x\!+\!iy)$ and $D\!=\!D_7(x\!+\!iy)$ by (14.56),(14.57) in the complex z-plane. [figco] [vladi07]

9 Comparison of approximations

On the base of representation of natural tetration through the Cauchy integral, the coefficients of various expansions of tetration are evaluated and the approximations with elementary functions are suggested. In this section, the mutual agreement or these representations is analysed.

The left hand side of figure 14.10 shows the agreement of approximation tai by (14.40) with approximation fima by (14.45):

$$D_6(z) = -\ln\left(\frac{|\mathrm{tai}(z) - \mathrm{fima}(z)|}{|\mathrm{tai}(z)| + |\mathrm{fima}(z)|}\right) \qquad \text{[vladiD6]} \quad (14.56)$$

The right hand side of figure 14.10 shows agreement of approximation tai by (14.40) with approximation maclo by (14.36)"

$$D_7(z) = -\ln\left(\frac{|\mathrm{tai}(z) - \mathrm{maclo}(z)|}{|\mathrm{tai}(z)| + |\mathrm{maclo}(z)|}\right) \qquad \text{[vladiD7]} \quad (14.57)$$

Figure 14.10 indicates, how to choose the appropriate approximation of tetration dependently on the imaginary part of the argument z at moderated values of $|\Re(z)|\!<\!1$. The boundary between the domains of the approximations should go through the loops, where $D > 14$. While $|\Im(z)| \le 1.5$, let approximation maclo be used; At $1.5 < \Im(z) \le 4.5$, let the approximation tai be used, and, at even larger values, let the evaluation be performed with approximation fima.

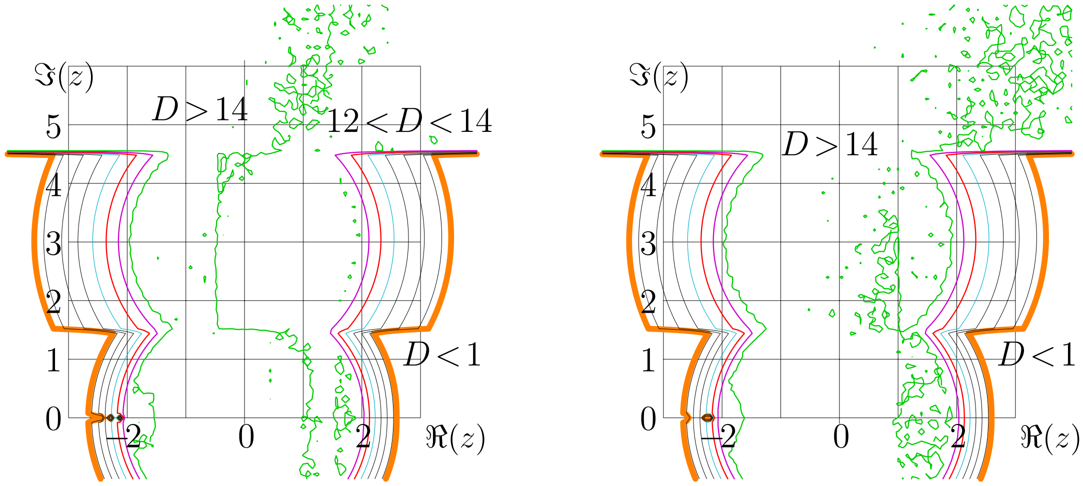

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Figure 14.11: Agreement $D = D_8$ by (14.59), left; the similar agreement for the contour integral with base domain shifted for -0.5 . [vladi08]

Looking at figure 14.10, I suggest the following approximation:

$$\mathrm{fse}(z) = \begin{cases} \mathrm{fima}(z) & , \quad 4.5 < \Im(z) \\ \mathrm{tai}(z) & , \quad 1.5 < \Im(z) \le 4.5 \\ \mathrm{maclo}(z) & , \, -1.5 \le \Im(z) \le 1.5 \\ \mathrm{tai}(z^*)^* & , \, -4.5 \le \Im(z) < -1.5 \\ \mathrm{fima}(z^*)^* & , \qquad\qquad \Im(z) < -4.5 \end{cases} \qquad \text{[fsexp]} \, (14.58)$$

This approximation can be compared to previous results. The left hand picture of figure 14.11 shows the agreement

$$D_8(z) = -\lg\left(\frac{|\mathrm{fse}(z) - F_4(z)|}{|\mathrm{fse}(z)| + |F_4(z)|}\right) \qquad \text{[DfseF4]} \qquad (14.59)$$

of approximation fse with the approximation F_4 obtained through the direct implementation of the contour integral.

Figure 14.11 reveals the defects of each approximation. The jumps at $\Im(z)\!=\!1.5$ and at $\Im(z)\!=\!2.5$ should be attributed to the transition from function maclo to function tai and from function tai to function fima in the combination FSE. Jumps at half-integer values of $\Re(z)$ should be attributed to the discontinuities of function F_4, which extends the approximation with the contour integral, valid for $|\Re(z)| < 1$, from the interval $|\Re(z)| \le 1/2$. The rounding errors appear as irregular dots. Within the strip $|\Re(z)| < 1.5$, the irregularities of all three approximations are of order of 10^{-14}.

The goal is to cover with efficient (fast and accurate) approximations at least the strip $\Re(z) \le 0.5$; then, values of natural tetration for the whole complex plane can be expressed through the transfer equation (14.1) in the right hand side of the complex plane, and through the ‘inverted” equation

$$\ln(\mathrm{tet}(z)) = \mathrm{tet}(z-1) \quad [\mathrm{rtanexp}] \tag{14.60}$$

in the right hand side. The left map in figure (14.11) indicates, that the goal is achieved; agreement with approximately 14 decimal digits takes place in significantly wider part of the complex plane. The approximations above are used for the fast implementation.

After to see the agreement discussed above, I had declared, that since now, the natural tetration can be evaluated so fast and so precisely, as other special functions, known since century 20. Then Henryk Trappmann asked me to make one additional numerical test. He vanted to see, wether the same tetration can be evaluated, if I misplpace the contour Ω in the Cauchy integral [103], moving it to the right. I recognised this as a trap (which would correspond to the last name of Henryk): if we displace the contour to the right, the derivatives of tetration becomes larger, and, with the same algorithm, we get lower precision. But I agreed to displace the contour for 1/2 to the left.

With the displaced contour, the same ab initio evaluation of tetration had been performed. Tetration $\mathrm{tet}(-1/2+iy)$ for real y had been evaluated; then, with the Cauchy integral and equations (14.1),(14.60), the approximation had been extended to the whole complex plane, in the similar way as with the first algorithm of evaluation of tetration [54]. The result is compared to the approximations with expansions in the way, similar to that of by (14.59); the new approximation is used instead of F_4. The resulting agreement is shown in the right hand side map in figure (14.11).

Figure 14.11 reveals defects of approximations mentioned above. The discontinuities in formula (14.58) are seen with horizontal jumps along lines $\Im(z) = 1.5$ and $\Im(z) = 4.5$, that are clearly shown with concentrated levels. Discontinuities of the initial, “primary” approximation appear with the vertical jumps along half-integer values of $\Re(z)$. The similar discontinuities are seen also for the evaluation with displaced contour at integer values of $\Re(z)$. All these jumps of the compared approximations are at the level of 10^{-14}, and this confirms the declared estimate of the precision of the evaluation of the natural tetration.

The agreement in the right hand side of figure 14.11 happened to be even better, than that in the left hand side. Henryk had been satisfied with that test. For the natural tetration, the new contour of integration happened to be a little bit better, than the initial choice. In such a way, the analogy with lazy engineer (or with stupid selfish tyrant), mentioned above, gets the confirmation: the initial contour of integration in the original publication [54] is not best. However, I still think, that the simplicity of that contour and the good agreement (figure 14.11) should be considered as some kind of excuse for the voluntaristic choice of the contour.

10 Implementation

After the tests, described in the previous section, for the numerical implementation, the following approximation is used: $\mathrm{tet}(z) \approx \mathrm{FSE}(z)$, with

$$\mathrm{FSE}(z) = \left\{ \begin{array}{ccccc} \mathrm{FIMA}(z) & , & 4.5 < & \Im(z) & \\ \mathrm{TAI}(z) & , & 1.5 < & \Im(z) & \le 4.5 \\ \mathrm{MACLO}(z) & , & -1.5 \le & \Im(z) & \le 1.5 \\ \mathrm{TAI}(z^*)^* & , & -4.5 \le & \Im(z) & < -1.5 \\ \mathrm{FIMA}(z^*)^* & , & & \Im(z) & < -4.5 \end{array} \right. \qquad \text{[FSE]} \quad (14.61)$$

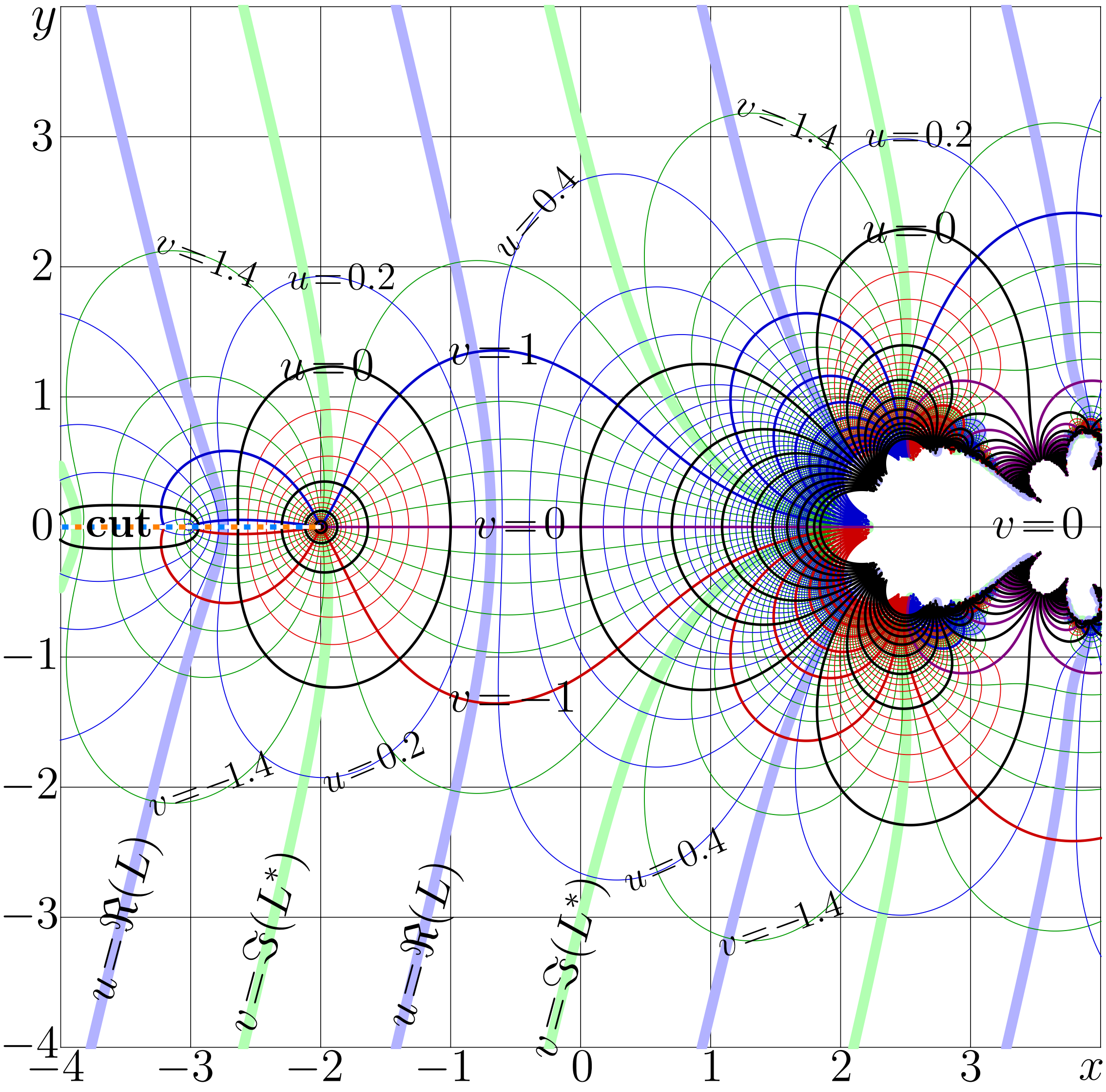
where

$$\mathrm{FIMA} = \left\{ \begin{array}{ll} \mathrm{fima}(z) & , \ \Im(z) > 4{+}0.2379\,\Re(z) \\ \exp(\mathrm{FIMA}(z{-}1)), & \Im(z) \le 4{+}0.2379\,\Re(z) \end{array} \right. \qquad (14.62)$$

$$\mathrm{TAI} = \left\{ \begin{array}{lccc} \mathrm{tai}(z) & , \ |\Re(z)| & \le & 0.5 \\ \log(\mathrm{TAI}(z{+}1)) & , \ \Re(z) & < & -0.5 \\ \exp(\mathrm{TAI}(z{-}1)) & , \ \Re(z) & > & 0.5 \end{array} \right. \qquad \text{[TAI]} \quad (14.63)$$

$$\mathrm{MACLO} = \left\{ \begin{array}{lccc} \mathrm{tai}(z) & , \ |\Re(z)| & \le & 0.5 \\ \log(\mathrm{MACLO}(z{+}1)), & \Re(z) & < & -0.5 \\ \exp(\mathrm{MACLO}(z{-}1)), & \Re(z) & > & 0.5 \end{array} \right. \qquad (14.64)$$

This approximation provides of order of 14 correct significant figures of the holomorphic tetration tet and agrees with the previous results [54]. Up to my knowledge, up to year 2016, function FSE above is the most precise and the fastest among ever reported approximations of the tetrational. Mnemonics of the name FSE is obvious: Fast Super Exponent. The C++ implementation of this algorithm is loaded as



http://mizugadro.mydns.jp/t/index.php/File:Tetma.jpg

Figure 14.12: $u+\mathrm{i}v=\mathrm{tet}(x+\mathrm{i}y)$ [tetmap]

http://mizugadro.mydns.jp/t/index.php/Fsexp.cin ; this approximation is used to plot the detailed map of tetration in figure 14.12, used also for the cover of this Book.

Many terms are kept in the approximations (14.40) and (14.36) in order to provide the wide range of the overlapping in figures 14.10 and 14.11. At the final step of the implementation, the number of terms can be reduced, boosting the algorithm, without loss of the precision. In particular, this applies to the evaluation of tetration along the real axis: it is sufficient to approximate $\mathrm{tet}(z)$ for $|z| \le 1/2$, using only a quarter of the radius of the precise approximation with function maclo.

For iterates of the exponent, the inverse function, id est, arctetration, or abelexponent, is also required. This arctetration is considered in the next chapter.

Chapter 15

Natural arctetration

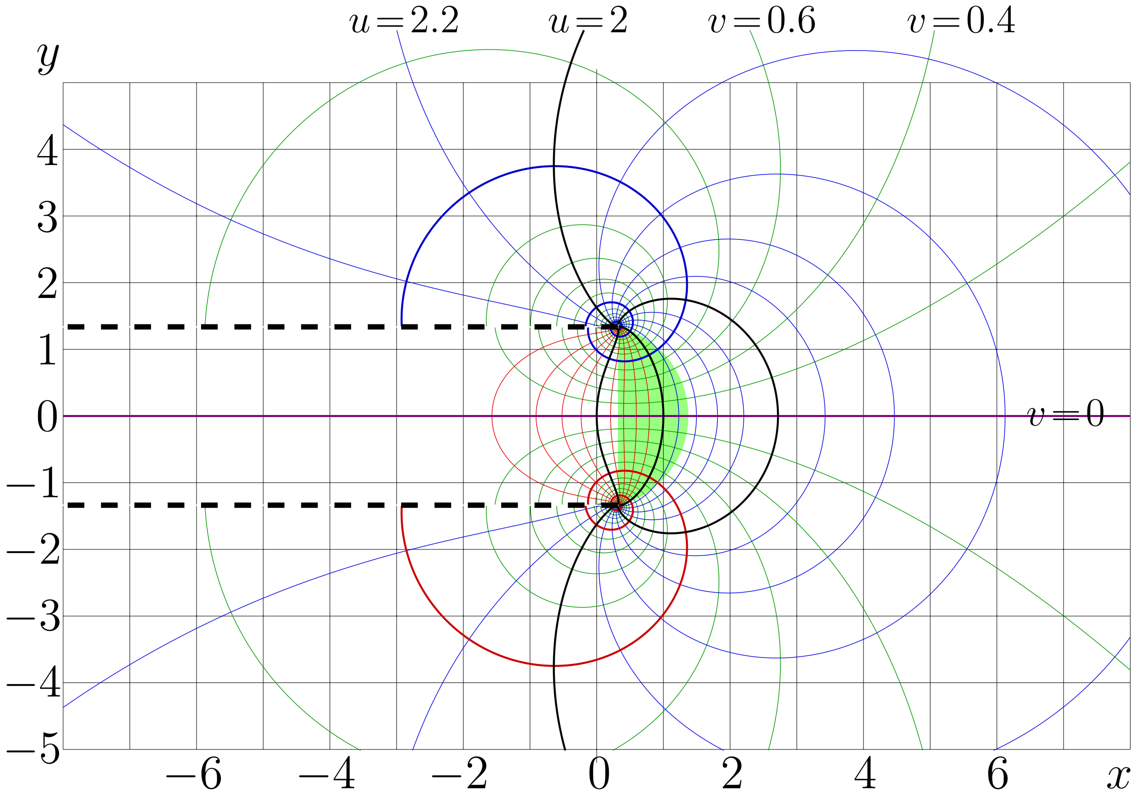


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Figure 15.1: $u+\mathrm{i}v = \mathrm{ate}(x+\mathrm{i}y)$ [vladi02c] [figsexpG]

The inverse function of tetration, id est, arctetration, or abelexponent, is denoted with name ate; ate $=$ tet^{-1}. Complex map of arctetration is shown in figure 15.1.

Arctetration satisfies the Abel equation

$$\mathrm{ate}\Big(\exp(z)\Big) = \mathrm{ate}(z) + 1 \quad \text{[abelate]} \tag{15.1}$$

and the additional condition

$$\mathrm{ate}(1) = 0 \quad \text{[abelate10]} \tag{15.2}$$

Properties of functions ate and the algorithm for the evaluation are described in this chapter.

1 Evaluation of arctetration

Arctetration can be evaluated as inverse function of tetration, using the Newton method. Function $\mathrm{ate}(z)$ appears as limit of sequence g_n with the recurrent relation

$$g_{n+1} = g_n + \frac{\mathrm{tet}(g_n) - z}{\mathrm{tet}'(g_n)} \quad \text{[atenewton]} \qquad (15.3)$$

The derivative of tetration can be approximated, differentiating the approximations of tetration with elementry functions described in the previous section. The representation through the Cauchy integral [54] also allows the straightforward differentiation. However, in this case, several iterates by (15.3) are required to evaluate the arctetration.

Evaluation of arctetration through tetration using equation (15.3) is significantly slower, than evaluation of tetration. In addition, the initial approximation g_0 should be specified. This specification should carry about the cutlines. In Figure 15.1, these cut lines are drawn parallel to the real axis. Over-vice, the recurrency by (15.3) returns a value from any of branches of the corresponding multivalued function, and the question about the range of holomorphism becomes difficult.

In order to get efficient approximation for the arctetration ate, I deal with the corresponding Abel equation (15.1), rather than with recurrences by (15.3). It worth to approximate arctetration with some function, which reproduce at least the leading terms of the asymptotic expansion of ate. This approximation is constructed below.

Arctetration, as solution of the Abel equation (15.1), should have singularities in the fixed points of logarithm L and L^*. From the precious chapter, we already know, that the dominant term of the asymptotic expansion appears as fixed point plus the corresponding exponential. This indicates, that the corresponding expansion of arctetration should begin with logarithm. The efficient approximation of arctetration can be obtained through the expansion of function h by

$$h(z) = \mathrm{ate}(z) - \frac{\ln(z-L)}{L} - \frac{\ln(z-L^*)}{L^*} \quad \text{[atelo]} \qquad (15.4)$$

Function h can be expanded to the Taylor series at unity. This expansion leads to the approximation

$$\mathrm{fsl}(z) = \frac{\ln(z-L)}{L} + \frac{\ln(z-L^*)}{L^*} + \sum_{n=0}^{N-1} u_n\,(z-1)^n \quad \text{[fsl]} \qquad (15.5)$$

$$\mathrm{ate}(z) = \mathrm{fsl}(z) + O(z-1)^N \qquad (15.6)$$

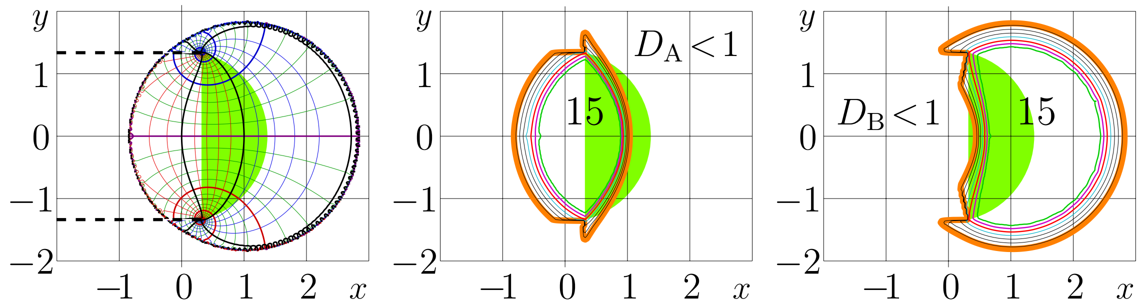

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Figure 15.2: $u+\mathrm{i}v=\mathrm{slo}(x+\mathrm{i}y)$ by (15.5), left, and the agreements by $D_{\rm A}=D_{\rm A}(x+\mathrm{i}y)$, $D_{\rm B}=D_{\rm B}(x+\mathrm{i}y)$, by formulas (15.7), (15.8)

Approximations of the first 30 coefficients of this expansion are shown in table 15.1. Complex map of function fsl by (15.5) at $N=70$ is shown in the left hand side picture of figure 15.2 in the same notations, as in figure 15.1. The central part of left map in figure 15.2 looks as a fragment from figure 15.1.

Table 15.1: Coefficients u_n in expansion (15.5).

n	u_n	n	u_n	n	u_n
0	1.41922521550451	10	0.00000003111805	20	0.00000000002293
1	−0.02606629029752	11	0.00000002940887	21	−0.00000000002462
2	0.00173304781808	12	−0.00000001896929	22	0.00000000000666
3	−0.00001952130725	13	0.00000000351784	23	0.00000000000322
4	−0.00006307006450	14	0.00000000204270	24	−0.00000000000354
5	0.00002567895998	15	−0.00000000171995	25	0.00000000000096
6	−0.00000559010027	16	0.00000000039882	26	0.00000000000051
7	−0.00000007279712	17	0.00000000019328	27	−0.00000000000055
8	0.00000065148872	18	−0.00000000019113	28	0.00000000000014
9	−0.00000027698138	19	0.00000000004947	29	0.00000000000009

Range of validity of approximation 15.5 is limited. In order to show this range, the central and right hand side maps in figure 15.2 show the agreements

$$D_{\rm A}(z) \;=\; -\lg\!\left(\frac{|\mathrm{fsl}(\exp(z))-1-\mathrm{fsl}(z)|}{|\mathrm{fsl}(\exp(z))-1|+|\mathrm{fsl}(z)|}\right) \qquad \text{[sloE]} \quad (15.7)$$

$$D_{\rm B}(z) \;=\; -\lg\!\left(\frac{|\mathrm{fsl}(\ln(z))+1-\mathrm{fsl}(z)|}{|\mathrm{fsl}(\ln(z))+1|+|\mathrm{fsl}(z)|}\right) \qquad \text{[sloL]} \quad (15.8)$$

Inside the inner loops in the central and right hand side pictures of figure 15.2, the agreement is better than 14. These domains are marked with symbol "15". Figure 15.2 indicates, that at $|z{-}1|<1.4$, the approximation $\mathrm{fsl}(z)$ provides of order of 14 significant figures. The precision of this approximation is a little bit worse in vicinity of fixed points L and L^*. This could be expected: First, there, the small variation of argument leads to the significant variation of value of function. Second, these points are at the edge of the range of convergence of expansion in equation (15.5).

For implementation of natural arctetration, it is sufficient to prepare the efficient primary approximation for the domain sickle, defined with

$$\mathrm{sickle} = \{z \in \mathbb{C} : \Re(z) \geq L, |z| < L\} \qquad [\text{sicle}] \qquad (15.9)$$

then, for other values of the argument, function can be expressed through the Abel equation (15.1). The region sickle is shaded in maps of figures 15.1, 15.2. This leads to implementation, denoted with FSL. I would like to check the self-consistency of implementation FSL with implementation FSE by (14.61). The numerical test of the relations below had been performed:

$$\mathrm{ate}(\mathrm{tet}(z)) = z \qquad [\text{atetet}] \qquad (15.10)$$

$$\mathrm{tet}(\mathrm{ate}(z)) = z \qquad [\text{tetate}] \qquad (15.11)$$

These relations are tested for the complex double implementations $\mathrm{tet}\approx$ FSE and $\mathrm{ate}\approx$FSL. Figure 15.3 shows the maps of the agreements

$$D_{\mathrm{at}}(z) = -\ln\left(\frac{|\mathrm{FSL}(\mathrm{FSE}(z)) - z|}{|\mathrm{FSL}(\mathrm{FSE}(z))| + |z|}\right) \qquad [\text{Dat}] \qquad (15.12)$$

$$D_{\mathrm{ta}}(z) = -\ln\left(\frac{|\mathrm{FSE}(\mathrm{FSL}(z)) - z|}{|\mathrm{FSE}(\mathrm{FSL}(z))| + |z|}\right) \qquad [\text{Dta}] \qquad (15.13)$$

In figure 15.3, the left hand side map shows $D = D_{\mathrm{at}}(x+\mathrm{i}y)$ and the right hand side map shows $D = D_{\mathrm{ta}}(x+\mathrm{i}y)$ in the x, y planes. The levels $D = \mathrm{const}$ are drawn with interval 2; as in other maps of agreement, the additional level $D = 1$ is shown with thick line. This line indicates the boundary of the range of validity of relation (15.10).

As for many other inverse functions, the range of validity to relation (15.10) is limited. Width of the strip, where the relation (15.10) takes place, is determined by the asymptotic periodicity of tetration in the upper and the lower parts of the complex plane. In vicinity of the real axis, the strip becomes wider, showing some kind of along the real axis to infinity. Thickness of this "beak" reduces quickly at the growth of the real part of the argument.

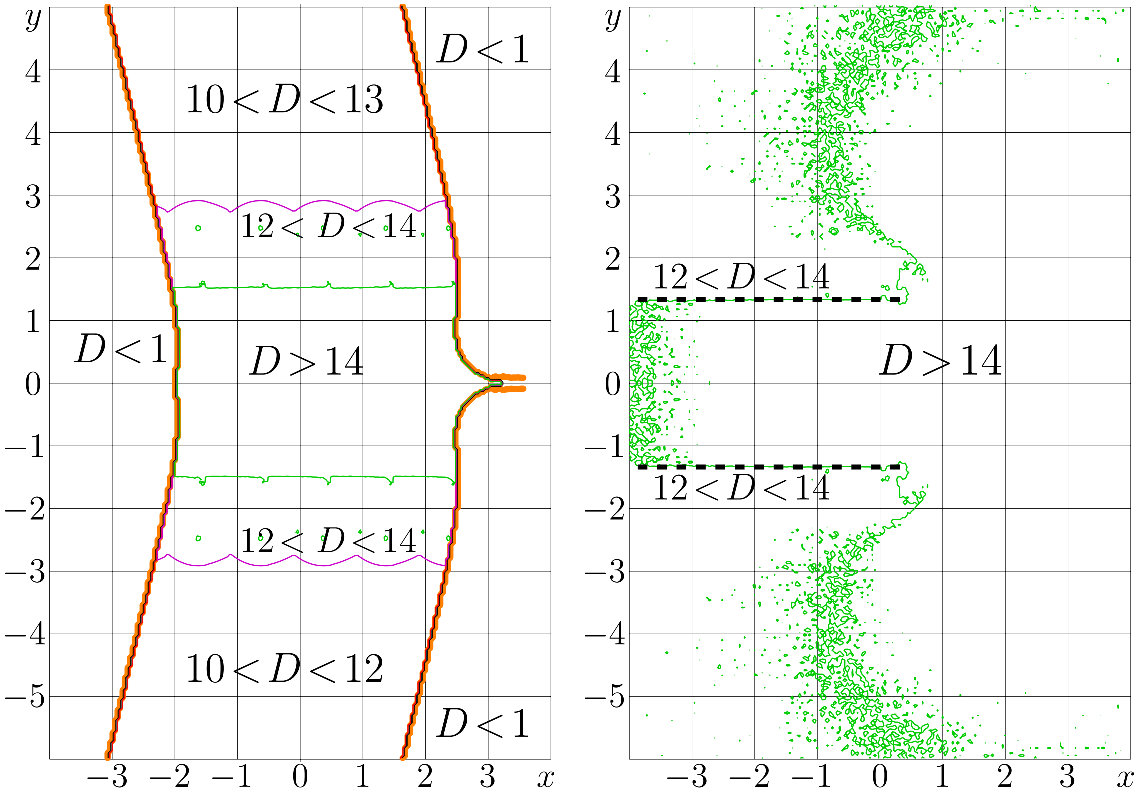

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Figure 15.3: $D = D_{\rm at}(x+{\rm i}y)$, left, and $D = D_{\rm ta}(x+{\rm i}y)$, right, by (15.12) and (15.13) [vladi11]

The numerical tests confirm, that the approximations of tetration and arctetration are self-consistent. The complex double implementations provide of order of 14 significant figures.

2 About names

The numerical implementations of algorithm FSE and FSL described above are loaded to TORI as
`http://mizugadro.mydns.jp/t/index.php/Fsexp.cin` and
`http://mizugadro.mydns.jp/t/index.php/Fslog.cin`
The names of these routines are discussed in this section.

Names of function FSEXP and FSLOG are historic. They may mean "Fast Super EXPonential" and "Fast Super LOGarithm. Hernryk Trappmann even wanted to add my last name to the identifier of each of these two functions. He had believed, that "my tetration" is not unique, and not so principal, as I claim, using names "tet" and "ate". Henryk wanted to use names "tet" and "ate" for the "true" tetration and arctetration, "more natural", more "true", than the functions I had constricted.

After the long discussion and the heavy deduction, Henryk had to agree, that the functions I have constructed are unique [73], and, in this sense, true and the only true tetration and arctetration. The routines FSEXP and FSLOG were already implemented that time, and I decided not to change the notations: the poor sistem of notation is still better and causes less confusions, than two "good" systems of notations.

I need to mention, that name FSLOG (Fast Super LOGarithm) is even more idiotic, than FSEXP. Name FSLOG makes impression, that it is superfunction of logarithm, while it is not really so. Superfunction of natural logarithm van be written as

$$z \mapsto \mathrm{tet}(-z) \tag{15.14}$$

I think, this function does not deserve to have a special name assigned.

I am far from eugenic ideas to refine the human rase, nor the system of notations. The best system of notations should vin the competition with other notations. All this should be considered as my excuse to keep names FSEXP and FSLOG for the approximations and the numerical implementations of tetration tet and arctetration ate.

Many superfunctions of natural exponent can be constructed with transformation (2.17), just misplacing the argument of tetration with some periodic real-holomorphic function. The range of holomorphism of these transforms is narrower, than the range of holomorphism of tetration. Now, I see no need to give them special names.

I expect, in future, even more efficient implementations for tetration will be suggested. Then, they may be called with the same names, as the name of the functions tet and ate, in the same way, as in the algorithmic languages the implementation of sin is denoted with the same name as the function.

After to eliminate the potential confusion with names, the tetration and arctetration can be used for the iterates of the exponent. These iterates are described in the next section.

3 Iterates of exponent

Tetration tet and arctetration ate, as superfunction and abelfunction of exponent, specify, determine the non-integer iterates:

$$\exp^n(z) = \mathrm{tet}(n + \mathrm{ate}(z)) \quad [\mathrm{expn}] \tag{15.15}$$

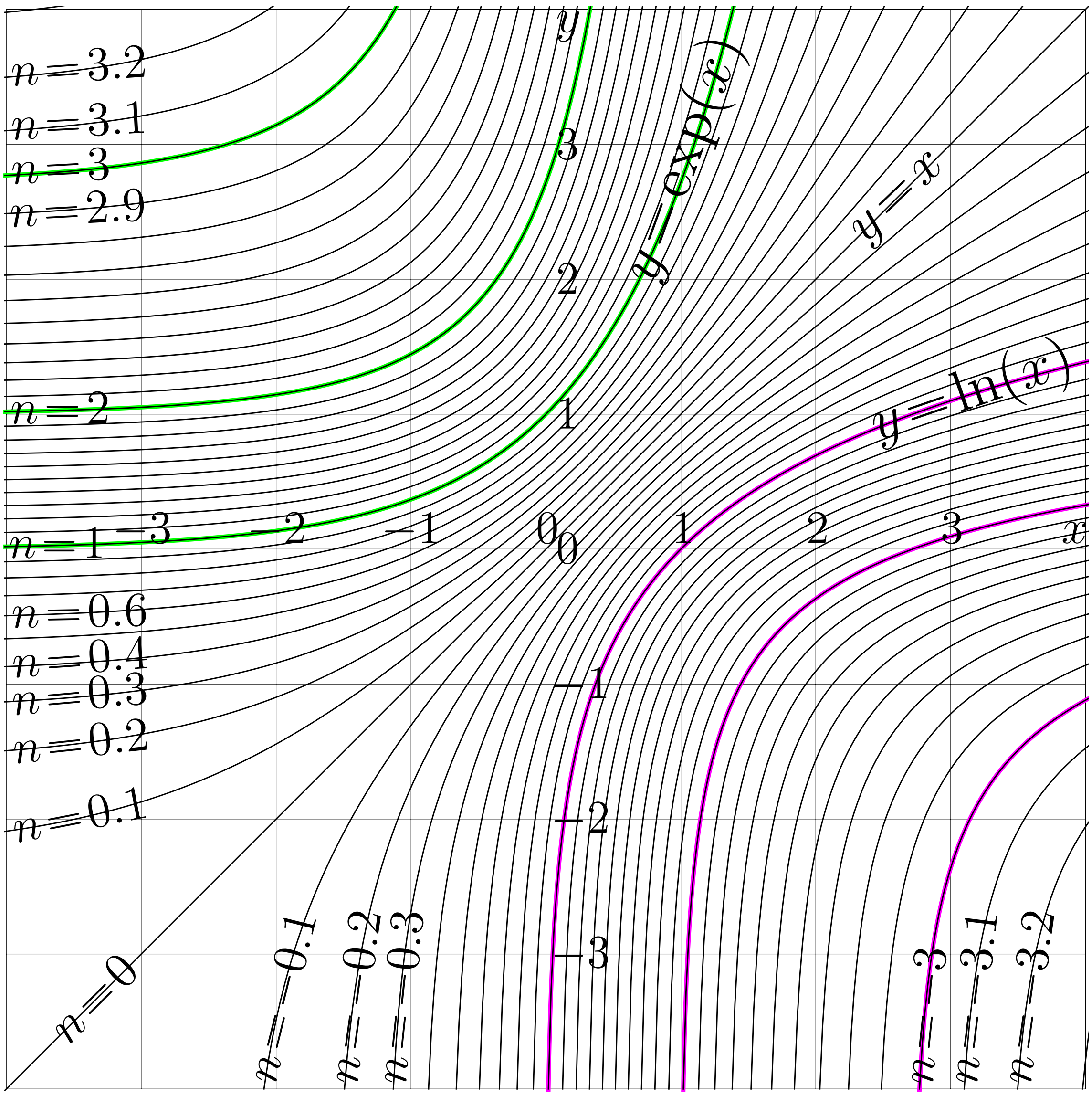

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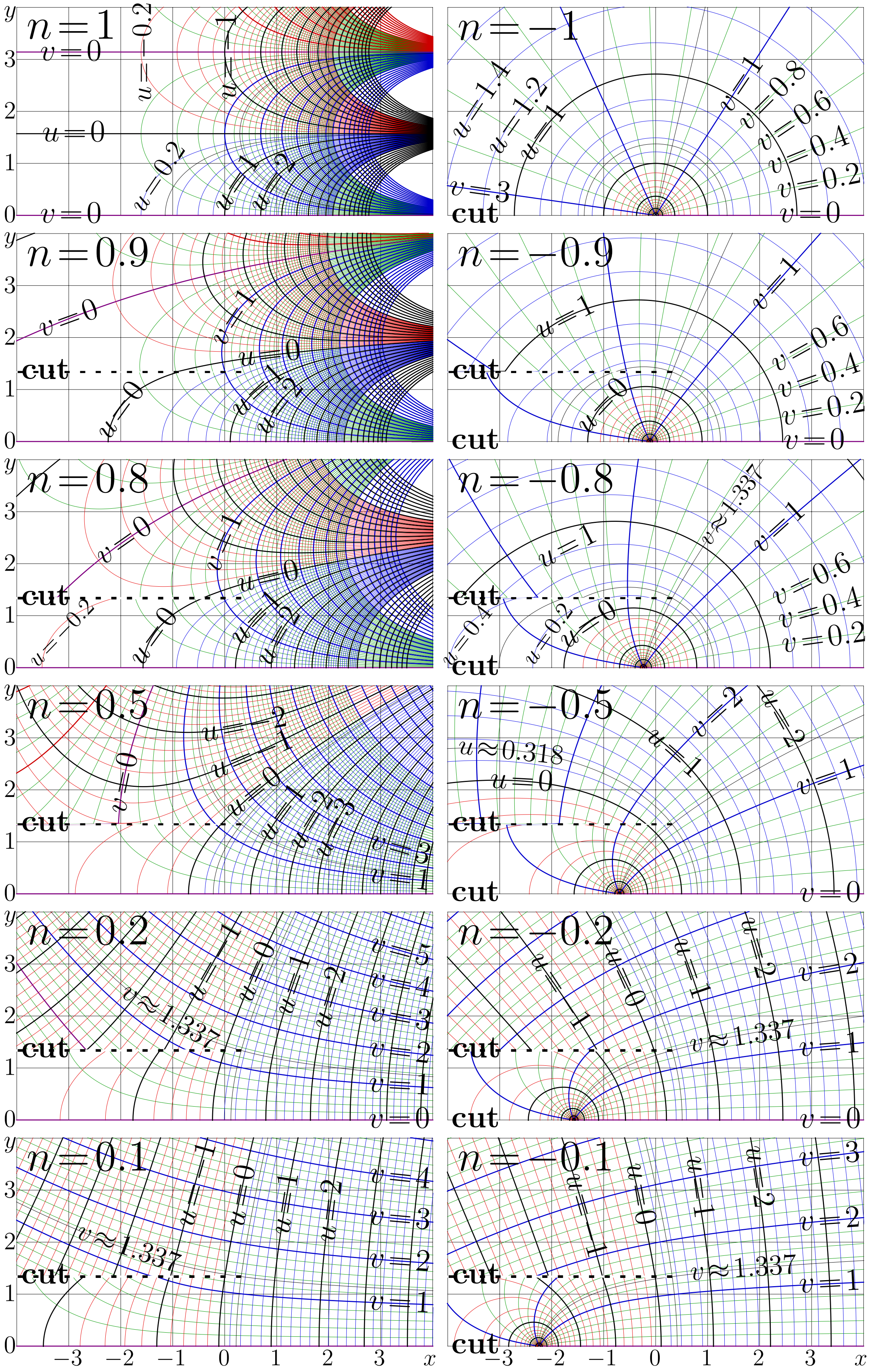
Figure 15.4: $y\!=\!\exp^n(x)$ by (15.15) for varuous n [expiteplot]

Here, number n of iterates, has no need to be integer (although, of course, can be integer too). For real values of argument, iterates of exponent by equation (15.15) are shown in figure 15.4, $y = \exp^n(x)$. Lines, that correspond to integer n (except $n = 0$), are thick. These lines correspond to $y\!=\!\exp(\exp(\exp(x)))$, $y\!=\!\exp(\exp(x))$, $y\!=\!\exp(x)$, $y\!=\!\ln(x)$, $y\!=\!\ln(\ln(x))$, $y\!=\!\ln(\ln(\ln(x)))$. Higher integer iterate happen to be out of range of the figure.

Complex maps of iterates of the exponent are collected in figure 15.5. Twelve maps are shown for

$$u\!+\!\mathrm{i}v = \exp^n(x\!+\!\mathrm{i}y) \quad \text{[uxexpxy]} \qquad (15.16)$$

with lines $u =$ const and lines $v =$ const in the x, y plane for various values of the number n of iterate; this n is printed with big font in the upper left corner of each map. The maps are symmetric with respect



http://mizugadro.mydns.jp/t/index.php/File:Expitemap.jpg

Figure 15.5: Maps of iterates of natural exponent by (15.16)

to reflection from the abscise axis (the only, the imaginary part of the function changes its sign). So, the only upper part of the complex plane is presented in each map.

Maps at the top of figure 15.5 correspond to $n=1$ and $n=-1$; these are complex maps of exponent and of logarithm. First of them reproduces part of figure 14.2. The exponent is holomorphic in the whole complex plane, but the logarithm has branch point at zero and the cut along the negative part of the real axis.

The second and following rows of the figure represent the non-integer iterates. These iterates have two additional cuts along the lines $y = \pm\Im(L)$; here $L \approx 0.3181315 + 1.3372357\,\mathrm{i}$ is fixed point of logarithm, id est, solution of equation $L = \ln(L)$. By default, all the cut lines are directed parallel to the real axis (axis x in the figure, abascissa) toward the negative direction of the real axis. In such a way, for negative non-integer n, the map has 3 cut lines (and that in the lower half-plane is not seen, as it is out of field of view of the map).

The thick lines in figure 15.5 corresponds to the integer values of u or v. The thin lines are drawn with interval 0.2; the additional lines $u = \Re(L)$ and $v = \Im(L)$ are also drawn. These lines always cross each other at the fixed point L.

Figure 15.5 shows the gradual transition of the map for the exponential (top of the left column) the map for the logarithm (top of the right hand side column). As the number n of iterate reduces from unity to zero, the web of the lines $u = \mathrm{const}$ and lines $v = \mathrm{const}$ rotate around the fixed point L, and become uniform rectangular grid at $n = 0$. At this value, the horizontal cuts along lines $\pm\Im(L)$ disappear, but they appear again, as n becomes negative non-integer. At $n < 0$, the additional branch point comes from $-\infty$ at the real axis and moves toward zero, as n becomes minus unity. With integer n, the branch points L and L^* disappear.

Maps of non-integer iterates can be plotted also for other transfer functions, considered in this Book. The readers are invited to download the implementations of the superfunctions and the abelfunctions, and plot the corresponding complex maps of the iterates.

4 Lessons of natural tetration and arctetration

On the base of representation of tetration through the Cauchy integral, through the solution of the integral equation (14.22), one can express also derivatives of this function; differentiation of the integrands in the right hand side of (14.22) is straightforward. Namely in this way, the derivatives of tetration for real and for pure imaginary values of the argument had been evaluated for the tables 1 and 2 in publication [54]. In this section, I suggest some philosophic speculations about tetration and arctetration.

Complex map of tetration tet is shown in figure 14.12, and its behaviour along the real axis is shown in figure 14.1. Properties (14.16)-(14.20) first were observed with various approximations of tetration with elementaty functions, and then postulated. The approximations reproduce values of tetration in vicinity of integer values of the argument,

$$\begin{aligned} \mathrm{tet}(-2+\varepsilon) &= \log(\varepsilon) + \mathrm{const} + O(\varepsilon) && (15.17)\\ \mathrm{tet}(-1) &= 0 && (15.18)\\ \mathrm{tet}(0) &= 1 && (15.19)\\ \mathrm{tet}(1) &= \mathrm{e} && (15.20)\\ \mathrm{tet}(2) &= \mathrm{e}^2 && (15.21) \end{aligned}$$

Then, the agreement at the substitution of the fitting function into the transfer equation (14.1) had been minimised for complex values of the argument.

The behaviour similar to properties (14.16)-(14.20) had been detected with various fitting functions. Then these properties were formulated as definition of tetration, id est, just postulated. First, I did not expect this set of postulates to be self-consistent. Contrary, I tried to find some contradiction; I expected to use such a contradiction as a proof of non-existence of holomorphic tetration. Such a non-existence would be an upgrade of the proof by M.Hooshmand [49], that uses the doubtful assumption about monotonous behaviour of the derivative of tetration; I tried to find a proof, that does not use this assumption. Expression of tetration through the Cauchy integral [54] allows to make the precise approximations [64], and no internal contradictions in the assumptions (14.16)-(14.20) had been detected. This leads to the conjecture about existence and uniqueness of tetration, that later had been confirmed with the careful analysis [73]. I show the first primitive approximations in figure 14.4, as they answer the frequent question by colleagues: "*How did you guess?*". I think this heuristic approach can be used also for analysis of other (and more complicated) functional equations.

Tetration and arctetration significantly extend the arsenal of functions, available for the description of physical phenomena. In particular, the non-integer iterates of exponent can be useful in description of processes, that grow faster than any polynomial, but slower than any exponent.

Following the TORI axioms, I formulate mainly the practical problems. From the point of view of applications, not the proof by itself is important, but the strong indication, that the system of postulates is not self-contradictory. The multiple (failed) attempts to reject the conjecture of existence and uniqueness can be considered as such indication.

Some "pure mathematicians" believe, that the only rigorous proof has a scientific value. In order to show, that actually it is not so, I suggest the example with the Euclid axioms of planimetry. Those axioms can be deduced from the properties of the coordinate plane. It is not so difficult, although first, one had to provide the accurate definitions of sin and cos as solution of the corresponding system of differential equations, check that their properties lead to the Pythagoras theorem and other properties, known as the Euclid Axioms. In the elementary school, however, till now, the teachers begin with the postulating the Euclid axioms. I believe, the superfunctions should become a pretty elementary tool, and their properties (including those of tetration) could be just postulated - in the similar way, as the Euclid axioms. If someone wants to reduce the amount of axioms, one may begin wight he Euclud axioms, having no need to deal with tetration and other superfunctions. I hope, one day, the beautiful, short, simple and rigorous proof of the existence and uniqueness will be formulated.

Form my side, I make all possible efforts in order to simplify refutation of my concept (for the case, f one day someone will be able to refute them). I load the figures from this book to my site as
`http://mizugadro.mydns.jp/t/index.php/Category:BookPlot`
`http://mizugadro.mydns.jp/t/index.php/Category:BookMap`
and I supply them with generators in C++ and Latex. Everybody can reproduce the figures, and plot the new figures, trying to find a hint to any internal contradictions in the concepts suggested. Of course, any other alternative hypothesis can be considered too, as it is shown in figure 15.6.

After to see, how the natural tetration comes from the Cauchy integral in a pretty natural way, I had constructed similar maps for other values of base b, namely, for $b\!=\!10$, $b\!=\!2$ and $b\!=\!1.5$, but I did not revealed any new property, that could be difficult to expect, looking at the natural tetration. The most of curves in figure 17.1, considered later, can be plotted with the Cauchy integral by very similar algorithms. I was sure,

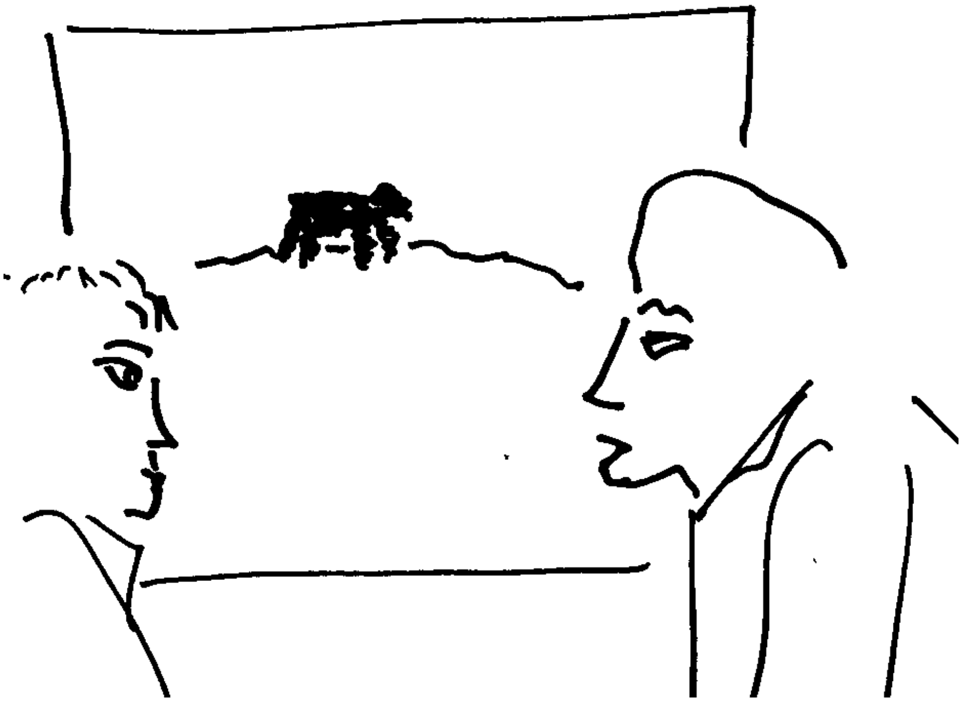


Figure 15.6: Two mathematicians go to the First International Congress on superfunctions, and discuss the color of the ship they see from the train: *Your assumption, dear colleague, seems to be not obvious, it is not supported with observations. Yet, all what we can conclude, that there is at least one ship in this country, and at least the right hand side of this ship is black.* [ship]

that my mission about tetration is finished. Then, Henryk Trappmann wanted still to reduce b; he asked me, wether I can evaluate in the similar way tetration to base $b = \sqrt{2}$. I had to confess, that I cannot. But I told, that I can do it by another way [61]. That "another way" happened to be even simpler, than application of the Cauchi integral; so, I described it in the previous chapters as "regular iteration". However, namely tetration to base $b = \sqrt{2}$ is not described above; this tetration is matter of the next chapter.

Chapter 16

Tetration to base $b\!=\!\sqrt{2}$

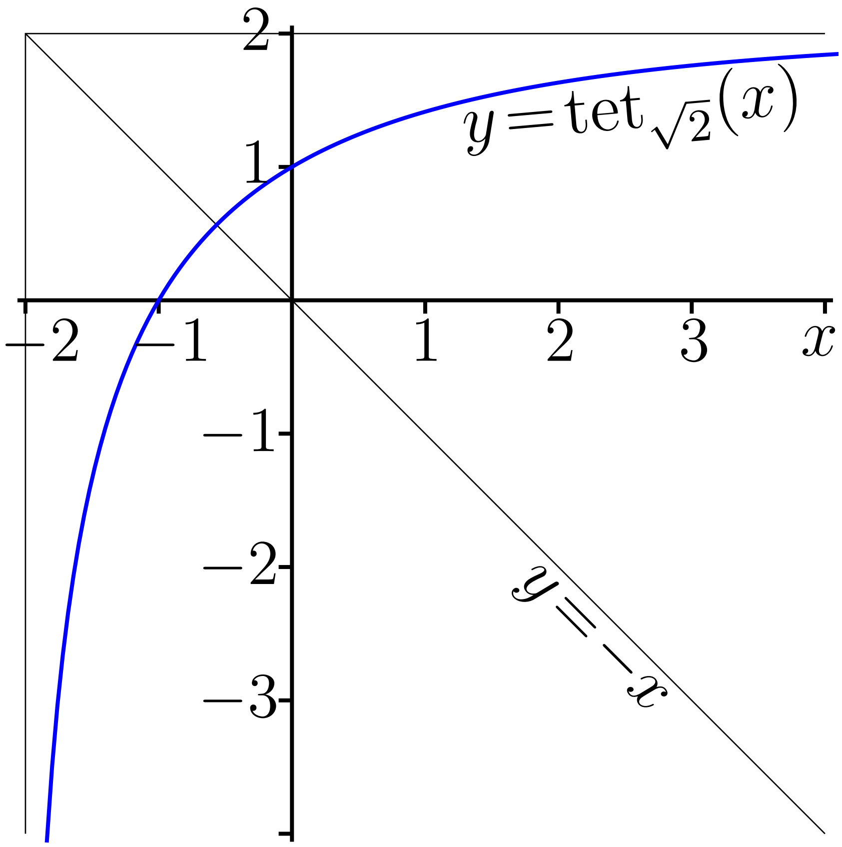


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Figure 16.1: $y = \mathrm{tet}_{\sqrt{2}}(x)$ [sqrt27a]

In chapter 13 above, the natural tetration is constructed and evaluated. I mean, tetration to base $b\!=\!\mathrm{e}\!\approx\!2.71$. For other bases, the definition of tetration should be generalised. This generalisation is suggested in this chapter. I try to follow the principles "from simple to complicated" and "from specific to general". First, I consider the specific base $b\!=\!\sqrt{2}$. For this base, the graphic of tetration is shown in figure 16.1. Namely for this base, the graphic looks especially symmetric. Below I show, that this is just visual impression, and the apparent symmetry $x \leftrightarrow -y$ is only approximation.

1 Definition

Tetration to the real base $b>1$ is real-holomorhpic function $f=\mathrm{tet_b}$, that satisfies the transfer equation

$$f(z+1) = b^{f(z)} \quad \text{[sqrt2transfer]} \tag{16.1}$$

at least for $\Re(z) > -2$, and is bounded at least in the range $|\Im(z)| \le 1$, and, in addition, the specific (the same for all b) value at zero is assumed:

$$f(0) \ = \ 1 \qquad \text{[sqrt2f01]} \tag{16.2}$$

Here, function $T=\exp_b$ appears as the transfer function, and tetration f as its superfunction.

The Reader is invited to check, that the natural tetration $\mathrm{tet} = \mathrm{tet_e}$, considered in chapter 14, also falls into into this definition. Below, the tetration to various bases is considered. In particular, this chapter deals with the special case $b = \sqrt{2}$. Exponential to this base is shown in figures 9.1 and 9.2. Namely for this case, in section 9, the growing super exponential $\mathrm{SuExp}_{\sqrt{2},5}$, is constructed; graphic $y=\mathrm{SuExp}_{\sqrt{2},5}$ is shown in figure 9.4.

Here, for the same base $b=\sqrt{2}$, id est, for the same transfer function, I describe another superexponent, namely, tetration. Its explicit plot is shown in figure 16.1 and it is pretty different from the plot of the growing superexponent $F=\mathrm{SuExp}_{\sqrt{2},5}$ shown in figure 9.4.

2 Again regular iteration

In this section, again I use the regular iteration, as in chapter 9, in other to construct another superfunction, namely, tetration, for the exponential to base $\sqrt{2}$ as the transfer function. This construction appears as an example to evaluation of tetration tet_b для $1<b<\exp(1/\mathrm{e})$.

In such a way, here I consider the case $b = \sqrt{2}$. Explicit plot of this tetration is shown in figure 16.1. The construction below is quite analogous to the construction of the growing super exponent to the same base, presented in chapter 9. Some formulas in this section are taken from publication [61].

The fixed points of exponent to base $b=\sqrt{2}$ are considered in chapter 9, see figure 9.1. These fixed points are 2 and 4. In that chapter, the superexponent $F_5 = \mathrm{SuExp}_{\sqrt{2},5}$ is built at the fixed point $L=4$. Along the real axis, that superexponential grows monotonously from 4 to infinity. For that super exponent, the equation $F_5(z) = 1$ has no real solution; that superexponent is not tetration. In order to get tetration, we should use the fixed point $L=2$. This case is considered in this chapter. The corresponding superfunction Φ can be expanded as follows:

$$f(z) = 2+\varepsilon+\sum_{m=2}^{M-1} v_m \varepsilon^m \quad \text{[sqrt2fas]} \tag{16.3}$$

$$\Phi(z) = f(z) + O(\varepsilon^n) \quad \text{[sqrt2Phi]} \tag{16.4}$$

where

$$\varepsilon = \exp(kz) \quad \text{[sqrt2ve]} \tag{16.5}$$

while increment k and coefficients v are constants. Substitution of the asymptotic expansion $F=\Phi$ in to the transfer equaiton

$$F(z+1) = \exp\left(\ln\left(\sqrt{2}\right) F(z)\right) \quad \text{[sqrt2transfereq]} \tag{16.6}$$

determines the increment

$$k = \ln(\ln(2)) \approx -0.3665129205816643 \quad \text{[sqrt2k]} \tag{16.7}$$

and leads to the chain of equations for coefficients v. I set $v_1=1$; then,

$$\begin{array}{lcl} v_2 = & \frac{\ln(2)/4}{\ln(2)-1} & \approx -0.56472283831773236365 \\ v_3 = & \frac{\ln(2)^2(2+\ln(2))/24}{(\ln(2)-1)(\ln(2)^2-1)} & \approx \;\;\, 0.33817758685118329988 \end{array} \quad \text{[sqrt2v2]} \tag{16.8}$$

Approximate values of coefficients v are collected in table 16.1.

At fixed number M of terms in the right hand side of equation (16.3), function f and be considered as approximation of superfunction with certain asymptotics, namely, that grows at infinity, approaching the fixed point 2. This superfunction appears as limit

$$\Phi(z) = \lim_{n\to\infty} T^{-n}(f(z+n)) = \lim_{n\to\infty} \log_b^n(f(z+n)) \quad \text{[sqrt2F]} \tag{16.9}$$

does not depend on the number M of terms in the right hand side of equation (16.3). However, at large M, the limit in the right hand side of (16.9) converges faster.

Table 16.1: Approximations of coefficients v and V in expansios (16.3), (16.16)

n	v_n	V_n
1	1.0000000000000000	1.0000000000000000
2	−0.5647228383177324	0.5647228383177324
3	0.3381775868511833	0.2996461813840881
4	−0.2103313021386278	0.1559323904892543
5	0.1344548790521098	0.0803518797481544
6	−0.0877843886012191	0.0411584960662439
7	0.0582880930830947	0.0209985209544120
8	−0.0392407117837278	0.0106825803202636
9	0.0267232860342981	0.0054228810223159
10	−0.0183765205976376	0.0027482526618683
11	0.0127420898467766	0.0013909151872678
12	−0.0088986329515697	0.0007031815862125
13	0.0062531995639749	0.0003551700677648
14	−0.0044181328624397	0.0001792537427482
15	0.0031365295362696	0.0000904088765718
16	−0.0022361213774487	0.0000455725430285
17	0.0016001999145218	0.0000229602263218
18	−0.0011489818761273	0.0000115627707503
19	0.0008274921384317	0.0000058201696570
20	−0.0005975832172069	0.0000029289688393

Tertation to base $b=\sqrt{2}$ that satisfies condition (16.2), appears as function Φ with displaced argument,

$$\mathrm{tet}_{\sqrt{2}}(z) = \Phi(x_1+z) \quad \text{[sqrt2tetF]} \tag{16.10}$$

where $x_1 \approx 1.25155147882219$ is solution of equation $\Phi(x_1)=1$. The readers are invited to verify, that this tetration satisfies the conditions, formulated in the section 1 of this chapter. For real values of the argument, graphic of this function is shown in figure 16.1. The complex map of tetration to base $b\!=\!\sqrt{2}$ is shown in figure 16.2.

Tetration by (16.10), is periodic; the period P is pure imaginary,

$$P = P(\mathrm{tet}_{\sqrt{2}}) = -\frac{2\pi \mathrm{i}}{\ln^2(2)} = -\frac{2\pi \mathrm{i}}{\ln(\ln(2))} \approx 17.14314817935485\,\mathrm{i} \tag{16.11}$$

I remind, the double logarithm $\ln^2(2)=\ln(\ln(2))$, but does not mean $\ln(2)^2$, according to notations declared at the beginning of this Book.

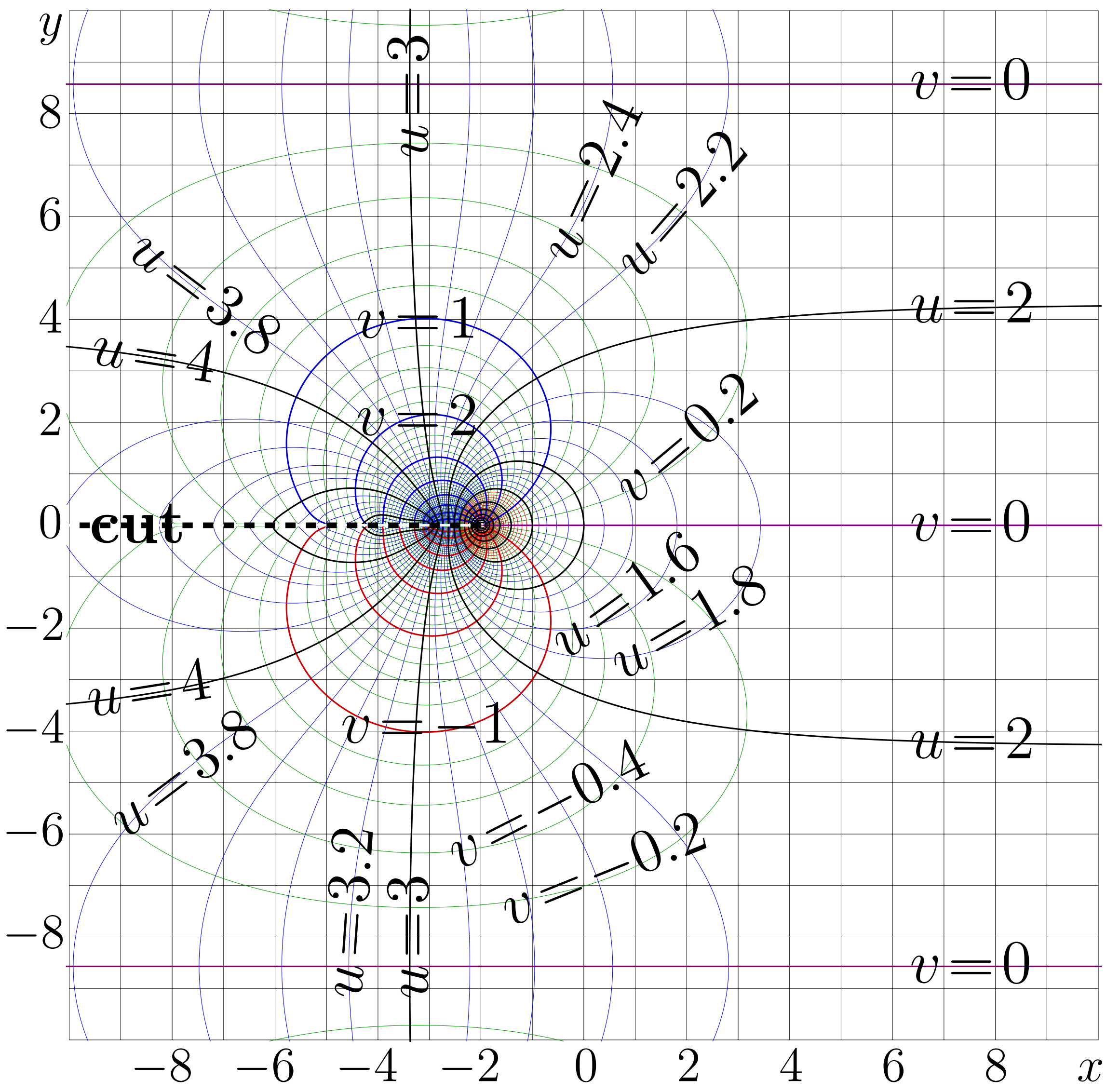

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Figure 16.2: $u+\mathrm{i}v = \mathrm{tet}_{\sqrt{2}}(x+\mathrm{i}y)$ [sqrt2tetmap]

As it is claimed above, tetration $\mathrm{tet}_{\sqrt{2}}(z)$ is holomorphic in the strip $|\Re(z)| \le 1$. The range of holomorphism is much wider than this strip. Tetration to base $\sqrt{2}$ is holomorphic in the whole complex plane, except the countable set of branch points and the corresponding cut lines

$$\{z \in \mathbb{C} : \Re(z) \le 2,\ \Im(z) = n\Im(P)\ ,\ n \in \mathbb{N}\} \qquad \text{[sqrt2tetCuts]} \quad (16.12)$$

Outside these cuts, tetration approaches the fixed points of the corresponding logarithm, to 2 or to 4, at the increase or decrease of the real part of the argument, respectively. For any real y,

$$\lim_{x\to+\infty} \mathrm{tet}_{\sqrt{2}}(x+\mathrm{i}y) \;=\; 2 \qquad \text{[sqrt2tetLim1]} \quad (16.13)$$

and for $y \ne \Im(T)n$, $n \in \mathbb{N}$,

$$\lim_{x\to-\infty} \mathrm{tet}_{\sqrt{2}}(x+\mathrm{i}y) \;=\; 4 \qquad \text{[sqrt2tetLim2]} \quad (16.14)$$

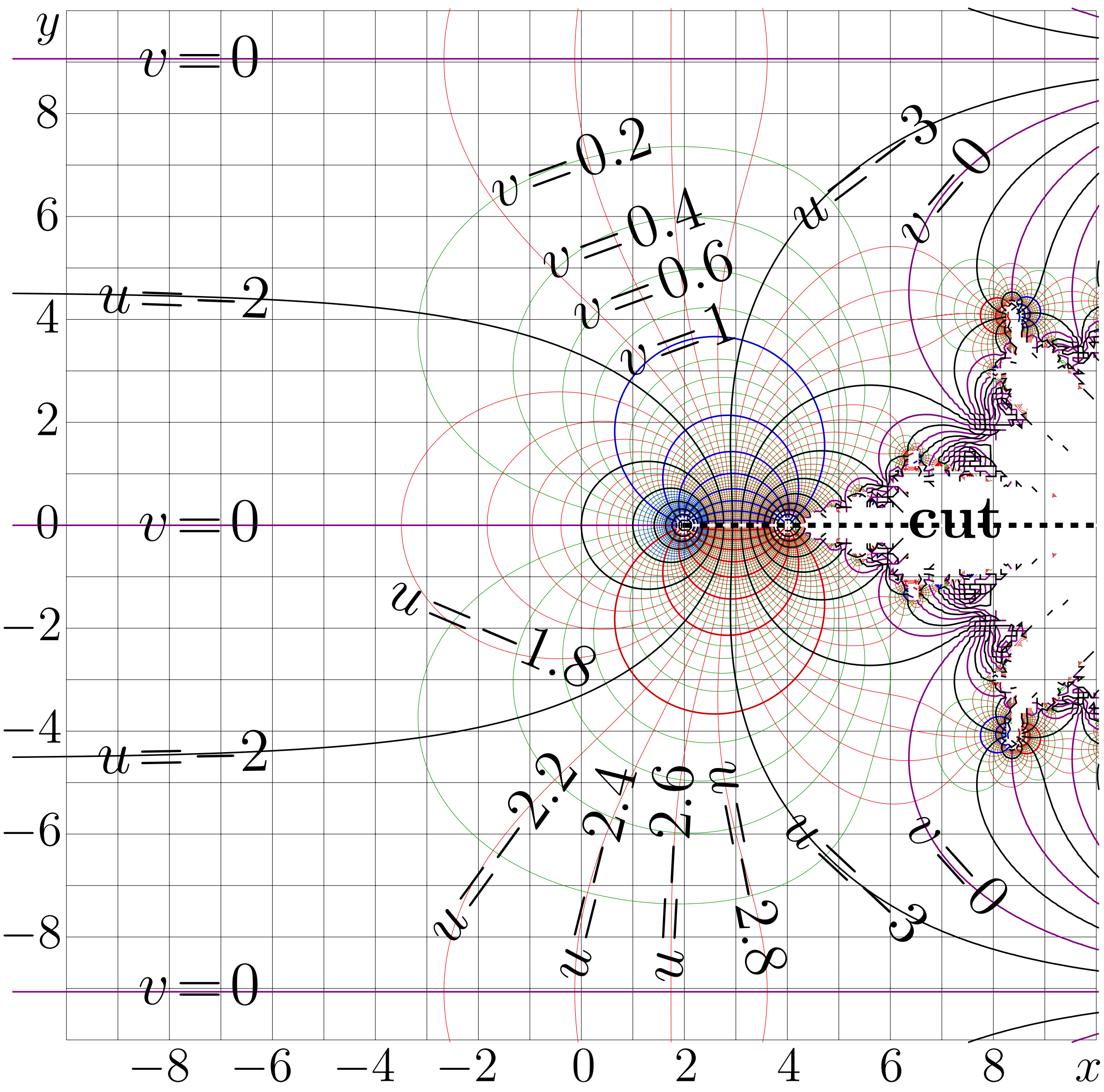

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Figure 16.3: $u+\mathrm{i}v = \mathrm{ate}_{\sqrt{2}}(x+\mathrm{i}y)$ [sqrt2atemap]

For real x, function $\mathrm{tet}_{\sqrt{2}}(x)$ is defined at $x > -2$. In point -2, the function has logarithmic singularity, as tetrations to other values of base. The function grows from $-\infty$ at -2, passes through points $(-1, 0)$ и $(0, 1)$, and continue to grow, approaching its limit value 2 at $+\infty$.

There are several reasons, why tetration to base $b = \sqrt{2}$ is especially interesting. Namely for this base $b = \sqrt{2}$, both real fixed points of $\log_b$ have integer values. In addition, namely for this base, the graphic of function in figure 16.1 seems to be symmetric with respect to reflection $x \leftrightarrow -y$. In order to stress this illusion, the additional line $y = -x$ is drawn in figure 16.1. For analysis of the illusion mentioned, the inverse function should be constructed; I mean, arctetration $\mathrm{ate}_{\sqrt{2}} = \mathrm{tet}_{\sqrt{2}}^{-1}$. The complex map of this arctetration is shown in figure 16.3. This function is described in the next section.

3 Arctetration to base $b=\sqrt{2}$

For the inverse function of tetration, I use name arctetration, in analogy with arcsin, arccos and arcBessel; $\mathrm{ate}_b = \mathrm{tet}_b^{-1}$. Arctetration is abelfunction of exponent and satisfies the corresponding Abel equation. For base $b=\sqrt{2}$, this equation can be written as follows:

$$G\Big(\big(\sqrt{2}\big)^z\Big) = G(z)+1 \quad \text{[sqrt2abeleq]} \qquad (16.15)$$

One of solutions of equation (16.15) is arctetration $G=\mathrm{ate}_{\sqrt{2}}$. Complex map of arctetration is shown in figure 16.3.

As other abelfunctions, arctetration to base $\sqrt{2}$ can be evaluated through its asymptotic expansion, using the Abel equation (16.3) to bring the argument into the range, where the fruncated expansion provides the required precision. The asymptotic expansion for the arctetration can be obtained with the asymptotic expansion of tetration. Also, the same expansion can be obtained directly from the Abel equation (16.3); we should add the constant to the solution in order to satisfy condition $\mathrm{ate}_{\sqrt{2}}(1)=0$.

Each of the two methods mentioned above lead to the same expansion

$$g(z) = \frac{1}{k} \ln\left(\sum_{n=1}^{M} V_n \cdot (z-2)^n\right) \quad \text{[sqrt2ateG]} \qquad (16.16)$$

where V are constant coefficients. Constant $k=\ln^2(2)$ is the same, as in the expansion (16.3)-(16.5), see equation (16.7). Then, asymptotic solution G of the Abel equation can be written as follows:

$$G(z) = g(z) + O(z-2)^{M+1} \quad \text{[sqrt2GO]} \qquad (16.17)$$

Substitution of this expansion into the Abel equation (16.15) leads to the chain of equations for coefficients V; in particular,

$$V_1 = \quad 1 \qquad (16.18)$$

$$V_2 = \quad -v_2 = \frac{1}{4}\frac{\ln(2)}{1-\ln(2)} \quad \approx 0.56472283831773236365 \qquad (16.19)$$

$$V_3 = \quad \frac{\ln(2)^2}{24}\frac{1+2\ln(2)}{(1-\ln(2))^2(1+\ln(2))} \quad \approx 0.29964618138408807683 \qquad (16.20)$$

Approximatioms of coefficients V are collected in the second (and last) column of table 16.1.

With asymptotic expansion (16.16), the solution G of the Abel equation (16.15) can be written as limit

$$G(z) = \lim_{n\to\infty} g\Big(\exp_{\sqrt{2}}^{n}(z)\Big) - n \quad \text{[sqrt2Glim]} \qquad (16.21)$$

Here, in the argument of function g, function $\exp_{\sqrt{2}}$ is iterated n times.

In order to make arctetration $\mathrm{ate}_{\sqrt{2}}$, not only the asymptotic properties should be taken into account, but also the value at unity. So, I define tetration as

$$\mathrm{ate}_{\sqrt{2}}(z) = G(z) - G(1) \approx G(z) - 1.25155147882219 \quad \text{[q2G1]} \quad (16.22)$$

that leads to the correct value $\mathrm{ate}_{\sqrt{2}}(1) = 0$. In previous publication [61], This arctetration is denoted with symbol ${F_{2,1}}^{-1}$; constant 2 in the subscript indicates the fixed point of the transfer function, at which the regular iteration is constructed, and content 1 in the subscript indicates its value at zero. As usually, the upper index indicates the number of iterate.

In figure 16.3, at the complex map of arctetration, its periodicty is seen. This periodicity follows from the representation of arctetration through limit in equation 16.3. The period P is determined by the period of the exponent to base $\sqrt{2}$;

$$P = P(\mathrm{ate}_{\sqrt{2}}) = \frac{4\pi \mathrm{i}}{\ln(2)} \approx 18.129440567308775239\,\mathrm{i} \quad \text{[sqrt2ateP]} \quad (16.23)$$

Imaginary part of this period is slightly greater, than that for tetration to the same base, see equation (16.11).

In figure 16.3, the isolines are reproduced at the translations along the ordinate axis for $\Im(P)$. In addition, due to the real-holomorphism, the maps of tetration and arctetration are symmetric with respect to reflection from the real axis, id est, with respect to the up side down flip.

In wide range of values of z, the identity

$$\mathrm{tet}_{\sqrt{2}}(\mathrm{ate}_{\sqrt{2}}(z)) = z \quad \text{[sqrt2tetatez]} \quad (16.24)$$

is valid. This range is shaded in figure 16.4. Technically, the shading is realised as complex map of the left hand side of equation (16.24), treated as function of z and plotted in coordinates $x = \Re(z)$ and $y = \Im(z)$. While the equation (16.24) holds, the levels of the constant real part and levels of the imaginary part are parallel to the coordinate axes and form the uniform rectangular grid, that at the poor resolution looks as shading. However, the relation (16.24) cannot hold in a strip wider than the period of function $\mathrm{ate}_{\sqrt{2}}$; so, the upper part and the lower part of the domain of the map are not shaded in this way. In addition, the range of

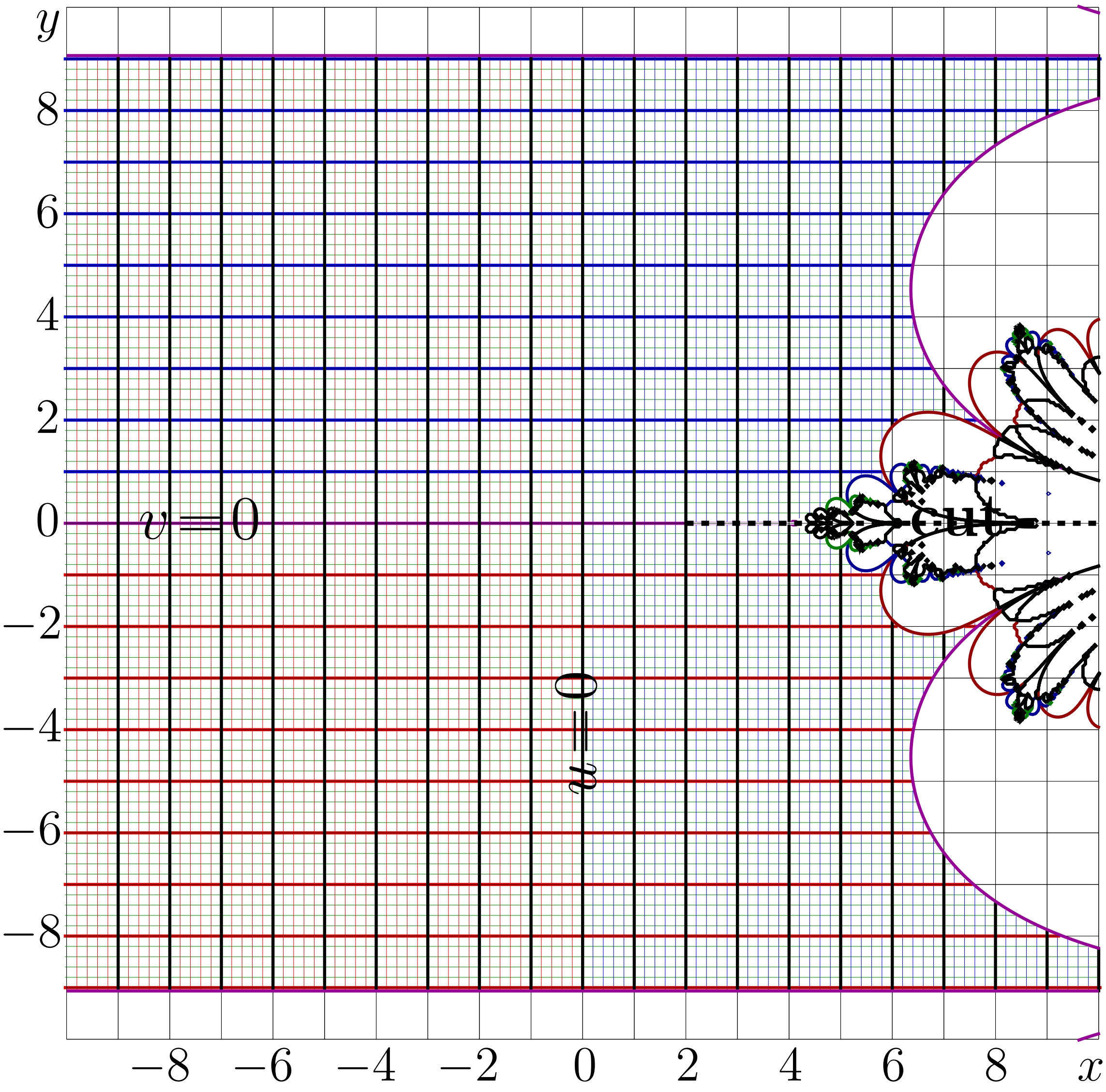

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Figure 16.4: $u+\mathrm{i}v = \mathrm{tet}_{\sqrt{2}}(\mathrm{ate}_{\sqrt{2}}(x+\mathrm{i}y))$ [sqrt2tetate]

validity of equation (16.24) is limited at the right hand side with levels

$$\Im\Big(\exp^n_{\sqrt{2}}(x+\mathrm{i}y)\Big) = \pm\frac{|P|}{2} = \pm\frac{2\,\pi}{\ln(2)} \quad \text{[sqrt2en]} \quad (16.25)$$

drawn above the map in figure 16.4 for integer $n = 0, 1, 2, 3, 4$. In the mentioned right hand side of the figure, relation (16.24) also is not valid.

Figure 16.4 can be considered as verification, validation, test of implementation of arctetration to base $\sqrt{2}$. These properties and the implementations of tetration and arctetration allow to analyse approximate the symmetry $y=-x$ of graphic $y = \mathrm{tet}_{\sqrt{2}}(x)$ shown in figure 16.1.

The apparent symmetry of the plot in figure 16.1 had been declared in the preamble of this chapter. The consideration had been postponed until tetration and arctetration to base $\sqrt{2}$ are described. The approximate symmetry mentioned means, that for $x > -2$,

$$\mathrm{tet}_{\sqrt{2}}(x) \quad \approx \quad -\mathrm{ate}_{\sqrt{2}}(-x) \qquad \text{[sqrt2approx]} \quad (16.26)$$

Properties of tetration and arctetration to base $\sqrt{2}$ indicate, that the exact equality un (16.26) can take only at the set of measure zero, due to very simple and pretty fundamental reason: tetration in the left hand side and atctetration in the right hand side of (16.26) have different (incompatible) periods.

Period of function in the left hand side of equation (16.26), see equation (16.11), is $P \approx 17.143\,\mathrm{i}$, while period in the right hand side of equation (16.26), see equation (16.23), is $P \approx 18.129\,\mathrm{i}$. The different periods indicates, that these are different functions. If two holomorphic functions coincide at the segment of finite length, they should coinside in the whole range of holomorphism. Hence, there is no exact equality in (16.26), for the most of z, nor for the exact symmetry in figure 16.1.

Deviation from the exact symmetry can be characterised with function

$$\mathrm{devia}(x) = \mathrm{tet}_{\sqrt{2}}(x) + \mathrm{ate}_{\sqrt{2}}(-x) \quad \text{[sqrt2simdevi]} \quad (16.27)$$

It is shown in figure 16.5 with dashed line. Where the symmetry be exact, the dashed curve should follow the abscise axis.

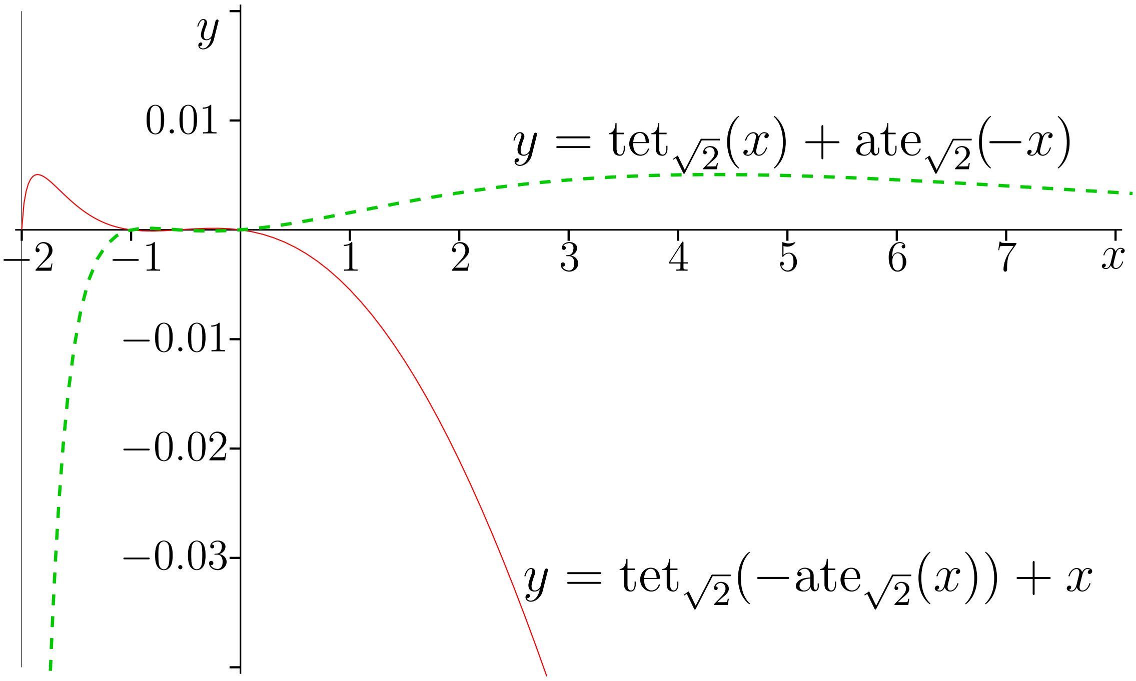

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Figure 16.5: Precision of "symmetry" of figure 16.1: $y = \mathrm{devia}(x)$ by (16.27), dashed, and $y = \mathrm{devib}(x)$ by (16.28), solid [sqrt27b]

Also, the deviation is characterized with function

$$\mathrm{devib}(x) = \mathrm{tet}_{\sqrt{2}}(-\mathrm{tet}_{\sqrt{2}}(x)) + x \quad \text{[q2tetatem]} \quad (16.28)$$

This dependence is shown in figure 16.5 with solid line. Expression $-\mathrm{tet}_{\sqrt{2}}(x)$ approximate function $\mathrm{ate}_{\sqrt{2}}(-x)$. Again, at the exact symmetry, the solid line would be just abscise axis. In such a way, figure 16.5 indicates the range of validity of the statement about the symmetry: it reproduce of order of 2 significant figures of tetration or arctetration to base $\sqrt{2}$; however, at the segment from -1 to 0 in figure 16.5, the "symmetry" holds with 4 signifivant figures; this is pretty sufficient to cause the illusion of symmetry in figure 16.1.

Similarity of dependences $y = \mathrm{tet}_{\sqrt{2}}(x)$ and $y = -\mathrm{ate}_{\sqrt{2}}(-x)$ for real x may look occasional. However, on the other hand, it is unavoidable for the following reasons. Every tetration to base $b > 1$ has logarithmic singularity at point -2; the graphics approach vertical line $x = -2$. Graphics of all these tetrations pass through points $(-1, 0)$ и $(0, -1)$, which correspond to the symmetry discussed. In addition, for $b < \exp(1/\mathrm{e})$, all the graphics have the horizontal asymptotic for large values of the augment, they approach some positive quantity (which is fixed point of logarithm). For some value of base, this quantity is 2, that corresponds to the apparent symmetry. This value of base is just $b = \sqrt{2}$, this base is chosen as an example in this chapter as illustration behaviour of tetration and arctetration to base b at $1 < b < \exp(1/\mathrm{e})$.

In years 2009-2010, the apparent symmetry of graphic in figure 16.1 caused hard discussion. The opponents had claimed, that the symmetry is obvious and does not require any verification. (Before, I had observed so strong believe in the wrong and absurd statements only in the USSR; Soviet veterans had insisted on concepts of sovetism, being unable to see internal contradictions of it.) To convince the opponents, Henryk and I had elaborated two independent demonstrations, that the exact symmetry cannot take place, without using of properties of these functions in the complex plane. Both these proofs are presented in publication [61].

Readers are invited to invent some real-holomorphic function with graphic that passes through points $(-1, 0)$ and $(-0, 1)$, and exponentially approach to the vertical line $x = -2$ and horizontal line $y = 2$. I suspect, such a function will be pretty similar to tetration to base $\sqrt{2}$.

With the suggestion above, I finish the description of arctetration to base $\sqrt{2}$. At lest in some vicinity of the half-line $z < 2$, relation (16.24) is valid, and the pair (tetration,arctetration) can be used to iterate the exponent. These iterates are considered in the next section.

4 Again iterate exponent to base $\sqrt{2}$

The real-holomorphic Iterates of exponent to base $\sqrt{2}$ are considered above, in Chapter 9, for large values of positive part of the argument, with functions $\mathrm{SuExp}_{\sqrt{2},5}$ and $\mathrm{AuExp}_{\sqrt{2},5}$. Those iterates are presented first (and shown in figure 9.8), because they look similar to iterates of other growing functions, considered in the first half of this Book. However, tetration and arctetration, constructed in this chapter, also can used to iterate the exponent to base $\sqrt{2}$. In this section, I show, that these iterates look similar in vicinity of the interval $(2,4)$; but far from this interval, the deviation becomes strong.

The "regular iteration" described above, allows to iterate the function, and iterates are regular in vicinity of the fixed point of the transfer function, used to construct the superfunction and the abelfunction. Bot these iterates may be not regular (have singularity, branch point) at the other fixed points of the same transfer function. Below, the illustration of this statement is presented.

Iterates of exponent to base $\sqrt{2}$ constructed with the infinitely growing superfunction $\mathrm{SuExp}_{\sqrt{2},4}$ by (9.11),(9.12),(9.13) are shown in figure 9.8. Similar iterates can be constructed also with tetration, described in this chapter,

$$\exp_{\sqrt{2},\mathrm{d}}^{\,n}(z) = \mathrm{tet}_{\sqrt{2}}\left(n + \mathrm{ate}_{\sqrt{2}}(z)\right) \qquad [\mathrm{sqrt2exptet}] \quad (16.29)$$

Here, symbol ",d" in the subscript indicates, that the lower, "down" fixed point of the transfer function is used for the asymptotic of the superfunction.

For real values of argument, iterates $\exp_{\sqrt{2},\mathrm{d}}^{\,n}$ by (16.29) are shown in figure 16.6 for various real values of n. This figure is analogy of figure 9.8, that represents the similar iterates built up with the infinitely growing superexponent $\mathrm{SuExp}_{\sqrt{2},5}$ and corresponding abelexponent $\mathrm{AuExp}_{\sqrt{2},5}$.

Graphics in figures 16.6 and 9.8 look similar. The thick curves, for the integer iterates are, indeed, the same. However, for the non-initeger n, the iterates also look similar, the curves in figure 16.6 seem to be just extension, continuation of those in figure 9.8. In the intermediate range, $2<x<4$, visually, the iterates $\exp_{\sqrt{2},\mathrm{d}}(x)$, evaluated through the tetration $\mathrm{tet}_{\sqrt{2}}$ and arctetration $\mathrm{ate}_{\sqrt{2}}$, seems to be the same, as iterates $\exp_{\sqrt{2},\mathrm{u}}(x)$, evaluated through the superexponent $\mathrm{SuExp}_{\sqrt{2}}$ and abelexponent $\mathrm{AuExp}_{\sqrt{2}}$. Then I saw this coincidence first time, it looked strange, counter-intuitive and therefore interesting. The matter is, that

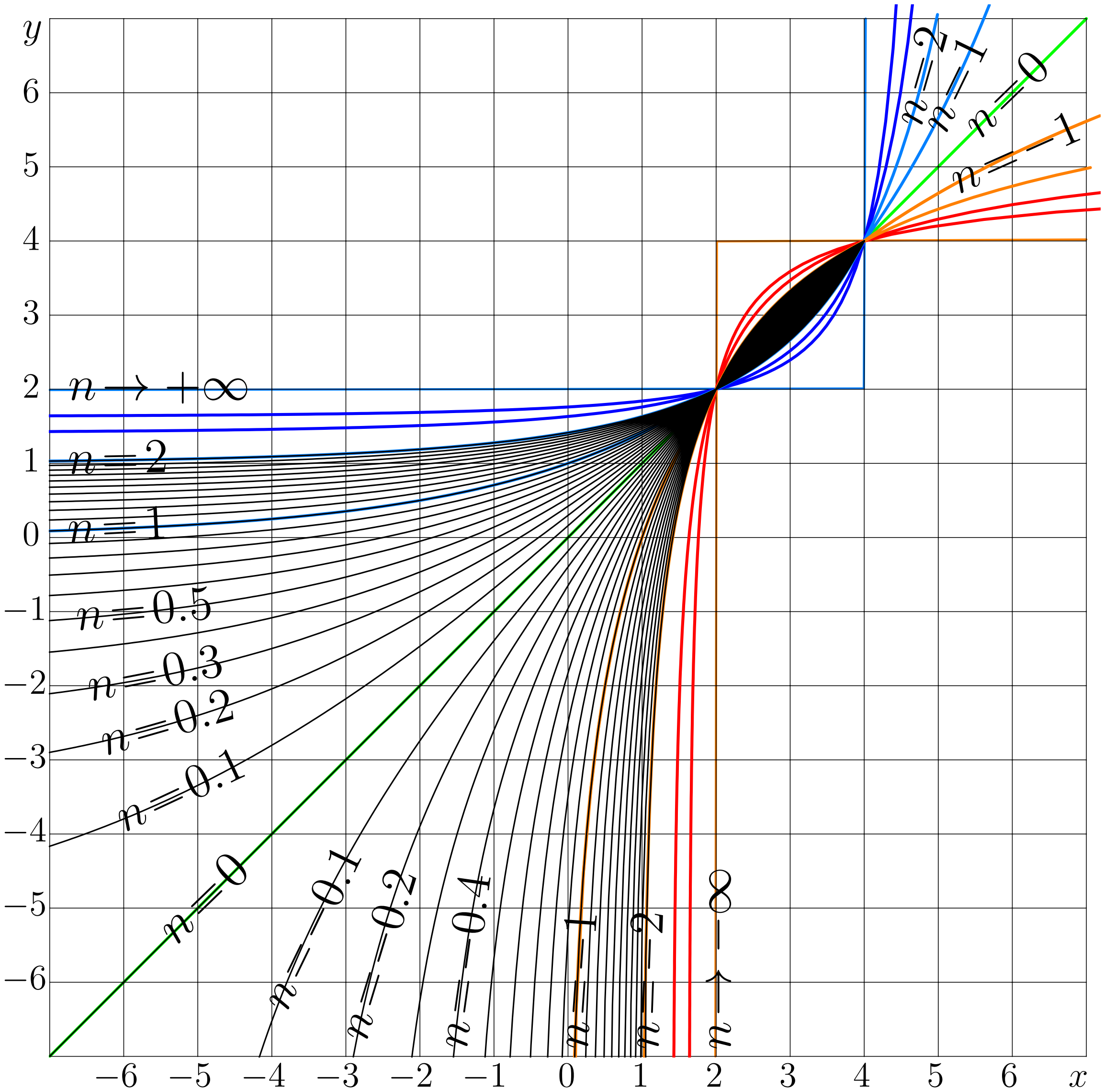

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Figure 16.6: $y = \exp^{n}_{\sqrt{2},\mathrm{u}}(x)$ for various n [sqrt2eitet]

two different holomorphic functions cannot coincide at the interval of finite length. If they are identical at the part of the real axis from 2 to 4, then they must coincide in the whole connected range of holomorphism.

As an example, I consider the case $n = 1/2$; id est, the iterates of exponent number half. The iterate constructed with growing exponent, id est, $\exp^{1/2}_{\sqrt{2},\mathrm{u}}$ had been shown earlier in figure 9.9. The iterate $\exp^{1/2}_{\sqrt{2},\mathrm{d}}$, constructed with tetration and arctetration, is shown in figure 16.7. These two maps are not the same. The second of them is periodic (with period $4\pi\mathrm{i}/\ln(2) \approx 18.12944\,\mathrm{i}$), while the first one is not. These two maps look similar only in vicinity of the interval (2,4) at the real axis.

I felt myself confused about the identical behaviours of the half iterates along the interval $(2, 4)$ of functions, that have different behaviour in

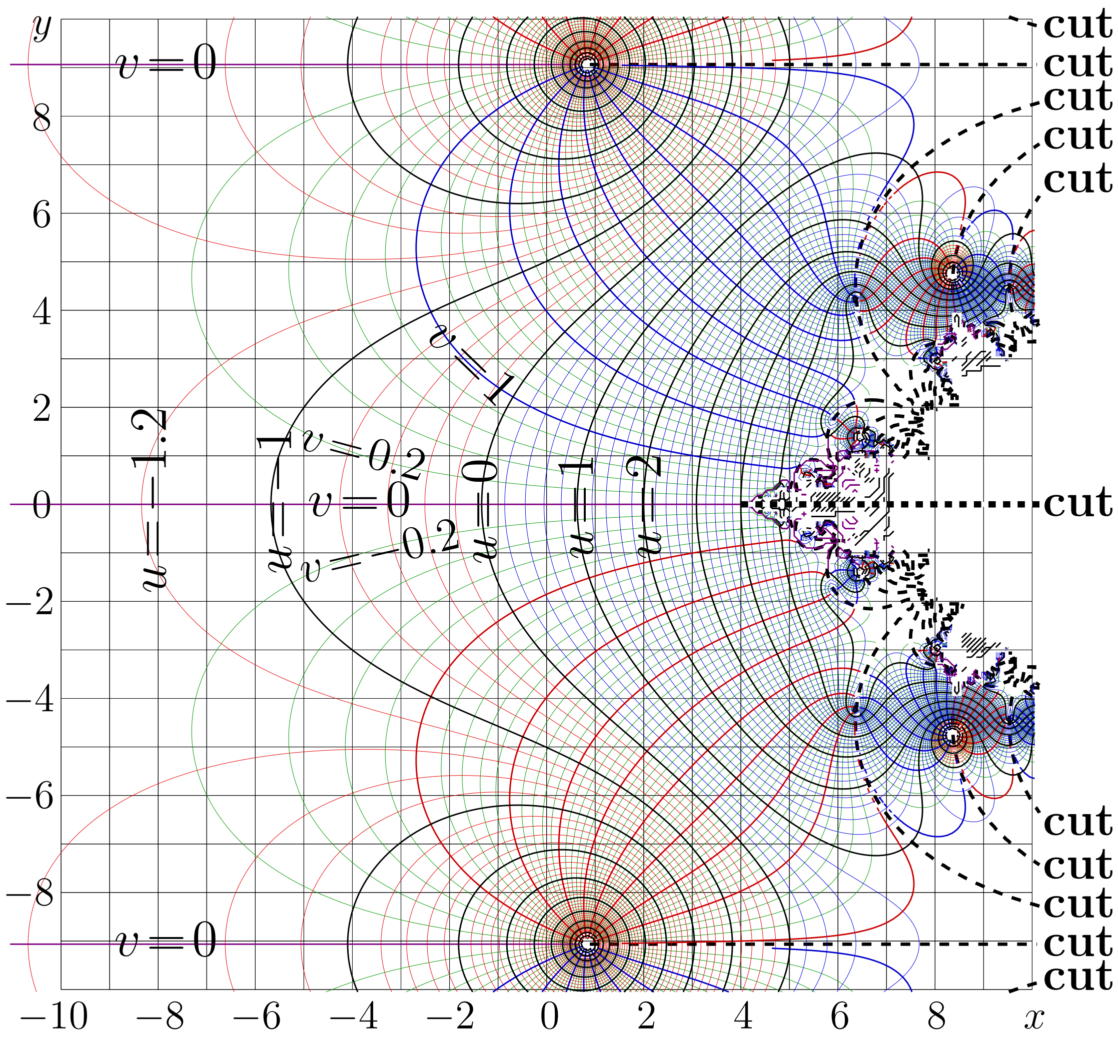

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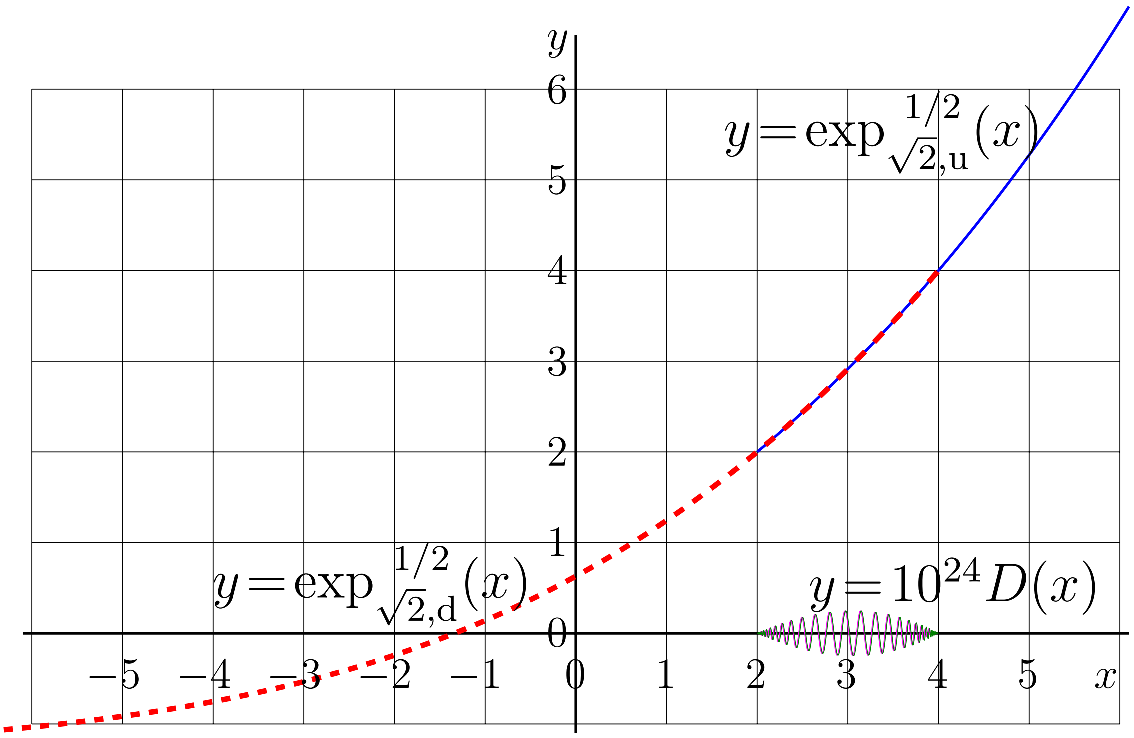
Figure 16.7: $u+\mathrm{i}v=\exp_{\sqrt{2},\mathrm{d}}^{1/2}(x+\mathrm{i}y)$ [sqrt2q2map]

the complex plane. I thought, that I made an error implementing these functions. I even considered the absolutely phantasmic hypothesis that I see the traces of the *Mizugadro number* [1], that reveals the internal contraction in the system of postulates of arithmetics (that is used in mathematical analysis and, in particular, in the theory of holomorphic functions). I had prepared the explicit plot the half iterates of the exponent to base $\sqrt{2}$, evaluated through the tetration and that evaluated through the super exponential $\mathrm{SuEx}_{\sqrt{2}}$; this plot is shown in figure 16.8, and looked at the zoom-in of the central part; then at the zoom-in of that zoom-in, and so on, but I could not see deviation of curve $y=\exp_{\sqrt{2},\mathrm{u}}(x)$ from curve $y=\exp_{\sqrt{2},\mathrm{d}}(x)$.

Searching for the error, as a working hypothesis, I assumed, that the precision of variables complex double (which is of order of 15 decimal

[1] http://mizugadro.mydns.jp/t/index.php/Mizugadro_number
http://budclub.ru/k/kuznecow_d_j/mizugade.shtml Mizugadro's number (2010-2011)



http://mizugadro.mydns.jp/t/index.php/File:Sqrt2srav.png

Figure 16.8: Comparison of the half iterations to base $\sqrt{2}$, constructed at fixed point 2 (dashed) and at fixed point 2 (solid curve). In the interval between these two points, the difference (16.30) is shown, scaled with factor 10^{24}. [sqrt2srav]

digits) is not sufficient to see the difference between $\exp^{1/2}_{\sqrt{2},d}$ by formula (16.29) and $\exp^{1/2}_{\sqrt{2},u}$ by formula (9.24). I begun to compute the iterate with a hundred decimal digits; the exact formulas and abilities of Maple and Mathematica allow this. The deviation happened to be in 25th digit. This deviation can be characterised with difference

$$D(x) = \exp^{1/2}_{\sqrt{2},\mathrm{u}}(x) - \exp^{1/2}_{\sqrt{2},\mathrm{d}}(x) \quad \text{[sqrt2D]} \qquad (16.30)$$

This difference is shown at the bottom of figure 16.8. In order to see it, I had to scale it with factor 10^{24}, as it is marked in the figure. For $|z-3|<2$, function $D(z)$ can be approximated with the 7 parameter fit

$$\tilde{D}(z)=2.48\cdot 10^{-25}(z-2)(4-z)\Big(1+0.120(z-3)+0.006(z-3)^2\Big)\times$$
$$\sin\Big(.747-.068(z-3)+0.007(z-3)^2+p_4\ln(4-z)+p_2\ln(z-2)\Big) \quad (16.31)$$

where $p_4 = 2\pi/\ln(2\ln(2)) \approx 19.23614904204285$
and $p_2 = -2\pi/\ln^2(2) \approx 17.14314817935485$
correspond to periods of the two superfunctions of $\exp_{\sqrt{2}}$, built up on the fixed points 4 and 3, see equations (9.6), (9.7) (16.11). This fit provides of order of two significant figures; at figure 16.8, the curves for D and for $\tilde{D}$ almost coincide.

After to plot the bottom curve in figure (16.8), I realised, that it is first case in my life, when the double precision, id est, 15 significant digits, happened to be not sufficient to see the difference between two functions, which have no small parameters. This example gives a good lesson: the numeral calculus with high precision serious evidence to refute, reject (or to verify) a conjecture. Whenever the rigorous proof is available of not, the numerical testing should be applied. The TORI axioms provide a good hint for the revision.

It should be noted, that my attempts to use the Maple-10 software for visualisation of difference D by (16.30) failed. I could not find way to pot graphics with precision better than just "float" [2]. In order to plot the beautiful figure (16.30), I had to save values of function D as a table, and then export this table to the C++ program. I hope, now there exist more straightforward ways to do the same; in particular, use of the fit provides the "quick and dirty" realisation. I expect, using the precise numerical solution, fit 4 can be significantly improved; the readers are invited to do this as an exercise.

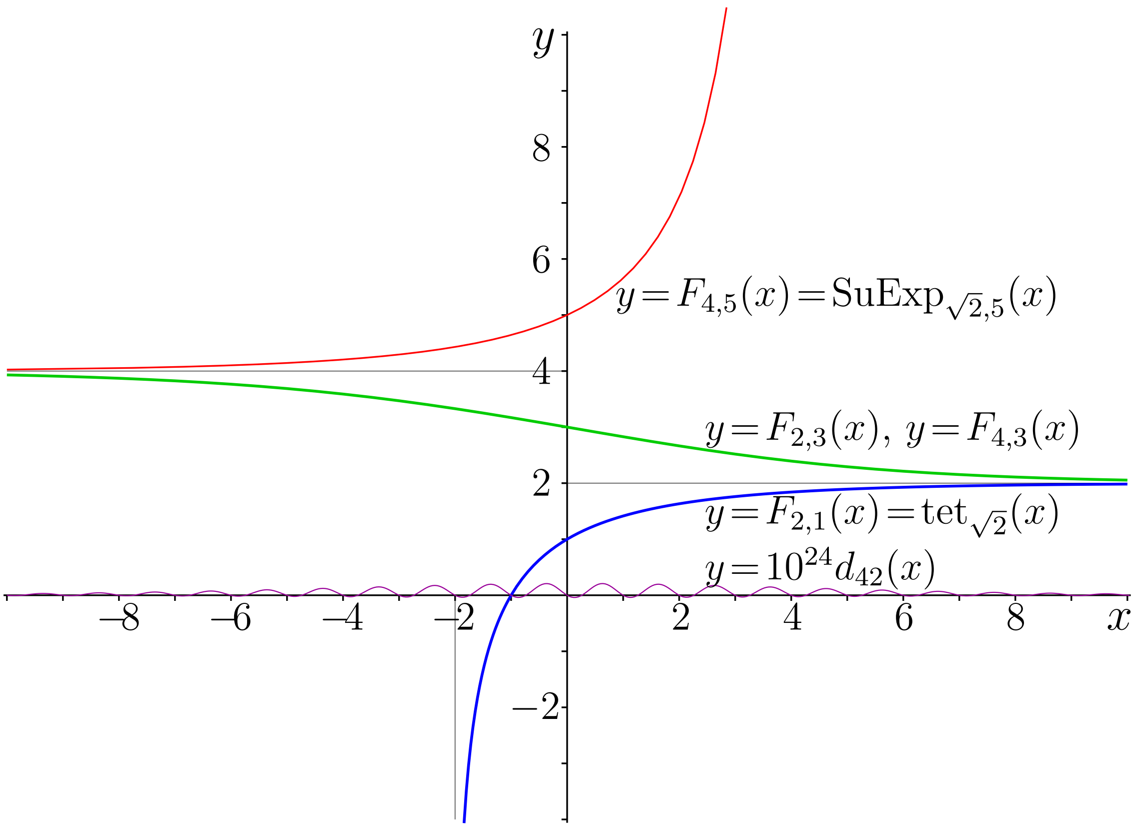
Iterate of a function, regular at some of fixed points, often is singular in another fixed point. Some exceptions, when the superfunction is expressed with elementary function, are mentioned in Chapter 4.

In such a way, for a given transfer function, there may exist many superfunctions, and some of them may be arguably declared as a "true" or "principal" superfunction. For the case of transfer function $T = \exp_{\sqrt{2}}$, the four real-holomorphic superfunctions with various exponential asymptotics are considered in the next section.

5 Four superexponents to base $\sqrt{2}$

On the base of consideration of previous chapters, one can built-up the four different real-holomorphic superfunctions for the transfer function

[2]The poor precision of the graphic procedures in Maple-10 is described in the poem `http://en.wikisource.org/wiki/Maple_and_Tea` Maple and tea. This is one few my texts, that are not yet removed from wikisource with pretext of protection of my author rights. This is common practice at wikisource and other sites of Wikimedia projects: the Soviet veterans promote sovietism and remove texts of anti-Soviet authors with any absurd pretext; often, the claim for the violation of the author rights is used, even if cases, when the author gave permission to publish the files providing the free ("copyleft") licence. The permission to use the author's file is usually removed together the file. While the soviet veterans act as trolls and vandals in wiki-projects, I loaded the copy also to `http://budclub.ru/k/kuznecow_d_j/maple.shtml`



`http://mizugadro.mydns.jp/t/index.php/File:Sqrt2sufuplot.png`

Figure 16.9: Four superexponents to base $\sqrt{2}$ [sqrt2sufuplot]

$T = \exp_{\sqrt{2}}$. These superexponentials are shown in figure 16.9 and discussed below.

Here I compare the four functions; in publication [61], they are are called $F_{2,1}$, $F_{2,3}$, $F_{4,3}$ and $F_{4,2}$. Each of them is real-holomrphic solution of the transfer equation

$$F(z+1) = \left(\sqrt{2}\right)^{F(z)} \tag{16.32}$$

The first number in the superscript indicates the limiting value, that the function approaches exponentially; it is any of the two fixed points of the $\exp_{\sqrt{2}}$, id est, either 2 or 4. The second number in the superscript indicates value of this function at zero.

Function $F_{2,1} = \mathrm{tet}_{\sqrt{2}}$ is tetration to base $\sqrt{2}$; the curve for $F_{2,1}$ is borrowed from the figure 16.1. Properties of this function are considered above in this chanter. It is superfunction of exponent to base $\sqrt{2}$, built up with the regular iteration at fixed point 2. As tetration to any other base, it takes value unity at zero.

Function $F_{4,5} = \mathrm{SuExp}_{\sqrt{2},5}$ refers to formula (9.13). This function, together with its inverse function $F_{4,5}^{-1} = \mathrm{AuExp}_{\sqrt{2},5}$ is used to built-up iterates of exponent to base $\sqrt{2}$; and these iterates grow up infinitely along the real axis. Fixed point 4 is used as the asymptotic value at minus infinity. With the appropriate translation along the real axis, the condition $F_{4,5}(0) = 5$ is achieved.

Function $F_{2.3}$ is tetration with displaced argument,

$$F_{2,3}(z) = \mathrm{tet}_{\sqrt{2}}(z+z_{2,3}) \tag{16.33}$$

$z_{2,3} = \mathrm{ate}_{\sqrt{2}}(3+\mathrm{i}o) \approx -3.3834692659172254 + 8.5715740896774228\,\mathrm{i}$

However, $F_{2,3}$ has the same periodicity, as tetration $F_{2,1}$.

Function $F_{2,3}$ is growing superexponent with displaced argument

$$F_{4,3}(z) = \mathrm{SuExp}_{\sqrt{2},5}(z+z_{4,3}) \tag{16.34}$$

$z_{4,3} = \mathrm{AuExp}_{\sqrt{2},5}(3+\mathrm{i}o) \approx 3.015784890490347{+}9.618074521021425\,\mathrm{i}$

Along the real axis, functions $F_{2,3}$ and $F_{4,3}$ decrease from 4 at minus infinity to 2 at plus infinity. In figure 16.9, curve $y\!=\!F_{2,3}$ overlaps well with curve $y\!=\!F_{4,3}$. The deviation is smaller than the thickenss of lines, and it is small compared to size of atoms, of which this book (or the screen where it is watched) is built. In otter to show the deviation, denote it with

$$d_{42}(z) = F_{4,3}(z) - F_{2,3}(z) \quad \text{[d42]} \tag{16.35}$$

Tn figure 16.9, the thin line shows $y\!=\!10^{24}d_{42}(x)$; I scale values of this difference for 24 orders of magnitude, to make it visible. This similarity takes place only in vicinity of the real axis. The functions have different periods, and one go them has singularities; so, they must be pretty different somewhere.

Similarity of functions $F_{4,3}$ and $F_{2,3}$ determines the similarities of corresponding iterates of exponential to base $\sqrt{2}$. These iterates are shown in figures 9.8, 9.9, 16.6 and 16.7. For real values of the argument, the half iterates are compared also in figure 16.8. I expect, for application in physics (where the precision usually does not exceed 20 decimal digits), any of the two iterates is declared as the “true iterate”. However, for some applications (for example, if the model refers to the complex numbers), the difference may be important, and the fixed point should be specified.

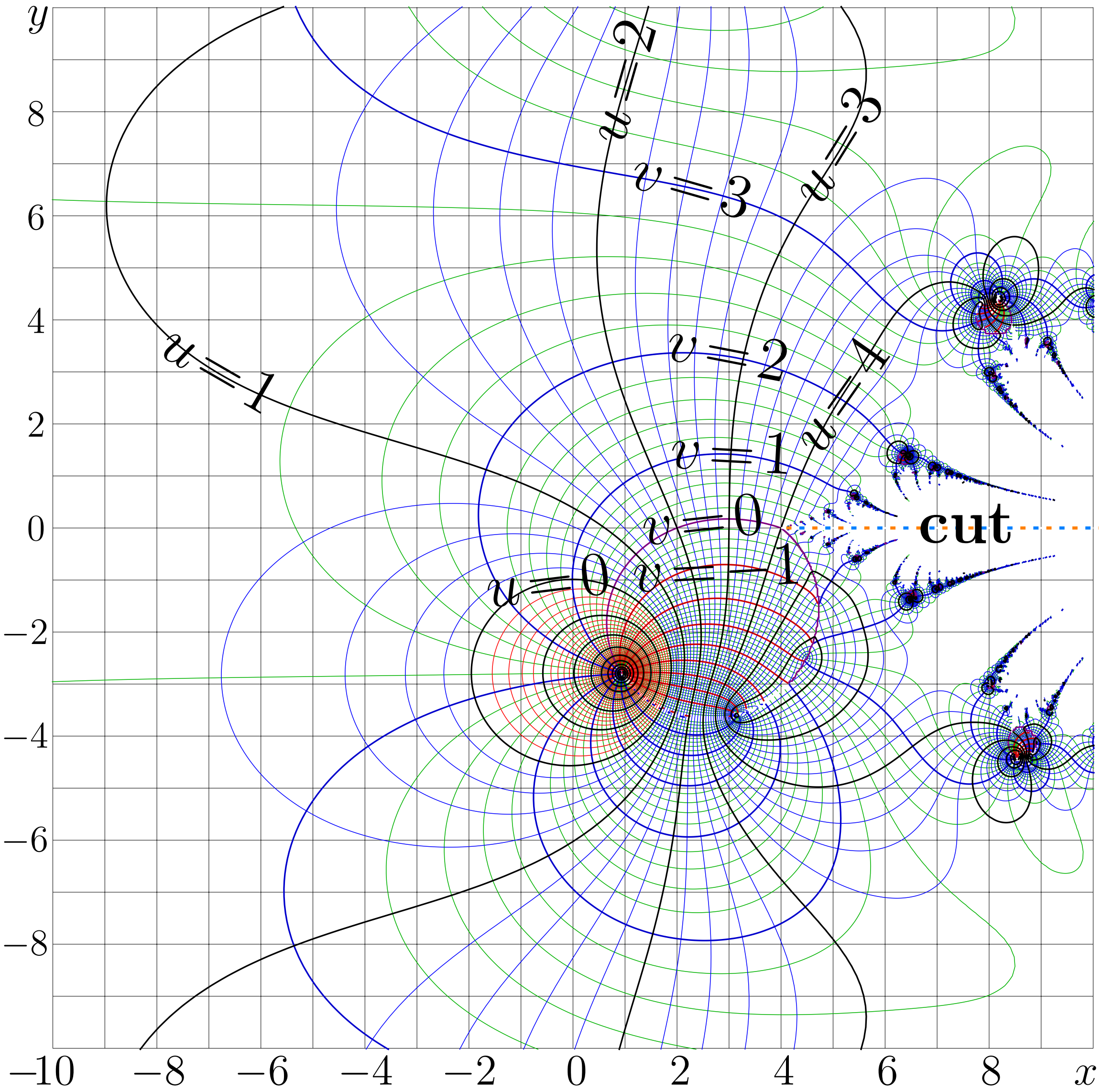


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Figure 16.10: $u+\mathrm{i}v=\exp_{\sqrt{2},\mathrm{d}}^{\mathrm{i}}(x+\mathrm{i}y)$ by (16.36) [sqrt2itemap1]

6 Complex iterates

When the draft of the Russian version of this Book had been completed, I found, that the book had no maps of the complex iterates. I had declared that I can calculate any real or even complex iterate, but all the examples refer to real iterate. I fill this gap in this section. Here I describe two iterates number i. As this i appear with Roman font; one may guess, that it is not variable, but a constant, square root of −1.

Figures 16.10 and 16.11 show the complex maps of iterates

$$\exp_{\sqrt{2},\mathrm{d}}^{\mathrm{i}}(z)=\mathrm{tet}_{\sqrt{2}}\Big(\mathrm{i}+\mathrm{ate}_{\sqrt{2}}(z)\Big) \quad \text{[sqrt2dii]} \qquad (16.36)$$

and

$$\exp_{\sqrt{2},\mathrm{u}}^{\mathrm{i}}(z)=\mathrm{SuExp}_{\sqrt{2},5}\Big(\mathrm{i}+\mathrm{AuExp}_{\sqrt{2},5}(z)\Big) \quad \text{[sqrt2uii]} \qquad (16.37)$$

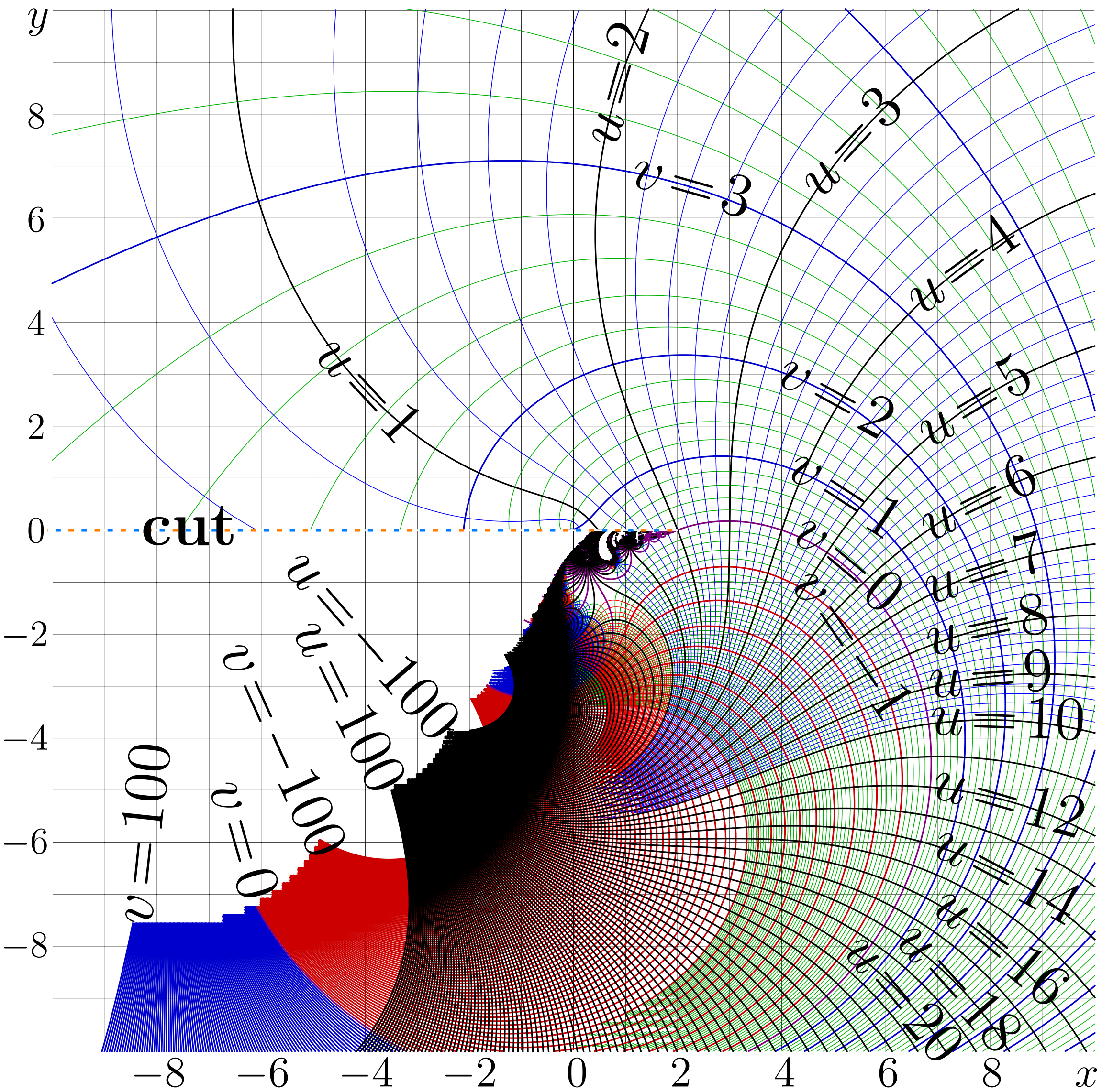

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Figure 16.11: $u+\mathrm{i}v = \exp_{\sqrt{2},\mathrm{u}}^{\mathrm{i}}(x+\mathrm{i}y)$ по формуле (16.37) [sqrt2itemap2]

Function $\exp_{\sqrt{2},\mathrm{d}}^{\mathrm{i}}$ by (16.36) is built up from the tetration and arctetration to base $\sqrt{2}$, considered in this chapter. Function $\exp_{\sqrt{2},\mathrm{u}}^{\mathrm{i}}$ by (16.37) is built up from the growing super exponent to base $\sqrt{2}$ and the corresponding abelexponent; these functions are considered in chapter 9. These pairs of functions look similar in vicinity of the interval $(2,4)$, but they are pretty different beeng evaluated far from this interval. As one could expect, the i th iterates, shown in figures 16.10 and 16.11, are also similar in vicinity of the interval mentioned, but far from this interval, they deviate strongly.

As in the case of real iterates, each of considered here complex iterates can be arguably qualified as “true”. In this sense, the are “equal”. In the similar sense, "all animals are equal" in the novel “Animal’s Farm” by George Orwell [3] In the movel, soon it happens that “All animals are equal, but some of them are more equal than others”. In the similar sense, one of iterates, either 16.10 or 16.11, may be “more equal”, if some additional criterion arises from some physical reasons, specifying, for example, the asymptotic behaviour of the iterate at infinity. If only one of iterates reproduces the required behaviour, this iterate immediately becomes “more equal” than another.

The example of this section once again confirms the general observation about non-integer iterates of a function, that have more than one fixed point: the regular iterate, holomorphic in vicinity of some fixed point, has no meed to be holomorphic at another fixed point. The choice to the fixed point (and choice of the iterate) should involve the additional requirement, that may arise from the applications. This general rule hold also for complex values of the number of iterate.

I expect, the integer iterates may appear more often that real; and the real iterate may appear more often than compex. On the other hand, the mathematical formalism should cover an area, which is wider, than that required for the today’s applications. For this reason I consider the case with complex number of iterate as an important example, that shows the power of the formalism of superfunctions.

Following the lessons I remember since the Soviet school, I wanted to say that “the formalism of superfunctions is omni-potnent, because it is true” [4]. On the other hand, the First TORI axiom prohibits consideration of omnipotent and almighty concepts in a scientific analysis; such doctrines and concepts are qualified as religious [68].

Various iterates are available for q transfer function with several fixed points. As soon, as the non-integer iterate of a holomorphic function with several fixed point is required, the additional conditions should be added to the formalism in order to decide, which of the iterates is “more equal than others”.

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[3] `http://msxnet.org/orwell/print/animal_farm.pdf` George Orwell. Animal Farm. 1945. .. *“All animals are equal, but some of them are more equal than others”*..

[4] `http://www.marxists.org/archive/lenin/works/1913/mar/x01.htm` V.I.Lenin. The Three Sources and Three Component Parts of Marxism. Lenins Collected Works, Progress Publishers, 1977, Moscow, Vol.19, p.21-28. .. *The Marxist doctrine is omnipotent because it is true.* ..

7 Not all is done

In this Book, I describe methods, that can be used to build-up (and evaluate) superfunctions, and the iterates that can be expressed through the superfunctions and the abelfunctions. The properties of superfunctons and those of the iterates appear as illustrations of the methods. I expect the Reader to use these methods not only for the transfer functions, considered in this book. In particular, this chapter indicates the interesting (from my point of view) branch for the future research, namely, comparison of superfunctions and the corresponding iterates, that are regular in various fixed points, and look for some general criteria: in which cases, the iterates, built up at different real fixed points of some real-holomorphic transfer function, behave in the similar way in the interval between these fixed points. They happened to be very similar in the case of exponent to base $\sqrt{2}$. How about other growing transfer functions with two real fixed points?

The Readers are invited to repeat the calculus, described in this Book (and in this Chapter) for other transfer functions. As an example, I would suggest to experiment with polynomial transfer function.

One example of the polynomial transfer function is shown in raw 5 of table 3.1, $T(z)\!=\!z^b$. The readers may confirm, that the primary approximation with the regular iteration at the fixed point $z\!=\!1$ gives the series of expansion of the superfunction, that is just expansion of the "outer" exponent in superfunction $\exp(b^z)$. Up to my knowledge, this is the only case, where the primary series by the regular iteration converges. The appropriate choice of the 0th approximation should cut the series at the first term. I expect, many of transfer functions can be treated in such a way, and the corresponding superfunctions and abelfunctions can be built-up. The goal of this Book not to describe all the examples, but to teach the Readers to to it by themselves. So, it is rather collection of tools, than collection of specific properties of the specific functions.

Consideration of tetration to base $b = \sqrt{2}$, presented in this chapter, can be generalised to other values of base; in particular, for real values $1\!<\!b\!<\!\exp(1/\mathrm{e})$. This generalisation is described in the next chapter.

Chapter 17

Tetration to base $b>1$

The previous chapters describe various methods of construction of superfunctions for the transfer functions that have real or complex fixed points. In particular, the examples of the exponential transfer function are considered, $z \mapsto b^z$ for the cases $b=\sqrt{2}\approx 1.44$, $b=\exp(1/\mathrm{e})\approx 1.46$ and $b=\mathrm{e}\approx 2.71$.
In this chapter, I combine methods, described in the previous chapters, and describe tetration to the real base $b>1$. For the real values of base, the interpretation of the superfunction, abelfunction and iterates of the exponent is especially explicit.

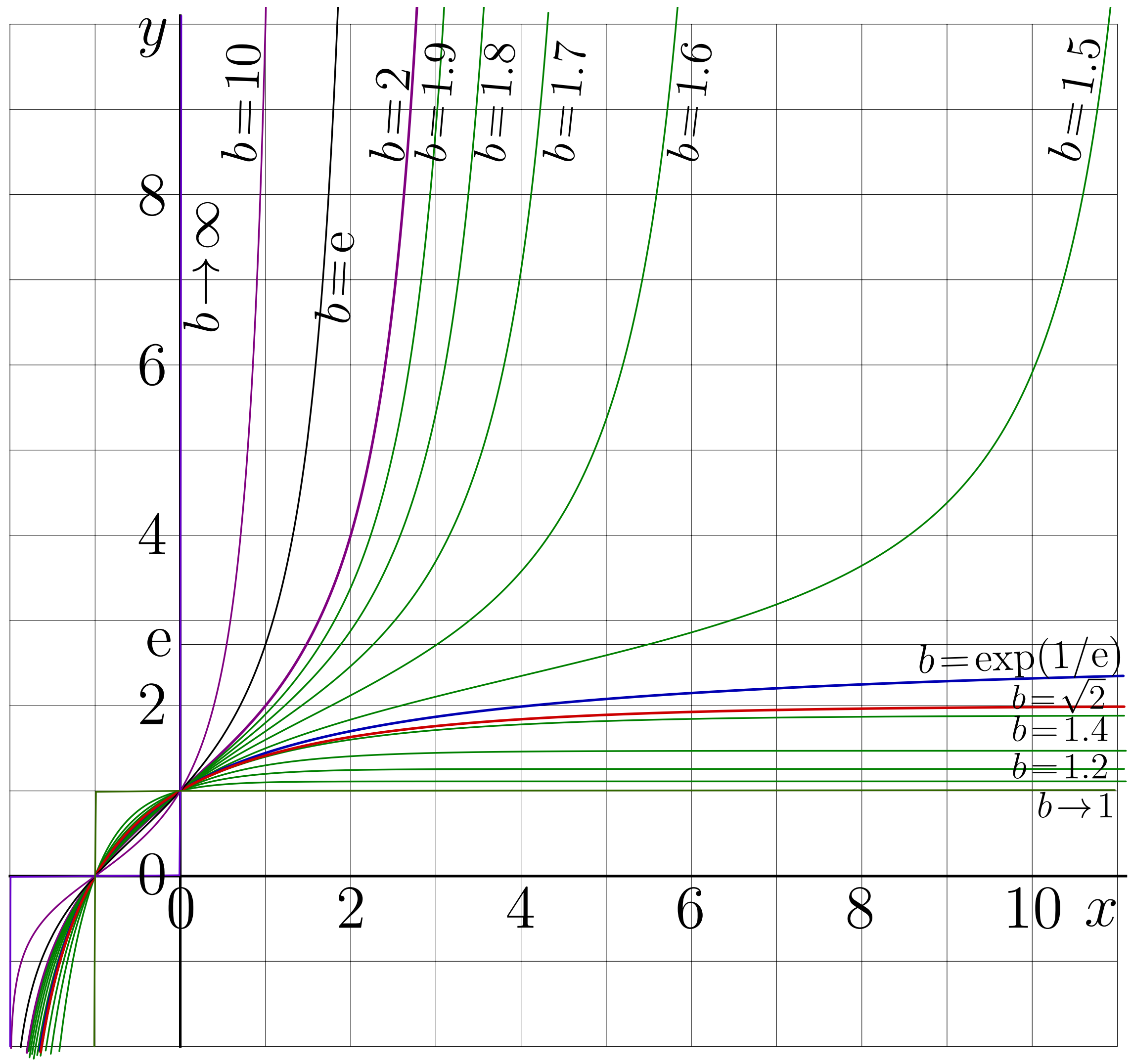
1 Approximation of tetration near zero

For the base b from the interval $1<b<\exp(1/\mathrm{e})$, tetration tet_b can be evaluated with regular iteration at the lowest (smallest) positive fixed point of the exponent to this base. For $b=\exp(1/\mathrm{e})$, it can be evaluated with the exotic iteration by equation 10.44. For $b > 1/\exp(1/\mathrm{e})$, the representation through the Cauchy integral can be used. In such a way, all the domain $b>1$ is covered with the efficient algorithms for evaluation of tet_b.

For real values of the argument, graphic $y=\mathrm{tet}_b(x)$ is shown in figure 17.1 versus x for various values of base $b>1$. Similar plot for arctetration is shown in figure 17.2.

Figure 17.1 for tetration and figure 17.2 for arctetration are generated, using the special approximation of tetration for $b<3$, by function fit1, defined with

$d=\ln(b)$



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http://en.citizendium.org/wiki/File:Tetreal10bx10d.png

Figure 17.1: $y = \mathrm{tet}_b(x)$ for various b [tet10bx]

$q=\sqrt{d}$

$c_0=-1.0018+\frac{0.1512848482(1.+33.0471529885q-3.51771875598d)q}{1+3.2255053261256337q}+\frac{\ln(2)-\frac{1}{2}}{d}$

$c_1=1.1-2.608785958462561(1-0.6663562294911147q)q-\frac{\ln(2)-\frac{5}{8}}{d}$

$c_2=-0.96+3.0912038297987596\frac{(1+0.60213980487853d)q}{1+4.24046755648d}+\frac{\ln(2)-\frac{2}{3}}{d}$

$c_3=1.2-10.44604984418533\frac{(1+0.213756892843q+0.369327525447d)q}{1+4.9571563666q+7.702332166d}-\frac{\ln(2)-\frac{131}{192}}{d}$

$$\mathrm{fit1}_b(z)=(1+c_0z+c_1z^2+c_2z^3+c_3z^4)(z+1)+\ln(z+2)-\frac{\ln(2)}{d}(1+z) \tag{17.1}$$

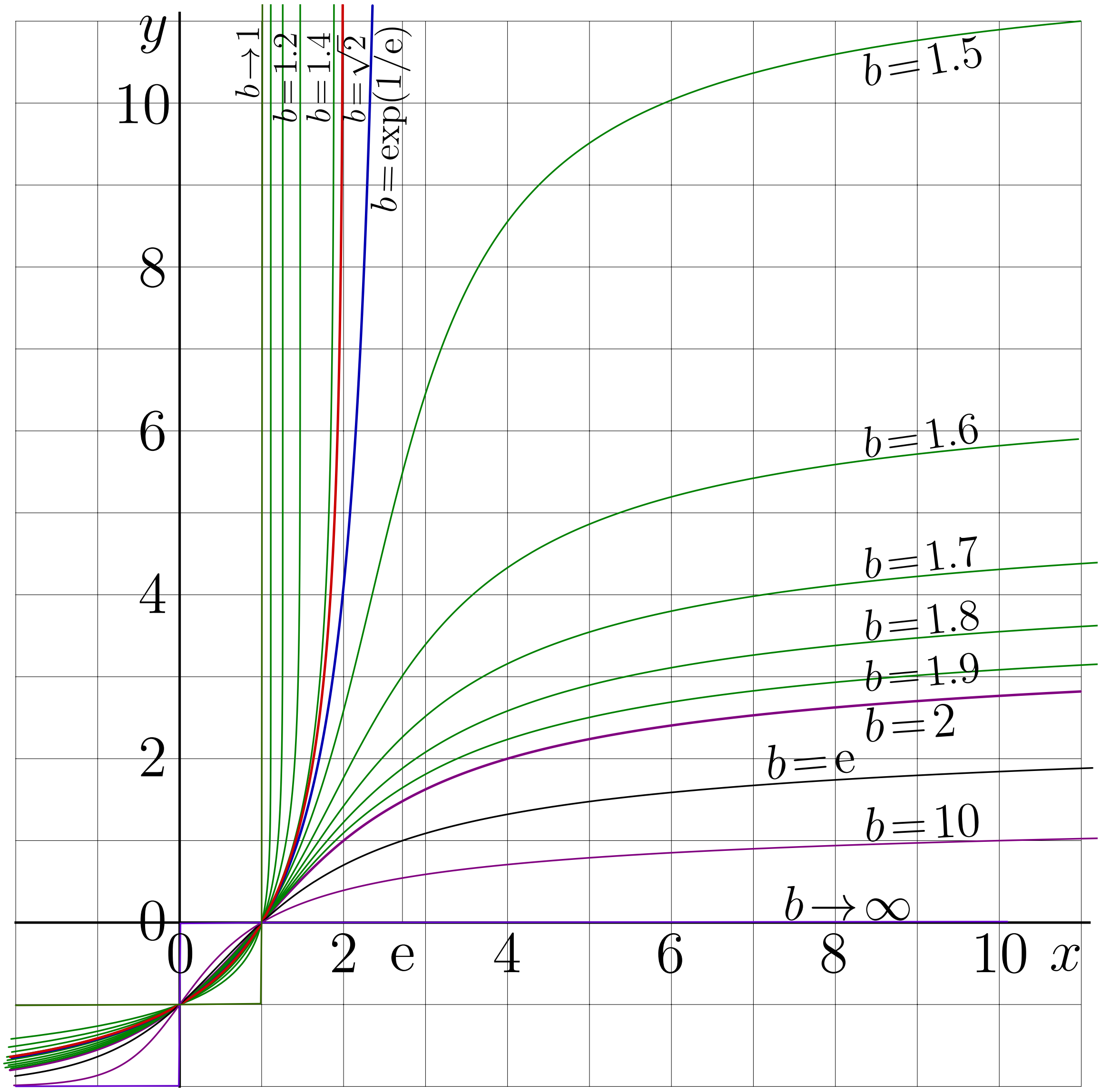

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Figure 17.2: $y = \mathrm{ate}_b(x)$ for various b [ate10bx]

In order to get approximation (17.1), I expanded the expression

$$\frac{\mathrm{tet}_b(z) - \ln(z+2) + \ln(2)/\ln(b)}{1+z} \tag{17.2}$$

into the Taylor series with powers of z for various values of (new) parameter $d = \ln(b)$. Coefficients of this expansion are approximated as functions of parameter d. Then, tetration is expressed through this expansion. The series is truncated; only few terms are taken into account. For $|z| \le 1/2$, the resulting approximation provides few correct decimal digits of $\mathrm{tet}_b(z)$. This approximation is used for $|\Re(z)| \le 1/2$; for other values of z, value of tetration is represented through its values at the appropriate argument, usng either

$$\mathrm{tet}_b(z) = b^{\mathrm{tet}_b(z-1)} \quad \text{[tetbminus]} \tag{17.3}$$

or

$$\mathrm{tet}_b(z) = \log_b\!\big(\mathrm{tet}_b(z+1)\big) \quad \text{[tetbplus]} \tag{17.4}$$

dependently on signum of $\Re(z)$. In such a way, tetration can be evaluated with few decimal digits for moderate values of $|\Im(z)|$ and $|\Im(b)|$. Yet, I have no similar approximation for larger values of $|\Im(b)|$, although, the Cauchy integral can be generalised for various, even complex values of b; the example is considered below in chapter 18. The range of validity of approximation $\mathrm{fit1_b}$ is wide; In particular, at $b<5$, the approximation provides of order of four significant figures. This is sufficient precision for plotting of the camera-ready pictures, the defects of this approximation are not seen in figure 17.1. However, at $b=10$, the approximation is a little bit worse; for this value, the primary representation through the Cauchy integral is used. This refers to only single curve in the picture.

Figure 17.1 shows behaviour of tetration of real argument at various values of base $b>1$:

In the interval $-2 \le x \le -1$, tetration $y=\mathrm{tet}_b(x)$ has negative values and grows while b increases.

In th interval $-1 \le x \le 0$, tetration $y=\mathrm{tet}_b(x)$ has positive value and grows with increase of b.

At $x>0$, tetration $y=\mathrm{tet}_b(x)$ has positive values and grows with increase of b.

In the limiting case $b\rightarrow 1$, the curve $y=\mathrm{tet}_b(x)$ approaches the asymptotics $x=-1$ and $y=1$.

In the limiting case $b\rightarrow\infty$, curve $y=\mathrm{tet}_b(x)$ approaches asymptotics $x=-2$ and $x=0$, and also $-2<x<0$ at the abscise axis.

At all $b>1$, tetration $\mathrm{tet}_b(x)$ is monotonic function, and curve $y=\mathrm{tet}_b(x)$ passes though points $(-1,0)$, $(0,1)$ and $(1,b)$.

At $b=\exp(1/\mathrm{e})$, the line $y=\mathrm{filog}(1/\mathrm{e})=\mathrm{e}$ becomes asymptotics. this line is added to the rectangular grid at integer values ob abscissas and ordinates.

At $1<b\le\exp(1/\mathrm{e})$, with grow of x the curve $y=\mathrm{tet}_b(x)$ approaches the horizontal asymptotics $y=\mathrm{filog}\big(\ln(b)\big)$.

Function filog expresses fixed points of logarithm as function of logarithm of its base. This function is considered in the next chapter. It is essential for evaluation of tetration of complex base. While, suggest to compare the results of the previous chapters for the real base. This is matter of the next section.

2 Various bases of exponent and the iterates

In this section, the iterates of exponent are compared. These iterates are considered in the previous sections. Below, I overview the results.

Iterates of exponent to bases $b = \mathrm{e}$, $b = \exp(1/\mathrm{e})$ and $b = \sqrt{2}$ are shown in figure 17.3 as functions of the real argument for various values of number n of iterate. The curves are drawn for $n = -2$, -1, -0.9, -0.5, -0.1, 0, 0.1, 0.5, 0.9, 1, 2.

The upper picture in figure 17.3 represents the case $b = \mathrm{e}$; iterates of the natural exponents are shown. These iterates are calculated through the natural tetration tet and natural arctetration ate, I repeat this formula once again:

$$y = \exp^n(x) = \mathrm{tet}\Big(n + \mathrm{ate}(x)\Big) \tag{17.5}$$

Similar pictures can be plotted also for other values of base $b >\exp(1/\mathrm{e})$. For these values of base, the real iterates are real-holomorphic functions at least in some vicinity of the real axis. In this area, the iterates of the exponent are so smooth, as the exponent itself.

As the base b, the width of the strip along the real axis (where the iterates are holomorphic) decrease; at base $b = \exp(1/\mathrm{e})$ all the curves for various iterates pass through the fixed point e. This case is shown in the central picture of figure 17.3. Then, the non-integer iterates of exponents at argument, larger than e, are not anymore holomorphic extensions of those for argument, smaller than e. In order to stress this, the curves, plotted through tetration and arctetration, are shown with dashed lines

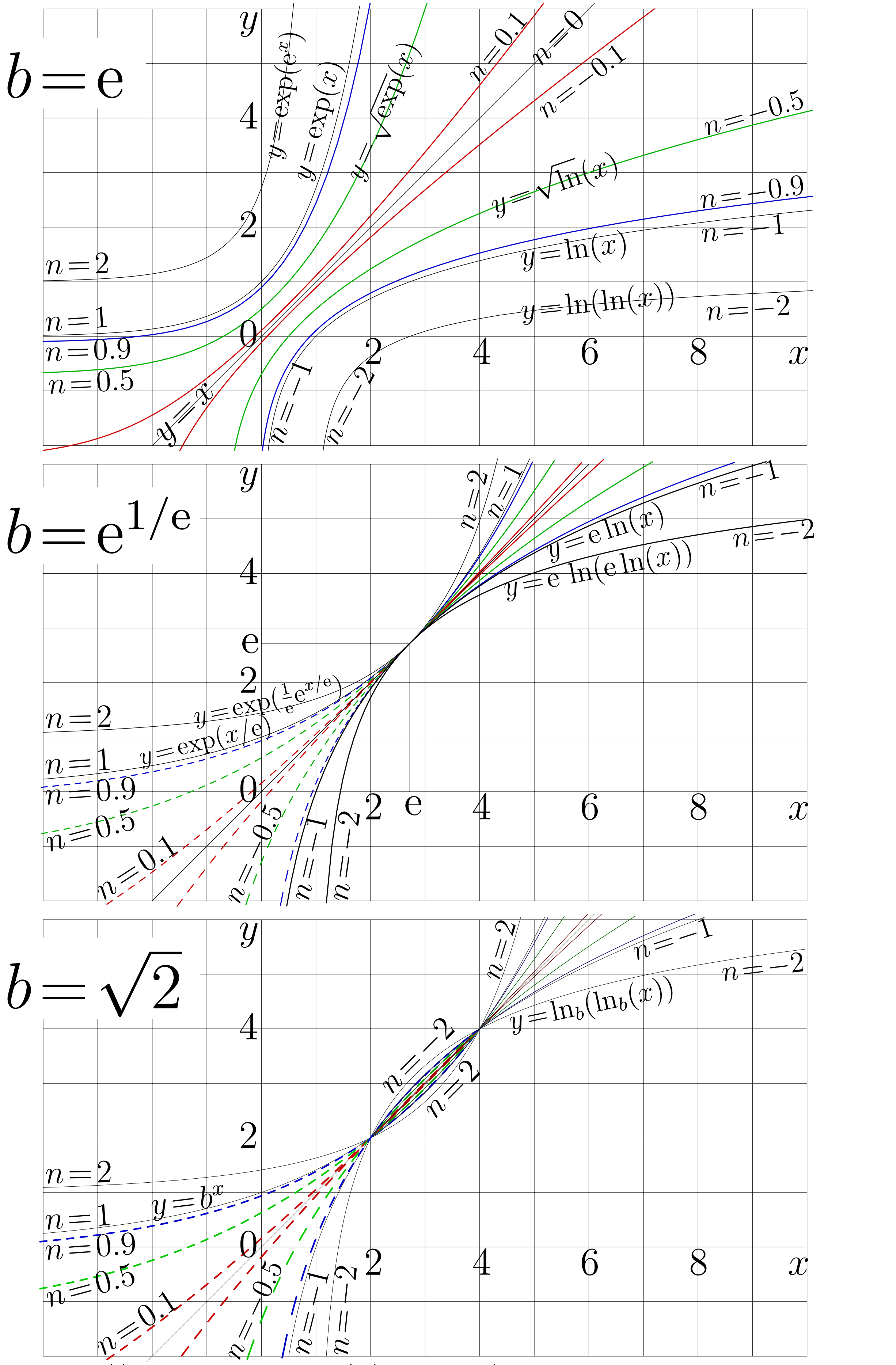
$$y = \exp_{b,\mathrm{d}}^n(x) = \mathrm{tet}_b\Big(n + \mathrm{ate}_b(x)\Big) \tag{17.6}$$

while the solid lines correspond to the iterates, expressed through the growing superexpponent SuExp,

$$y = \exp_{b,\mathrm{u}}^n(x) = \mathrm{SuExp}_{b,3}\Big(n + \mathrm{AuExp}_{b,3}(x)\Big) \tag{17.7}$$

Tetration tet_b to base $b = \eta = \exp(1/\mathrm{e})$ is determined with (10.44), and the arctetration to this base can be evaluated through (10.49). In the similar way, the growing superexpenent is determined by (10.45) with $\mathrm{SuExp}_{b,5} = F_3$, and the corresponding abelexponent can be evaluated with (10.50) at $\mathrm{AuExp}_{b,3} = G_3$.

For the real base b, at $1 < b < \exp(1/\mathrm{e})$, The exponent has two real fixed points (See figures 9.1, 10.1). Each to these fixed points can be used for



`http://mizugadro.mydns.jp/t/index.php/File:E1e14z600.jpg`

Figure 17.3: $y=\exp_b^n(x)$ for various n at $b=\mathrm{e}$, $b=\exp(1/\mathrm{e})$, $b=\sqrt{2}$ [e1e14]

the regular iteration. However, the non-integer iterate, regular at one fixed point, is singular at another one. At the bottom picture of figure 17.3, value $b=\sqrt{2}$ is chosen.
Dashed lines corresponds to

$$y = \exp_{b,\mathrm{d}}^{n}(x) = \mathrm{tet}_b\Big(n + \mathrm{ate}_b(x)\Big) \tag{17.8}$$

while the solid curves refer to

$$y = \exp_{b,\mathrm{u}}^{n}(x) = \mathrm{SuExp}_{b,5}\Big(n + \mathrm{AuExp}_{b,5}(x)\Big) \tag{17.9}$$

For non-integer number n of iterate, the fixed point 2 or 4 limits the range of holomorphism of each iterate. In the interval from 2 to 4, each of the two iterates is holomorphic, and the difference between these iterate is very small, of order of 10^{-24}, see figure 16.8. Due to this smallness, the dashed lines at the bottom picture of figure 17.3 seem to coincide with the corresponding solid lines.

Similar illusion takes place for the central part of figure 17.3; the dashed lines seem to be continuations of the corresponding solid lines. The example with exponent show, that, in order to specify the non-integer iterate, one should choose, establish the asymptotic behaviour of the superfunction in the complex plane. Over-vice, there may exist various solutions, and each of them arguably can be declared as the "true" one. In order to specify the superfunctions, they are considered for the complex argument.

3 Dependence of tetration on its base

Graphics of tetration of real argument, shown in figure (17.1), allow to guess, that the dependence of tetration on the base (at fixed argument) is continuous (and, perhaps, even holomorphic) function. In order to show, that this refers not only to the real values of the argument, figure 17.4 shows the complex maps of tetration for $b=1.5$ at left, for $b=\exp(1/\mathrm{e})\approx 1.44$ at center, and for $b=\sqrt{2}\approx 1.41$ at right.

All the 3 maps in figure 17.4 look similar, although different algorithms are used for evaluation of tetration. In principle, tetration to base $b=\exp(1/\mathrm{e})$, for moderate values of the imaginary part of the argument, could be evaluated also through tetration with a little bit smaller or a little bit larger values of the base b, as limit $b\to\exp(1/\mathrm{e})$, using the corresponding representation through the integral Cauchy (for bigger values) or with regular iteration (for smaller values). The Readers are invited to calculate tetration to base $b\approx\exp(1/\mathrm{e})$ and estimate, how

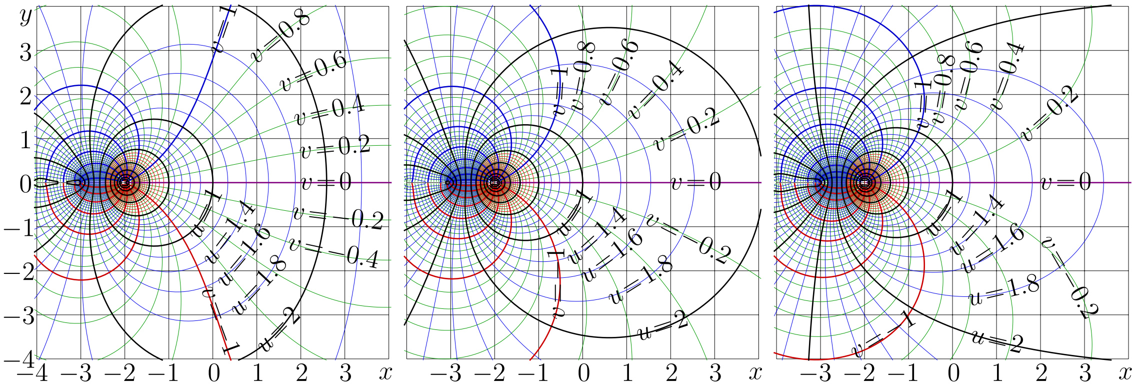

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Figure 17.4: $u+\mathrm{i}v=\mathrm{tet}_b(x+\mathrm{i}y)$ for $b=1.5$, $b=\exp(1/\mathrm{e})$ and $b=\sqrt{2}$ [e1e09]

many significant figures can one achieved in such a way, assuming, that the arithmetics with finite precision is used.

At $b \to \exp(1/\mathrm{e})$, the efficiency of evaluation of tetration (both through the Cauchy integral and through the regular iteration) reduces. For this reason, Henryk Trappmann had expected, that the tetration is not continuous function of b at point $b = \eta = \exp(1/\mathrm{e})$. For this reason, Henryk wanted the asymptotic expansion namely for $\eta = \exp(1/\mathrm{e})$, and it had been done [1]. I had suggested the expansion (10.36) and plotted pics for $b = \exp(1/\mathrm{e})$, and Henryk had arranged a lot of mathematical deduction around it [79].

I hope, the Reader already understands, how to guess the expansion for the exotic iterates, and can write the similar expansion of superfunction for any other transfer function, as soon as such an superfunction will be requested for any application. The inversion of the series gives the expansions (and the precise approximations) for the corresponding abelfunctions.

I hope, with the tools above, the colleague can evaluate any iterate of any holomorphic transfer function, not only real, but also complex. The examples with iterates number i are hown in figures 16.10 and 16.11 for the transfer function $T = \exp_{\sqrt{2}}$.

After publications of the results presented above, the colleagues at the Henryk's forum had agreed, that the complex iterates, and in particular, those of the exponent to various real base $b > 1$, can be evaluated in a pretty regular way. However, There were some doubts about iterates of the exponent to the complex base. This case is considered in the next chapter.

[1] This was soon after the article about four real-holomorphic superexponentials to base $\sqrt{2}$ had been submitted to Mathematics of computation [61]; as usually, the appetite comes while eating.

Chapter 18

Tetration to complex base

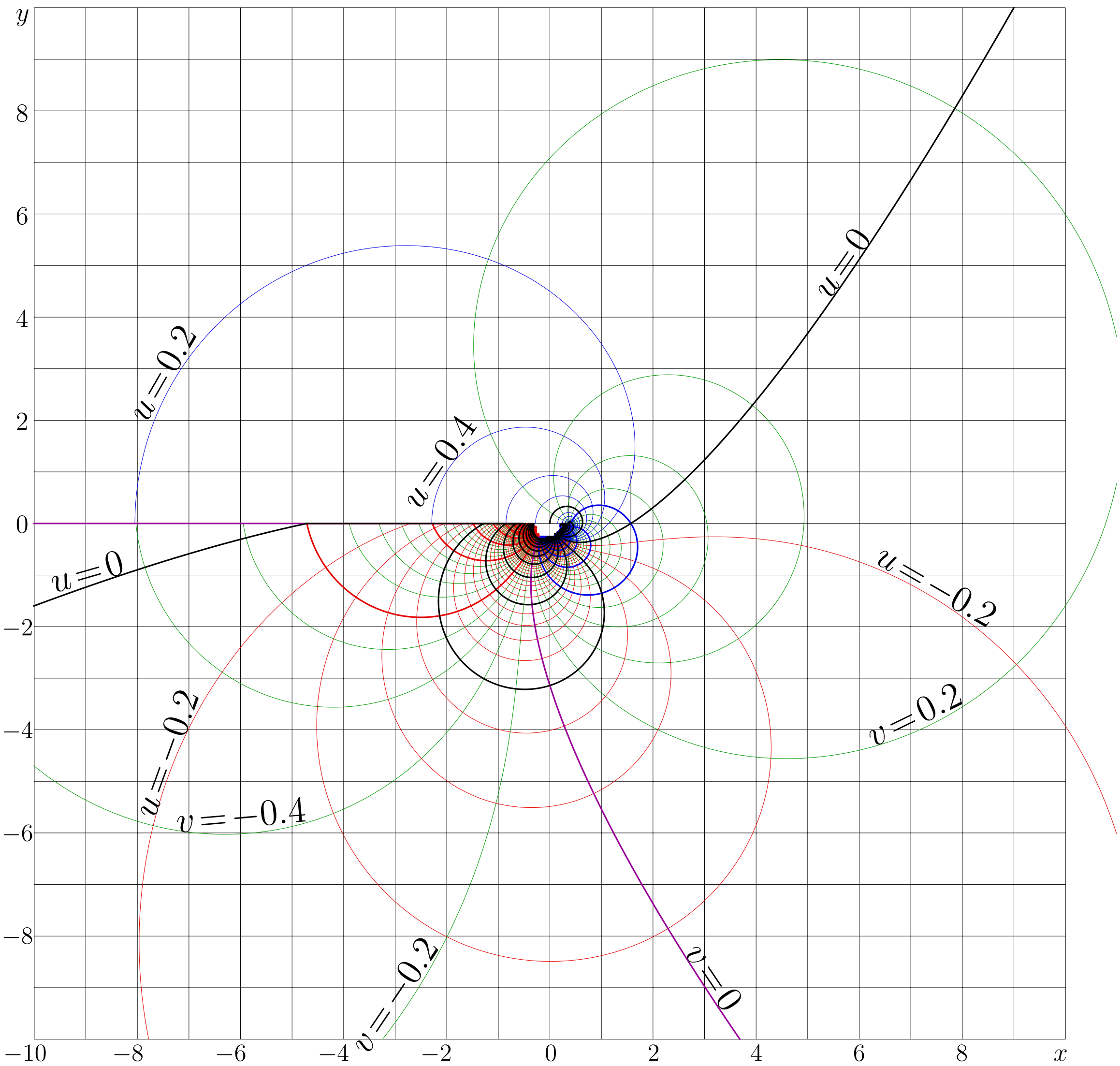
Here, tetration to complex base is considered. Id est, superfunction for the transfer function $T(z) = b^z$, where base b is not real. In order to hande this case, I need an additional special function "flog"; it is also described in this chapter.

In principle, the superfunction of the exponent to complex base can be constructed with regular iteration, in the similar way, as the tetration to base $\sqrt{2}$ is constructed in Chapter 16. However, the important question is, at which of the fixed points, the superfunction should be regular, and which of possible superfunction should be qualified as tetration.

In addition, at some values of the base, the exponential asymptotic solution has real part of the increment zero or close to zero; this makes the application of regular iteration difficult, if at al. One of such cases is considered below in more details, as an example. The representation of tetration through the Cauchi integral is not sensitive to the real parti of the asymptotic increment of the solution; so, such a representation gives the efficient way of evaluation of tetration to complex base. However, the asymptotic behaviour of tetration in the upper and in the lower complex plane should be specified.

The main idea of this chapter is to make superfunction that approaches one of the fixed points of the exponent at the upper half of the complex plane, and to another fixed point at the lower part of the complex half plane, using the assumption, that, for the complex base, the imaginary part of the asymptotic increment is not zero.

Question about the fixed points is important (as in the case of any other superfunction), and it should be considered. This consideration is presented in the following section.



http://mizugadro.mydns.jp/t/index.php/File:Filogbigmap100.png

Figure 18.1: $u+\mathrm{i}v=\mathrm{filog}(x+\mathrm{i}y)$, as solution $L=\mathrm{filog}(B)$ of (18.1)

1 Fixed points of logarithm

This section is dedicated to relation between the base b of the exponential and its fixed point L. Let $L = \mathrm{filog}(B)$ be solution of equation

$$\ln(L) = BL \quad \text{[LL]} \tag{18.1}$$

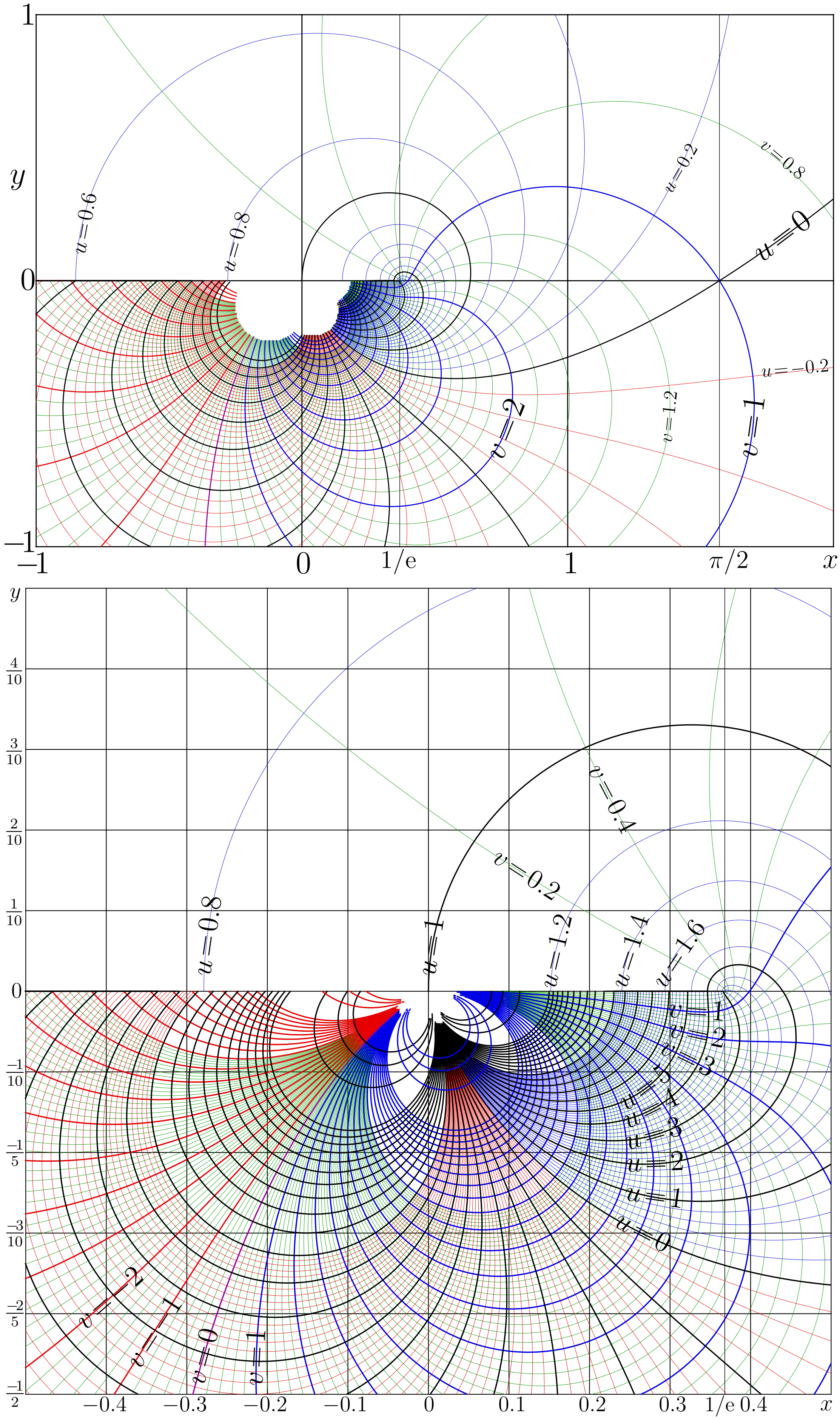
Complex map of function filog is shown in figure 18.1. The zoom-in of the central part of this map is shown in figure 18.2. Let

$$B = \ln(b) \quad \text{[Bb]} \tag{18.2}$$

Then filog(B) determines the fixed point L_1 of logarithm to base b; another fixed point L_2 is determined with the complex conjugation:

$$L_1 = \mathrm{fllog}(B) \quad \text{[L1filog]} \tag{18.3}$$

$$L_2 = \mathrm{filog}(B^*)^* \quad \text{[L2filog]} \tag{18.4}$$



http://mizugadro.mydns.jp/t/index.php/File:Figlogzo2t.jpg

Figure 18.2: $u+\mathrm{i}v = \mathrm{filog}(x+\mathrm{i}y)$, zoom-in from the central part of figure 18.1 [filogzo2]

Function filog can be expressed through the special function Tania by (5.3) as follows:

$$\mathrm{filog}(z) = \frac{\mathrm{Tania}\big(\ln(z) - 1 - \pi \mathrm{i}\big)}{-z} \tag{18.5}$$

Note, that here, namely Tania is used, but not WrightOmega, which looks similar to Tania in vicinity of the real axis. The Readers are invited to try to reproduce figure 18.1, using WrightOmega instead of Tania, and look, what does it give instead of the beautiful map.

Function filog determines the fixed points of logarithm (which are also fixed point of the exponent) to base b:

$$L_1 = \mathrm{filog}\big(\ln(b)\big) \qquad \text{[L1filob]} \tag{18.6}$$

$$L_2 = \mathrm{filog}\big(\ln(b^*)\big)^* \qquad \text{[L2filob]} \tag{18.7}$$

At $\pm \mathrm{i}\infty$, tetration should approach these fixed points. This assumption to use the Cauchy integral for the definition (and evaluation)

2 Tetration to the Sheldon base

This section describes the tetration to the Sheldon base,

$$b = 1.52598338517 + 0.0178411853321\,\mathrm{i} \qquad \text{[sheldonS]} \tag{18.8}$$

This number is named after Sheldon Levenstein. In 2015, Sheldon had expected, that the namely this base causes difficulties at evaluation of tetration. It was the only request from colleagues to evaluate tetration to the specific complex base; and this request had been fulfilled.

Complex map of tetration to the Sheldon base is shown in the top picture of figure 18.3. Explicit plot of this function is shown in the central picture of that figure. The bottom picture shows the explicit plot of tetation to Sheldon base along the imaginary part. This tetration has complex values; so, the graphics are drawn for the real and for the imaginary parts.

In this section I assume, that value of b is determined with equation (18.8). I consider this as an example; tetration to other values can be calculated in the similar way.

It is convenient to define $B = \ln(b)$. Then, the fixed points of logarithm to base b, id est, solutions L of equation $\ln_b(L) = L$, can be expressed through function filog, described in the previous section:

$$L_1 = \mathrm{filog}(B) \approx 2.0565398441043761 + 1.1445267140098765\ \mathrm{i} \tag{18.9}$$

$$L_2 = \mathrm{filog}(B^*)^* \approx 2.2284359658711805 - 1.3507994961102865\ \mathrm{i} \tag{18.10}$$

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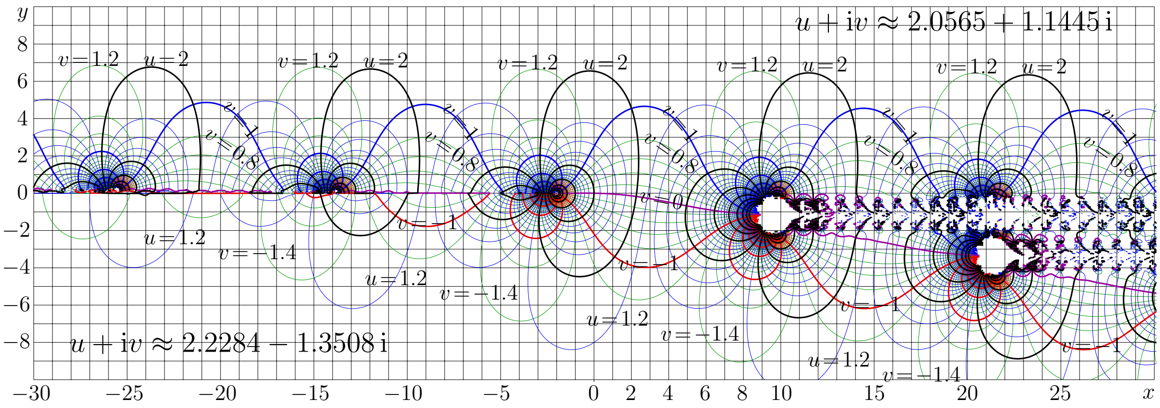

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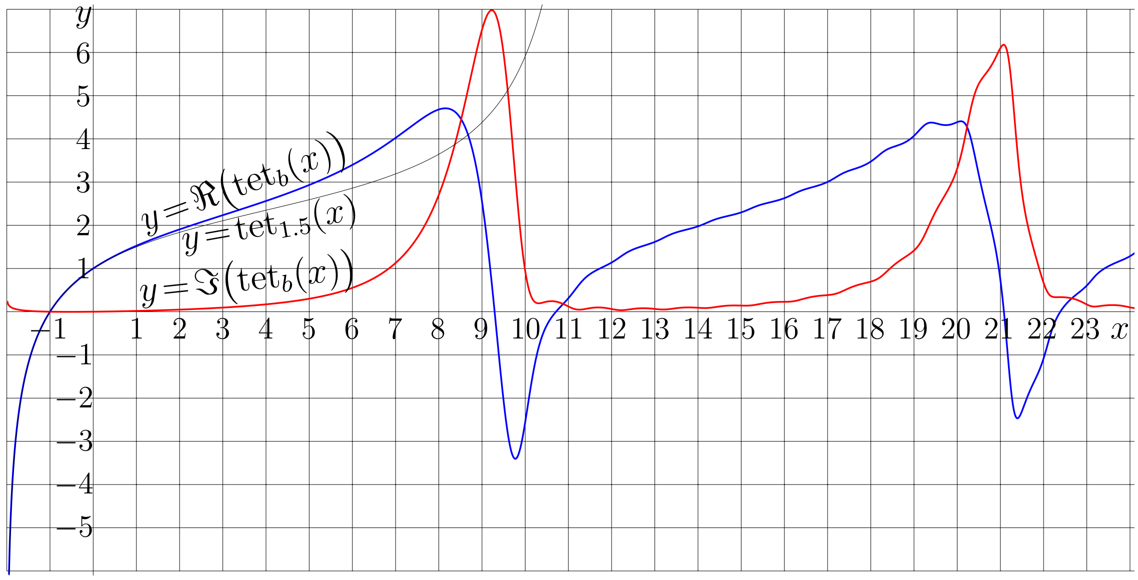

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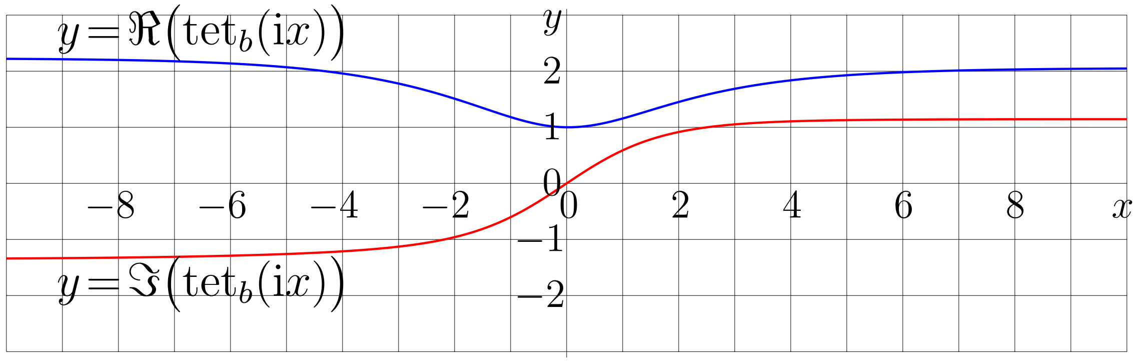

Figure 18.3: Tetration to base $b=1.52598338517+0.0178411853321\,\mathrm{i}$:
Top map: $u+\mathrm{i}v=\mathrm{tet}_b(x+\mathrm{i}y)$;
Intermediate plot: $y=\Im(\mathrm{tet}_b(x))$ and $y=\Re(\mathrm{tet}_b(x))$, thick curves;
the thin curve shows $y=\mathrm{tet}_{1.5}(x)$ from figure 17.1 ;
Bottom plot: $y=\Im(\mathrm{tet}_b(\mathrm{i}x))$ and $y=\Re(\mathrm{tet}_b(\mathrm{i}x))$ [sheldonmap]

where constant B is evaluated as follows:

$$B = \ln(b) \approx 0.4227073870410604 + 0.0116910660021443\mathrm{i} \quad (18.11)$$

Solution F of the transfer equation $b^{F(z)} = F(z{+}1)$ with asymptotics

$$F(z) = L_1{+}\exp(k_1 z{+}\phi_1) + \mathcal{O}(\exp(2k_1 z)) \text{ при } \Im(z){\rightarrow}\infty \quad (18.12)$$
$$F(z) = L_2{+}\exp(k_2 z{+}\phi_2) + \mathcal{O}(\exp(2k_2 z)) \text{ при } \Im(z){\rightarrow}{-\infty} \quad (18.13)$$

can be considered in the similar way, as for tetration to real base, larger than the Henryk base $\eta = \exp(1/\mathrm{e})$. Even the same contour of integration can be used. Subslitution of the asymptotic solutions (18.12), (18.13) into the transfer equation determines the increments

$$k_1 = \ln(L_1 b) \approx -0.0047589243931785{+}0.5354935770338939\,\mathrm{i} \quad (18.14)$$
$$k_2 = \ln(L_2 b) \approx \quad 0.0970758595007548{-}0.5172895961559847\,\mathrm{i} \quad (18.15)$$

The solution has quasiperiod

$$P_1 = \frac{2\pi\mathrm{i}}{k_1} \approx 11.7325200133916496{-}0.1042667514229599\,\mathrm{i} \quad (18.16)$$

in the upper part of the complex plane, and quasiperiod

$$P_2 = \frac{2\pi\mathrm{i}}{k_2} \approx 11.7331504449085493{-}2.2018723603861230\,\mathrm{i} \quad (18.17)$$

in the lower part of the complex plane,

The properties above are sufficient to express the solution $F(z)$ of the transfer equation

$$F(z{+}1) = \exp\Big(B\,F(z)\Big) \quad \text{[sheldonTra]} \quad (18.18)$$

through the Cauchy integral. This construction is quite analogous to that for the tetration to base e, described in chapter 14; therefore I do not repeat here the description of the contour of integration nor the iterational procedure, that provides the approximations of the solution. The tetration is expressed through the solution F in the following form:

$$\mathrm{tet}_b(z) = F(z_1 + z) \quad \text{[sheldonTetDef]} \quad (18.19)$$

where z_1 is solution of equation $F(z_1) = 1$. Using equation (18.18), the solution can be extended at least to the right hand side of the complex plane. As for the left hand side, in the Second quadrant of the complex plane, the branch points and the cutlines appear. These cutlines appear at the use of the logarithmic function to extend the solution to the direction of negative values of the real part of the argument.

Unfortunately, the most of cut lines do not fit the frame of map in figure 18.3, althogh I make it in the whole width of the page. The asymptotic (18.12) indicates, that these cuts are unavoidable, while the imaginary part of quasi period P_1 is negative. Each time, when the tetration takes value zero, there is branch point at value of the argument, for unity smaller. These cuts are unavoidable also for other tetrations, while formula

$$F(z) = \log_b{}^m(F(z+m)) \tag{18.20}$$

is used for some integer m such that value $z+m$ belongs to the strip $\{z \in \mathbb{C} : |\Re(z)| \leq 1/2\}$. In this sense, the tetration to the Sheldon base is similar to tetration to the real base.

3 Shell-Thron region

For moderate values of argument z, tetration $\mathrm{tet}_b(z)$ looks as a smooth function of base b. At large values of $\Re(z)$, dependently on the base, tetration either has complicated, quasi-chaotic behaviour, or approaches some of the fixed point of the corresponding logarithm. At the site "Eretrande", the range of values of base b, for which the tetration approaches its limiting value, is called Shell-Thron region [1]. Such a name seems to be commonly accepted, and I even suggest the Russian literal translation "Область Тронной Ракушки" for the Russian version of this Book. However, yet, it is difficult to predict, how convenient and stable are these names.

In this Book, I am interested mainly the tools for evaluation of superfunctions, and resolving the paradoxes, that are discussed among colleagues. Consideration of many examples, that cause no doubts, fall out of scope of this Book.

In principle, at the iterates of exponent to complex base, we have to deal with the 6-dimensional space. Coordinates of this space are the real and imaginary parts of base b (or real and imaginary parts of its logarithm, $B = \ln(b)$), the real and imaginary parts of the argument,and the real and imaginary parts of the number of iterate. There may be many interesting effects hidden in the 6-dimensional space. The detailed description of these effects may require a special book, dedicated namely

[1] Shell-Thron region:
http://math.eretrandre.org/hyperops_wiki/index.php?title=Shell-Thron_region.

to tetration to complex base. Since year 2010, Henryk Trappmann tries to compose such a book [66] (I had promised to provide all the algorithms and pictures he needs for this, and all the pictures requested up to date are already plotted an included there). I hope, one day Henryk gets good grant for this activity and will be able to polish the text to the state he considers as satisfactory. While I still see no principal problem, that cause serious difficulty at the evaluation of the corresponding iterates; so, I do not include parts of [66] to this Book. For this reason, analysis of the Shell-Thron region is presented here in a declarative form.

I hope, the Readers can plot all the pictures that are necessary for illustration of tetration to the complex base, using the tools described above. However, if any difficulties of paradoxes appear in the analysis, I shall try to consider and to resolve them. This point of view is described by the Russian writers Arkadi & Boris Strugatski in the novel "Monday begins on saturday" [14]: .. *It's nonsense to look for a solution if it already exists. We are talking about how to deal with a problem that has no solution. ..*

I hope, the Readers can evaluate tetration to other bases by themselves, using the tools any examples above. Following the idea mentioned, I continue to deal with cases that are believed to have "no solution". One of the such cases refer to the superfunction of tetration; let it be called "pentation". In order to bring is to the textbook case, as it is shorn in figure 18.4, this function, among other ackermanns, is considered in the next chapter.

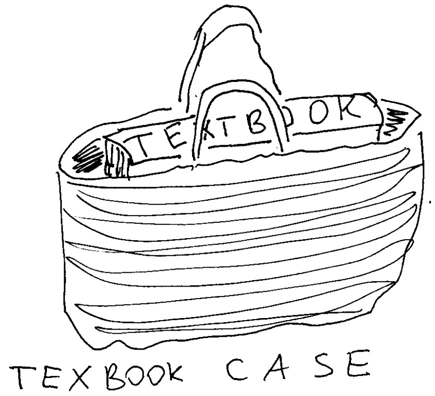

Figure 18.4: Textbook case

Chapter 19

Ackermanns

Figure 19.1: W.Ackermann

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— Do you know, owr man asks hoggible questions!
— Who is that man?
— He is called Asker-man!

Here I retell some results for holomorphic ackermanns, tetration and pentation [89]. For base $b > 1$, tetration $f = \mathrm{tet}_b$ is solution of equations

$$f(z+1) = b^{f(z)} \quad , \quad f(0) = 1 \qquad \text{[tetb]} \tag{19.1}$$

holomorphic at least in range

$$z \in \mathbb{C} : \Re(z) > -2 \qquad \text{[htetb1]} \tag{19.2}$$

that is limited in range

$$z \in \mathbb{C} : |\Re(z)| \le 1 \qquad \text{[htetb12]} \tag{19.3}$$

Equation (19.1) appears as special case of the Ackermann equations

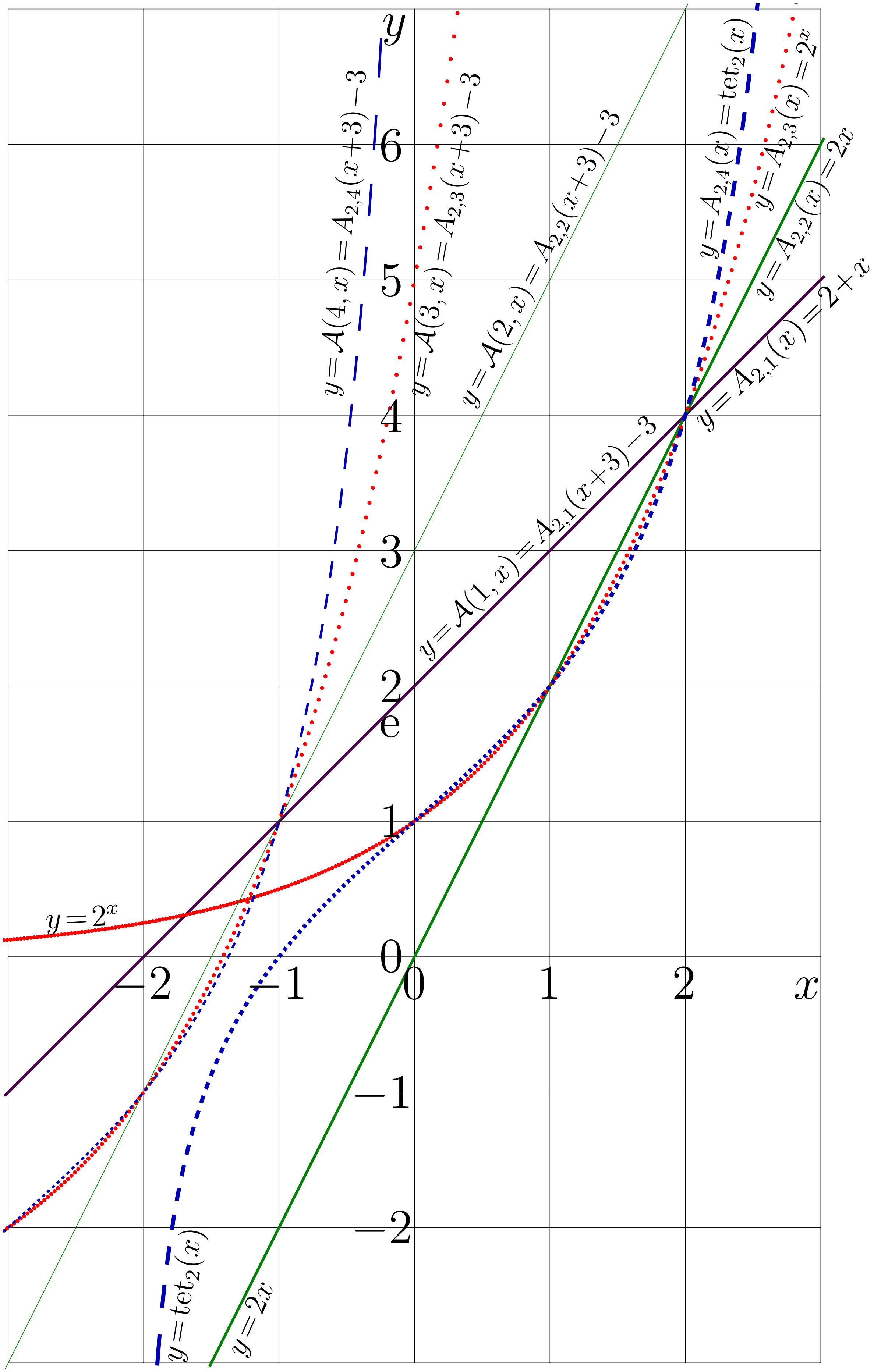
$$A_1(z) \;=\; b + z \;,\; z \in \mathbb{C} \qquad \text{[Acker1]} \tag{19.4}$$

$$A_n(1) \;=\; b \;,\; n \in \mathbb{N}, n \ge 1 \qquad \text{[Acker2]} \tag{19.5}$$

$$A_n(z+1) = A_{n-1}\big(A_n(z)\big) \;,\; n \in \mathbb{N}, n > 1 \qquad \text{[Acker3]} \tag{19.6}$$

I call functions A "ackermanns" in order to avoid confusion with mathematician Wilhelm Ackermann, shown in figure 19.1. His last name, to make difference from the name of the function, is written with Capital letter [7]. Ackermanns A are subject of this chapter. However, for the highest ackernanns, the range of holomorphism has no need to be the same as condition (19.2) for tetration. This range, as well as the asymptotic behaviour of highest ackermanns should be specified. I cannot yet provide the general specification; below, I suggest only the first approach to the problem.

For base $b = 2$, the explicit plots of the first four ackermanna are shown in figure 19.2 with thick lines, solid, solid, dotted and dashed. The thin lines refers to another (and more conventional) system of notations, described in the next section.



http://mizugadro.mydns.jp/t/index.php/File:Acker2t400.jpg

Figure 19.2: Comparison of definitions for binary ackermanns [acker2t]

1 Binary ackermanns

In century 20, the ackermanns are considered mainly (or even exclusively) for the base $b=2$, and mainly (or even exclusively) for integer values of the argument. I call ackermanns to base $b=2$ as "binary" ackermanns.

For the binary ackermanns, the special notations are used. The number of ackermann is written as additional, first argument, and the base b is not indicated at all; there is no need to indicate it, as it remains to be two. Relation of the classical (and usual for today) notations for the ackermanns by (19.4) (19.5) (19.6) can be expressed with the simple formula

$$\mathcal{A}(m,z) = A_{2,m}(z+3) - 3 \quad \text{[asa3]} \tag{19.7}$$

In particular,

$$\begin{aligned}
A_{2,1}(x) &= 2+x = \mathcal{A}(1,x{+}3)-3 = \mathcal{A}(1,x) && (19.8)\\
A_{2,2}(x) &= \ 2\,x \ = \mathcal{A}(2,x{+}3)-3 && (19.9)\\
A_{2,3}(x) &= \exp_2(x) = \mathcal{A}(3,x{+}3)-3 = 2^x && (19.10)\\
A_{2,4}(x) &= \mathrm{tet}_2(x) = \mathcal{A}(3,x{+}3)-3 && (19.11)
\end{aligned}$$

Figure 19.2 shows relation between the single-argument function A with subscripts and conventional $\mathcal{A}$ without subscript, but with two arguments. Four ackermanns are plotted as functions of real argument; $y = A_{2,m}(x)$ for $m=1,2,3,4$ are shown with thick lines (solid, solid, dotted and dashed); dependences $y=\mathcal{A}(m,x)$ for $m=1,2,3,4$ are shown with thin lines (solid, solid, dotted and dashed);

Due to relation (19.7), the thick lines in Fig.19.2 can be obtained from thick lines by translation for 3 units along x axis and for the same along the ordinate axis; the only straight line for the First ackermann stays at its place at such a transform.

Especially for the binary ackermanns, the system of equations (19.4), (19.5), (19.6) can be a little but extended, in order to include the "zeroth" ackermann. Equations for $\mathcal{A}$ can be written as follows:

$$\begin{aligned}
\mathcal{A}(0,z) &= z+1 && (19.12)\\
\mathcal{A}(m{+}1,0) &= \mathcal{A}(m,1) && (19.13)\\
\mathcal{A}(m{+}1,z{+}1) &= \mathcal{A}\Big(m,\mathcal{A}(m{+}1,z)\Big) && \text{[ackerbi]}\ (19.14)
\end{aligned}$$

Displacemenr of both, argument and the function, by formula (19.7) can be qualified as conjugation. The transfer equation (19.14) for the binary ackermann in the "classcal" notations has the same form, as the transfer equation (19.6).

One of applications of the ackermanns is to denote the huge numbers (for real moderate values of the argument). Due to the displacement of the argument, the canonical ackermanns can make illusion of a little bit more fast growth, than the ackermanns by (19.4), (19.5), (19.6). However, for the applications, this "acceleration" is not important. I think, the notations in formulas (19.4),(19.5),(19.6) are better than the canonical notations. In notations (19.4), (19.5),(19.6), the third ackermann happens to be just exponent, the fourth - just tetration, and so on. In addition, I consider ackrmanns as superfunctions, holomorphic with respect to the last argument; so, it is good, to keep this argument single, The base b and the number of the ackermann appear as parameters; they can be specified in the subscript. I hope, the Reader will meet no problems using relation (19.7) for the conversion from one system of notations to another one.

2 Names and notations

As I had mentioned above, in century 20, the functions of Ackermanns are considered for base $b\!=\!2$ and only for integer values of the argument z. While, I see no fundamental limits, that would prohibit existence and evaluation of these functions for various, including complex, values of b and z; however, the appropriate requirement on the range of holomorphism should be formulated.
I hope, for all ackermanns, we may require the real holomorphism at least in some vicinity of at least some part to the real axis. In general, the solution of the transfer equation is not unique; so, we should indicate also the way of construction of each ackermann, or to guess (invent, postulate) its behaviour in the complex plane, following the idea, used to construct natural tetration described in chapter 14.

Several ackermanns already have special names:
$A_{b,1} \;=\; z \mapsto b\!+\!z$, addition of constant b,
$A_{b,2} \;=\; z \mapsto bz$, multiplication by constant b,
$A_{b,3} \;=\; \exp_b \;=\; z\!\mapsto\!\mathrm{pow}(b,z) \;=\; z\!\mapsto\! b^z$, exponent to base b,
$A_{b,4} \;=\; \mathrm{tet}_b \;=\; z \mapsto \mathrm{tet}_b(z)$, tetration to base b,
$A_{b,5} \;=\; \mathrm{pen}_b \;=\; z \mapsto \mathrm{pen}_b(z)$, pentation to base b.

The following functions can be denoted with sumbols hex_b, hep_b, oct_b,.. and be called, accordingly, with terms "hexation", "heptation", "octation" and so on. These notations appear at the truncation of latin names of numbers to three characters.

Initially, the formalism of sulerfunctions had been developed for tetration, id est, for superfunction of the exponent, for the Fourth ackermann. However, this formalism can be applied also for other functions, and, in particular, for various ackermanns. As an example, in the next section I consider the 5th ackermann, the pentation.

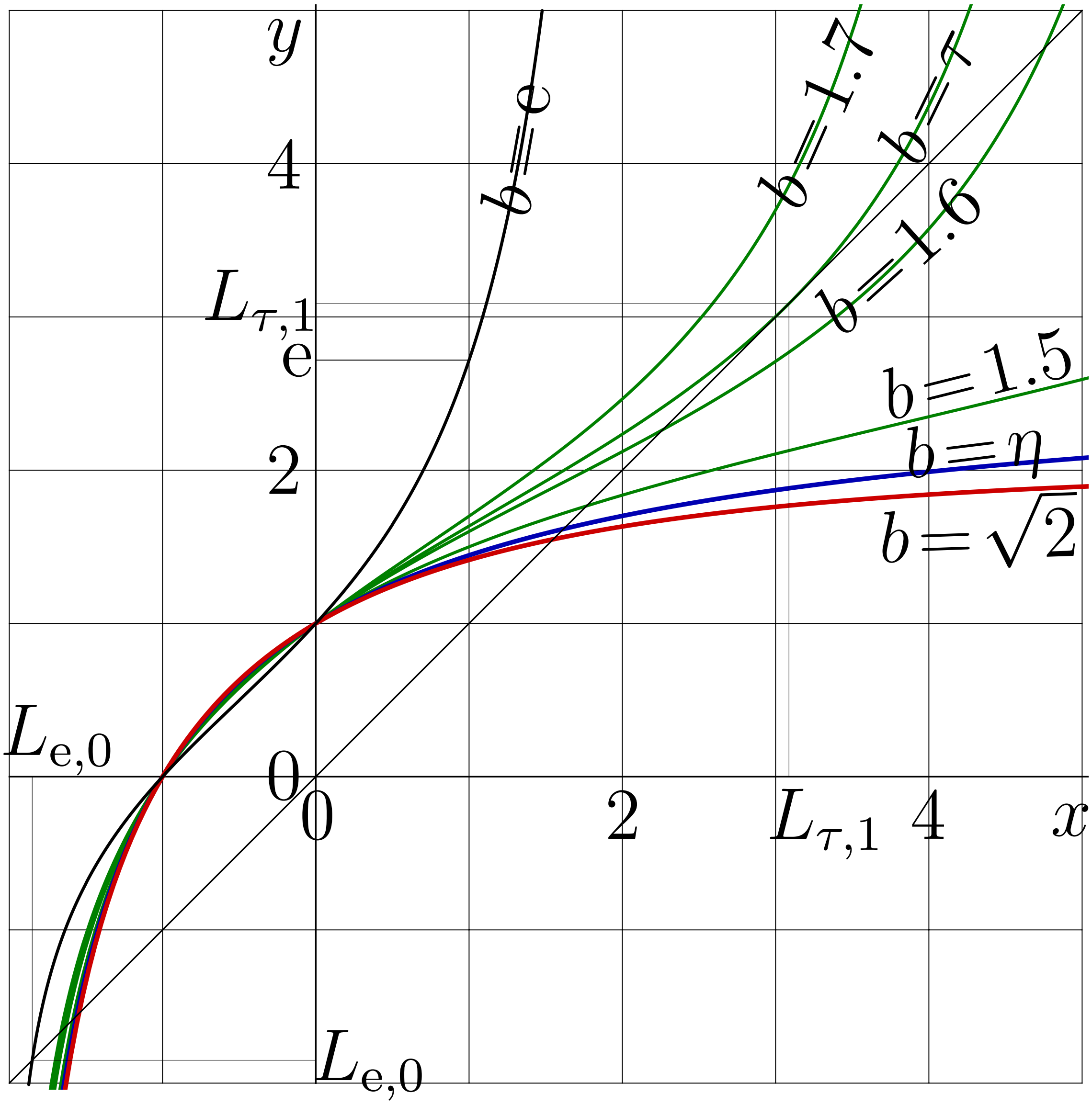
3 Pentation

If you give to some true Mathematician the teapot, the gas stove, the matches and water tap, and ask to prepare tea, the Mathematician puts water into the teapot, ignites the gas and put the teapot on the fire. If the Mathmatician finds the tea powder, then, perhaps, he or she even drop it into the hot water after to see the boiling and switch off the gas. But if, after that, the Mathematic again is asked to prepare the tea, while the water is already in the tea pot, and the gas fire at the stove is ready.. Ooh.. You may guess, that the true Mathematician shuts down the gas, drops the water from the teapot and spells the magic sentence: **"The problem is reduced to the previous one!"**.

I hope, with methods, described above, and especially, keeping in mind the previous paragraph, the Reader already can calculate the superfunctions, just following the general algorithms from this Book. On the other hand, I am more physicists, than mathematician. (The mathematician colleagues have no doubt in this, although some physicists express the opposite opinion.) Theredore, instead of to spell the magic sentence **"The problem is reduced to the previous one!"**, in this chapter, I suggest one more example of evaluation of ackermann. This example refers to the natural pentation, id est, $\mathrm{pen} = \mathrm{pen}_{\mathrm{e}} = A_{\mathrm{e},5}$.

At the construction of a superfunction, the key question is about the fixed points of the transfer function. For pentation, the transfer function is tetration, considered in chapter 14. The real fixed points of tetration are shown in figure 19.3. This is modification of figure 17.1: some lines are removed, and the new curve for tetration to base $b\!=\!\tau\!\approx\!1.63532$ is added. Here, τ is specific values of base, at which the curve $y\!=\!\mathrm{tet}_\tau(x)$ touches the line $y\!=\!x$. The point of touching has coordinates $(L_{\tau,1}, L_{\tau,1})$, and $L_{\tau,1} \approx 3.087$. For this value, the additional grid lines are added in figure 19.3.

At base $b > \tau$, tetration tet_b has the only one real fixed point. In

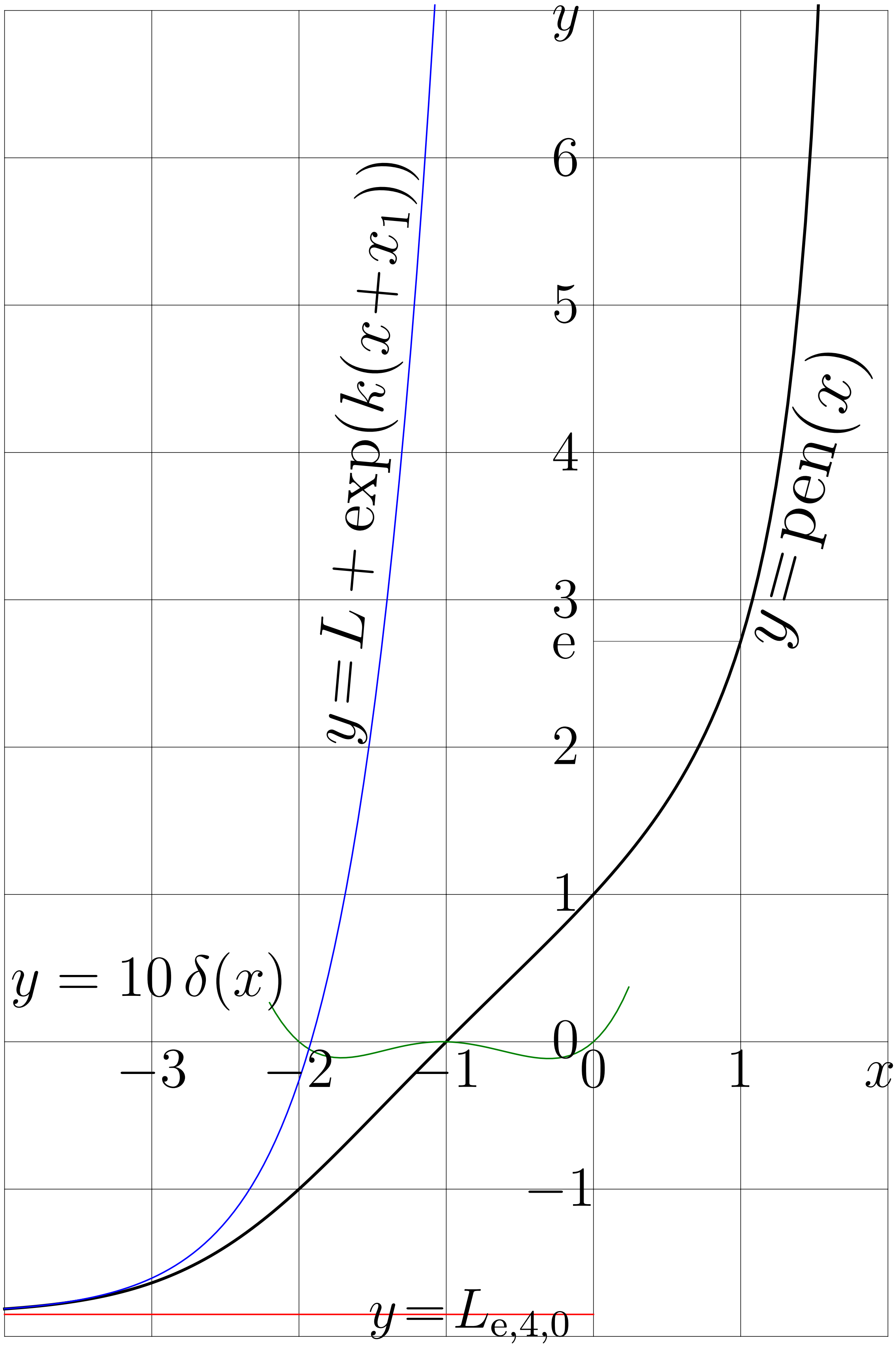


`http://mizugadro.mydns.jp/t/index.php/File:Tet5loplot.jpg`

Figure 19.3: $y = \mathrm{tet}_b(x)$, fragment of figure 17.1; line $y=x$ and $y=\mathrm{tet}_\tau(x)$ are added

particular, this refers to the case of natural tetration tet = tet_e. For natural tetration, this point is $L = L_{\mathrm{e},0} \approx -1.85035452902718$, and for this value, also the additional grid lines are shown in figure 19.3. Namely this point is chosen to built-up the natural pentation pen, id est, for the fifth ackermann to base $\mathrm{e} = \exp(1) \approx 2.71$. As the Reader can see, not so many options we have in this case.

The growing real-holomorphic superfunction of natural tetration, that is constructed with regular iteration at the fixed point $L_{\mathrm{e},0}$ and approches this fixed point at $-\infty$, I call the fifth ackermann, or pentation. Graphic of pentation is shown in figure 19.4. I describe the construction below.



`http://mizugadro.mydns.jp/t/index.php/File:Penplot.jpg`

Figure 19.4: $y=\mathrm{pen}(x)$ by (19.24), its asymptotic (19.25), and deviation of the linear approximation by (19.27)

For superfunction F of transfer function tet, the transfer equation is

$$F(z{+}1) = \mathrm{tet}\Big(F(z)\Big) \tag{19.15}$$

I construct the growing along the real axis solution F by the regular iteration at the fixed point of tetration $L{=}L_{\mathrm{e},5,0}{\approx}{-1.85035452902718}$; I mention the key point of the construction below.

For some natural number $M > 1$, I search the solution F of equation (19.15) in the following form:

$$F(z) = f(z) + O(\varepsilon^M) \quad \text{[penF]} \tag{19.16}$$

where

$$f(z) \;=\; L_{\mathrm{e},4,0} + \sum_{m=1}^{M-1} a_m \varepsilon^m \qquad \text{[penf]} \tag{19.17}$$

$$\varepsilon \;=\; \exp(kz) \qquad \text{[penepsilon]} \tag{19.18}$$

Here, the positive constant k has sense of the increment of the growth of superfunction at large negative values of the argument, and a are real coefficients. For simplicity, I set $a_1{=}1$. Substitution of representations (19.16), (19.17) to the transfer equation (19.15) and the asymptototic analysis with small parameter ε give

$$k = \ln\Big(\mathrm{tet}'(L)\Big) \approx 1.865733 \tag{19.19}$$

and the coefficients a; in particular,

$$a_2 = \frac{\mathrm{tet}''(L)/2}{\Big(\mathrm{tet}'(L){-}1\Big)\mathrm{tet}'(L)} \approx -0.6263241 \tag{19.20}$$

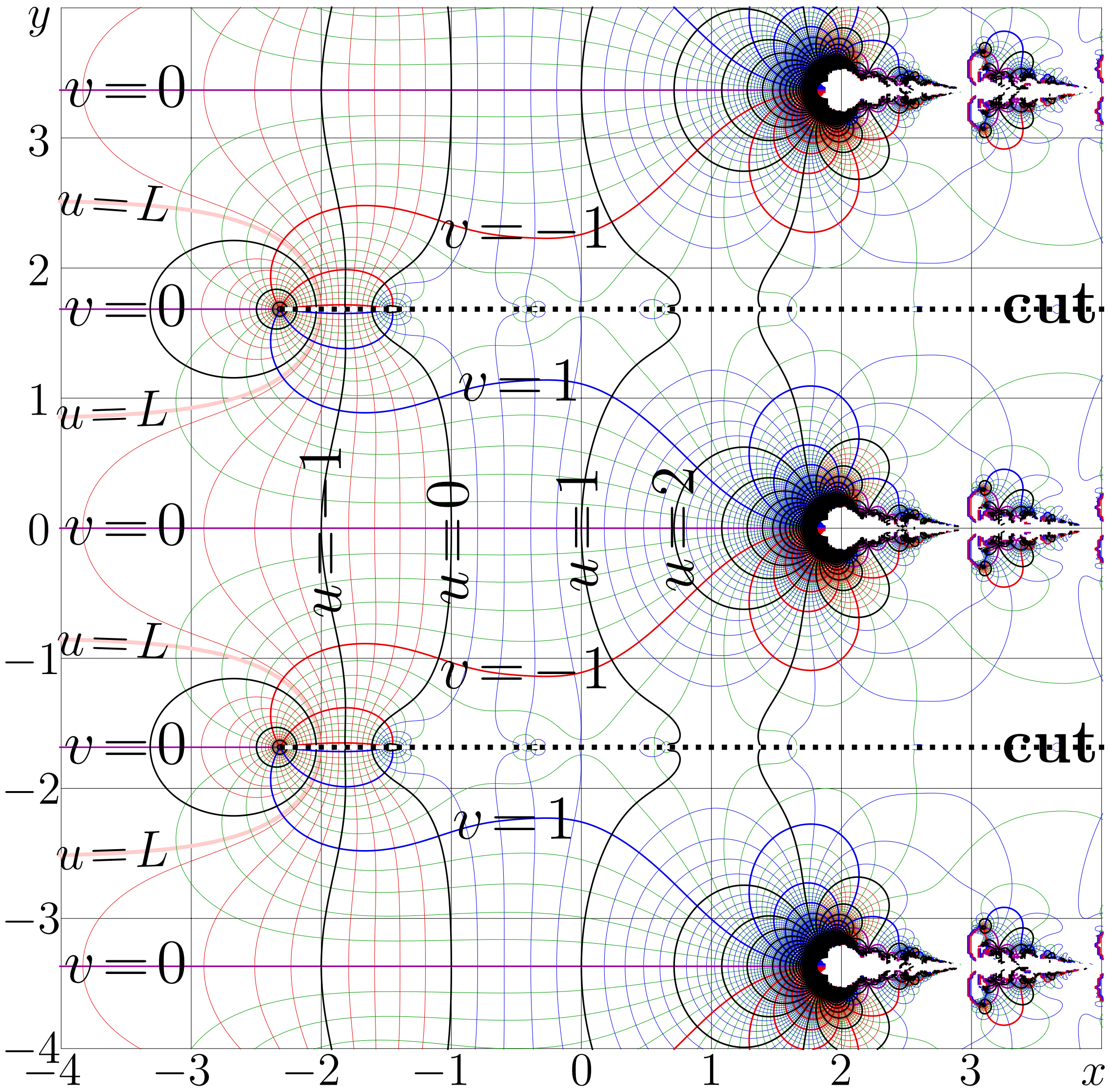
$$a_3 = \frac{\mathrm{tet}''(L)a_2 + \mathrm{tet}'''(L)}{\Big(\mathrm{tet}'(L)^2{-}1\Big)\mathrm{tet}'(L)} \approx \;\; 0.4827 \tag{19.21}$$

For the numerical implementation, in (19.17), I choose $M = 4$; this is sufficient to evaluate pentation with 14 significant figures and to plot all the figures of this article in real time. This approximation is good for large negative values of the real part of argument of supertetration. Then, for integer n, I define

$$F_n(z) = \mathrm{tet}^n(f(z-n)) \quad \text{[pentalim]} \tag{19.22}$$

The exact superfunction F appears as limit

$$F(z) = \lim_{n\to\infty} F_n(z) \quad \text{[flim]} \tag{19.23}$$



`http://mizugadro.mydns.jp/t/index.php/File:Penmap.jpg`

Figure 19.5: $u+\mathrm{i}v = \mathrm{pen}(x+\mathrm{i}y)$ по формуле (19.24) [penmap]

The limit does not depend on the chosen number M of terms in the asymptotic expansion. However, the larger is M, the faster the limit in (19.23) does converge.

The pentation appears as superfunction F with displaced argument:

$$\mathrm{pen}(z) = A_{\mathrm{e},4}(z) = F(x_5+z) \quad \text{[pendef]} \qquad (19.24)$$

where $x_5 \approx 2.24817$ is solution of equation $F(x_5)=1$. Complex map of this pentation is shown in figure 19.5.

The real-real plot of pentation by (19.24) is shown in figure 19.4 with thick curve. The additional horizontal grid line $y = L_{\mathrm{e},4,0}$ shows the asymptotic at large negative values of the argument. The thin curve shows the more advanced asymptotic

$$y = L_{\mathrm{e},4,0} + \exp(k(x+x_5)) \quad \text{[pen01e]} \tag{19.25}$$

Pentation is holomorphic at least for $|\Re(z)| < |P|/2 \approx 1.6838$, where $P = 2\pi\mathrm{i}/k$ is period; pentation, as exponential, is periodic function. A little bit more than two periods are covered by the range of the map at figure 19.5. Pentation has the countable set of cut lines, parallel to the real axis. In figure 19.5, these cuts are marked with dashed lines.

In vicinity of the segment of length 2 at the negative part of the real axis, pentation can be approximated with the linear function,

$$\mathrm{pen}(x) \approx 1 + x \quad \text{[penlin]} \tag{19.26}$$

At $-2.1 < x < 0.1$, approximation (19.26) provides two significant figures. Deviation of this approximation from pentation pen can be expressed with function

$$\delta(x) = \mathrm{pen}(x) - (1+x) \tag{19.27}$$

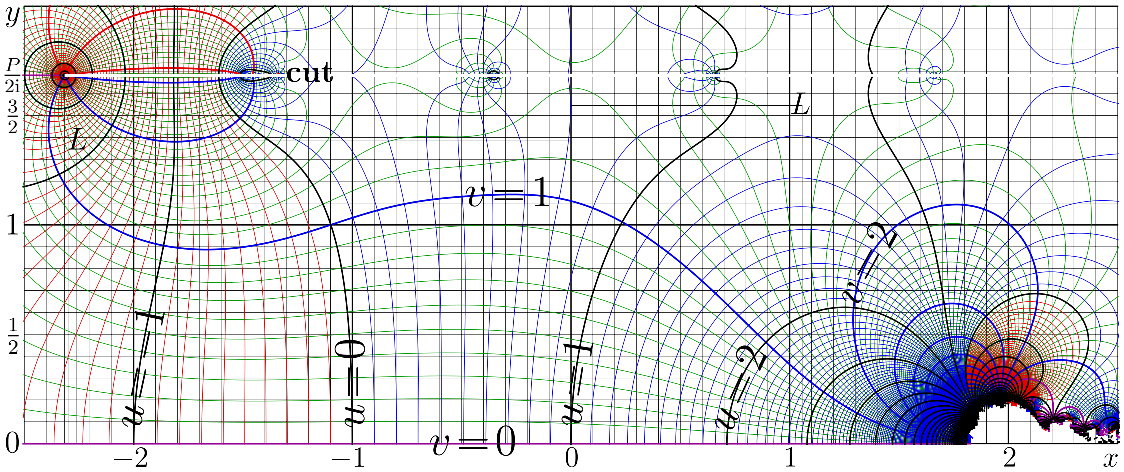
This deviation is shown in figure 19.4 with thin line; it is scaled with factor 10; curve $y = 10\,\delta(x)$ is drawn.

The linear function in the right hand side of equation (19.26) approximates also the previous ackermann, id est, tetration; its graphic is show in figure 14.1. For tetration, the function in the right hand side of formula (19.26) also gives of order of two significant figures; however, the range of this approximation for pentation is twice wider, than for tetration.

Complex map of pentation by (19.24) in figure 19.5 demonstrates, that $\mathrm{pen}(z)$ is holomorphic at least for $\Re(z) < -2.5$. As the real part of the argument approaches minus infinity, pentation exponentially approaches the limiting case $L = L_{\mathrm{e},4,0} \approx -1.850354529$, shown in figure 19.3. In order to show this explicitly, the light strip in figure 19.5 indicates the additional level $u = L_{\mathrm{e},4,0}$.

Pentation is periodic; its period P is determined by the increment k, id est, by the derivative of tetration at its fixed point $L_{\mathrm{e},4,0}$:

$$P = \frac{2\,\pi\,\mathrm{i}}{k} \approx 3.36767615657879\,\mathrm{i} \quad \text{[penP]} \tag{19.28}$$



http://mizugadro.mydns.jp/t/index.php/File:Penzoo25t400.jpg

Figure 19.6: $u+\mathrm{i}v = \mathrm{pen}(x+\mathrm{i}y)$ by (19.24), zoom-in from figure 19.5 [penzoo]

The period is pure imaginary; the complex map of pentation reproduces itself at translations for integer factor of the period. The cuts of the range of holomorphism are also reproduced.

Along the real axis, pentation shows very fast growth, faster, than that of natural tetration. As maps of many other fastly growing functions, the map of pentation has complicated structure in vicinity of the real axis. Pentation varies with huge derivatives, that correspond to enormous density of the levels at the complex map. The plotter could not draw the huge amount of lines, and so, the narrow region in vicinity of the real axis in figure 19.5 is left empty. The same applies to the translations for integer factor of period P. In order to shown behaviour of pentation in vicinity of the real axis and in vicinity of the cut line, figure 19.6 shows the zoom-in from the figure 19.5.

General methods of construction of superfunctions can be used to build-up the ackermanns. If the growth of tetration happens to be not sufficiently fast, the pentation described above, can be used. The highest ackermanns can be constructed in the similar way. In particular, the fixed points of pentation indicate the way to build-up its superfunction, id est, the next ackermann. Fixed points of tetration are considered in the next section.

4 Fixed poins of pentation and future work

In this section I suggest some hint, how can one built-up the next ackermann, pentation, using the fixed points of tetration. Actually, here I do not construct pentation (because, anyway, I have to stop somewhere); I only mention the way to do it.

As usually, one should begin with the fixed points, id est, solutions L of

$$\mathrm{pen}(L) = L \quad \text{[penLeq]} \tag{19.29}$$

Some of the solutions are shown in figure 19.6, these solutions are marked with character L. They correspond to

$$L = L_{\mathrm{e},5,0} \approx -2.260 + 1.384\,\mathrm{i} \tag{19.30}$$
$$L = L_{\mathrm{e},5,1} \approx 1.057 + 1.546\,\mathrm{i} \quad \text{[penL1]} \tag{19.31}$$

There are also solutions in vicinity of the real axis

$$L = L_{\mathrm{e},5,2} \approx 3.43 + 0.07\,\mathrm{i} \tag{19.32}$$
$$L = L_{\mathrm{e},5,3} \approx 4.39 + 0.11\,\mathrm{i} \quad \text{[penL3]} \tag{19.33}$$

but they do not fit the frame of the map on figure 19.6. The readers are invited to solve numerically equation (19.29) and to adjust values of $L_{\mathrm{e},5,0}$ $L_{\mathrm{e},5,1}$. I expect, one can find the real-holomorphic solution F to equation

$$F(z\!+\!1) = \mathrm{pen}\Big(F(z)\Big) \tag{19.34}$$

with additional conditions

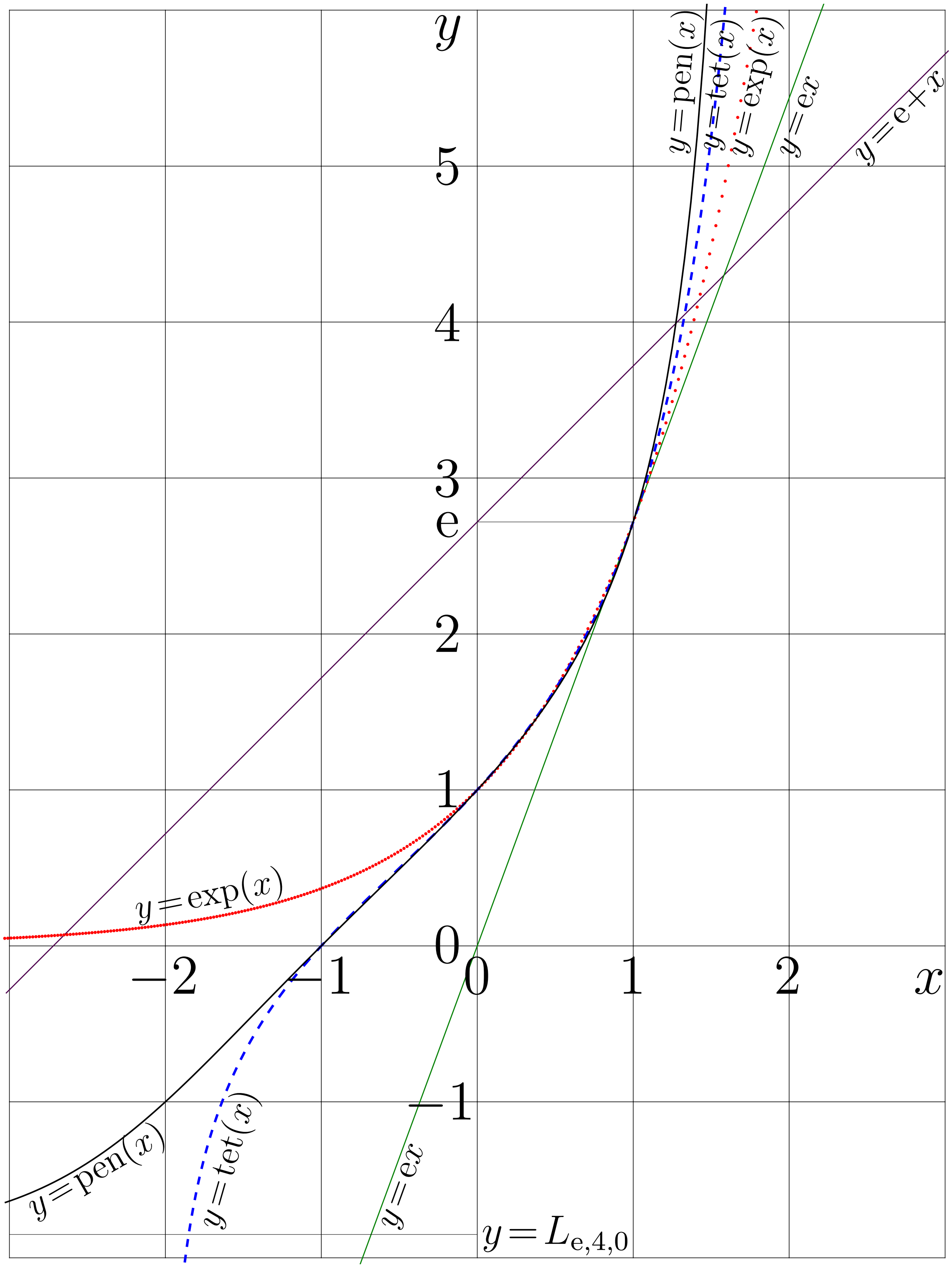
$$F(0)\!=\!1~, \quad F(\mathrm{i}\infty)\!=\!L_{\mathrm{e},5,0}~, \tag{19.35}$$

The readers are invited to find this solution and interpret it as hexation, id est, the 6th ackermann.

At this point, I stop constructing new ackermanns. I want to compare the first 5 ackermanns. It is mater of the next section.

5 Comparison of natural ackermanns

In this section I compare the first 5 sclermanns for the natural base $b = \mathrm{e}$. In such a way, I overview results for ackermanns. For real values of argument, these functions are plotted in figure 19.7.



http://mizugadro.mydns.jp/t/index.php/File:Ackerplot400.jpg

Figure 19.7: $y = A_{\mathrm{e},n}(x)$ for $n = 1, 2, 3, 4, 5$ [ackerplo]

Functions plotted in figure 19.7 are:
$A_{e,1}(z) = e+z$, addition of constant e,
$A_{e,2}(z) = e\, z$, multiplication to constant e,
$A_{e,3}(z) = \exp(z) = e^z$, natural exponent ,
$A_{e,4}(z) = \mathrm{tet}(z)$, tetration to base e,
$A_{e,5}(z) = \mathrm{pen}(z)$, pentation to base e.
These notations are used to mark curves $y = A_{e,n}(x)$ at figure 19.7. As usually, if the base is not specified at the subscript, it is assumed to be e, base of natural exponent and that of natural logarithm.

Tetration and pentation at the segment $[-1, 1]$ look similar. However, at the printing with good resolution, the deviation is seen, it exceeds the width of the lines in figure 19.7.

I expect, the tetration already has the growth fast enough to describe the fastest function that may appear in the applications. However, the main property of the scientific revolution is that they are unexpected. If for some case, the growth of tetration is not fast enough, the pentation or even higher sckermanns can be implemented, using the tools from this Book.

When I plotted the pentation, I ask myself: "Why not to do in the similar way the next ackermann?" The answer is simple: I already know, how to do it. And the Reader, if reached here, also knows. Anyway, the Book should be finished at some moment, see figure 19.8.

Instead of to add more examples for the tools described above, I think, it is more important to consider at least one example, when the tools above do not work. This example correspond to the transfer function, that has no fixed points at all - neither real, nor even complex. Such an example is considered in the next chapter, and, while typing this, I believe, that will be the last example in this Book.

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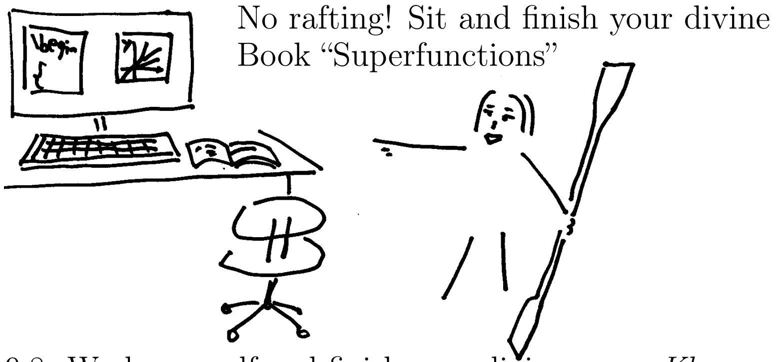


Figure 19.8: Wash yourself and finish your divine opera *Khovanshina*! [15]

Chapter 20

Without fixed points

In this chapter I describe the last (for this book) example of the transfer function, I consider function without fixed points. I suggest, that the Reader tries to invent, to guess, to write-out some entire function without fixed points, before to look at the formula below.

For the transfer function without fixed points, neither method of regular iteration, nor the representation through the Cauchy integral can be applied as is for the primary evaluation of superfunction. For this reason, this example is interesting and deserves the special chapter. Here, I retell results published recently [88].

1 Trappmann function

I give name 'Trappmann function" to the elementary function

$$\mathrm{tra}(z) = \exp(z) + z \tag{20.1}$$

Function tra has no fixed point; equation $\mathrm{tra}(L) = L$ has no solution. Function tra is compared to exponent in figure 20.1.

Function tra had been expected to be a trap, trump, to catch me on my claim, that I can build-up a superfunction for any holomorphic transfer function. Henryk Trappmann had suggested this function; so, I use first three caracters of his family name to denote it.

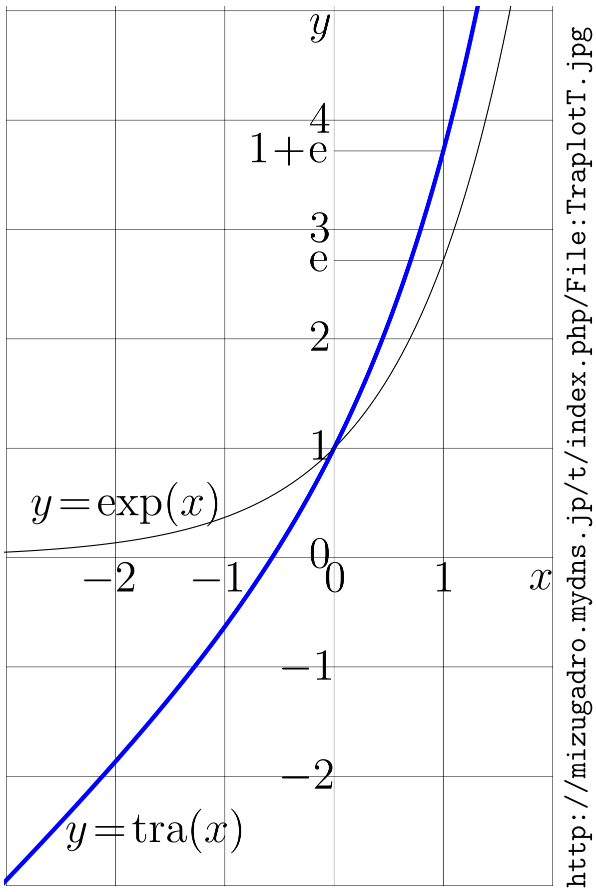
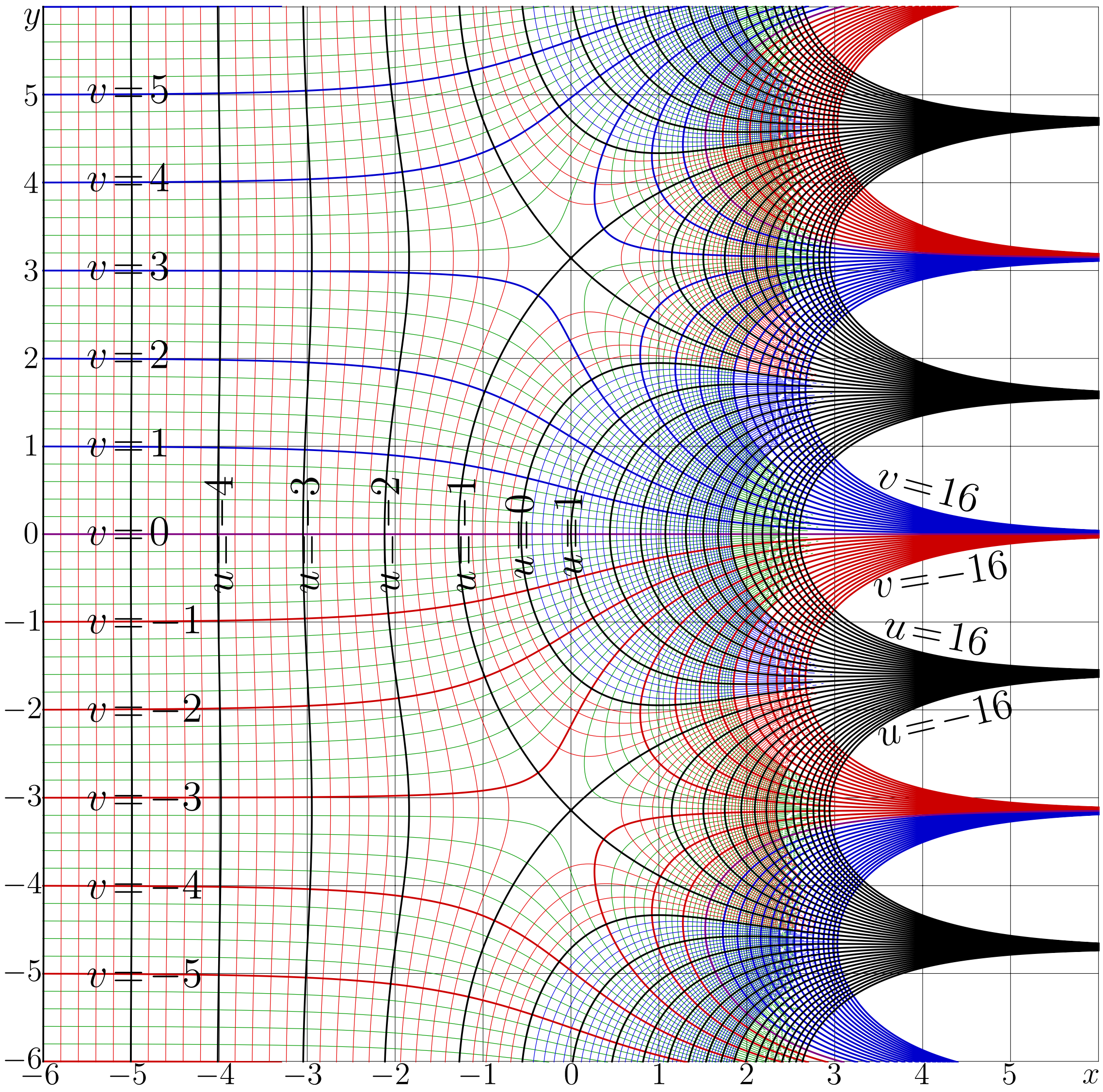


Figure 20.1: $y = \mathrm{tra}(x)$ and $y = \exp(x)$



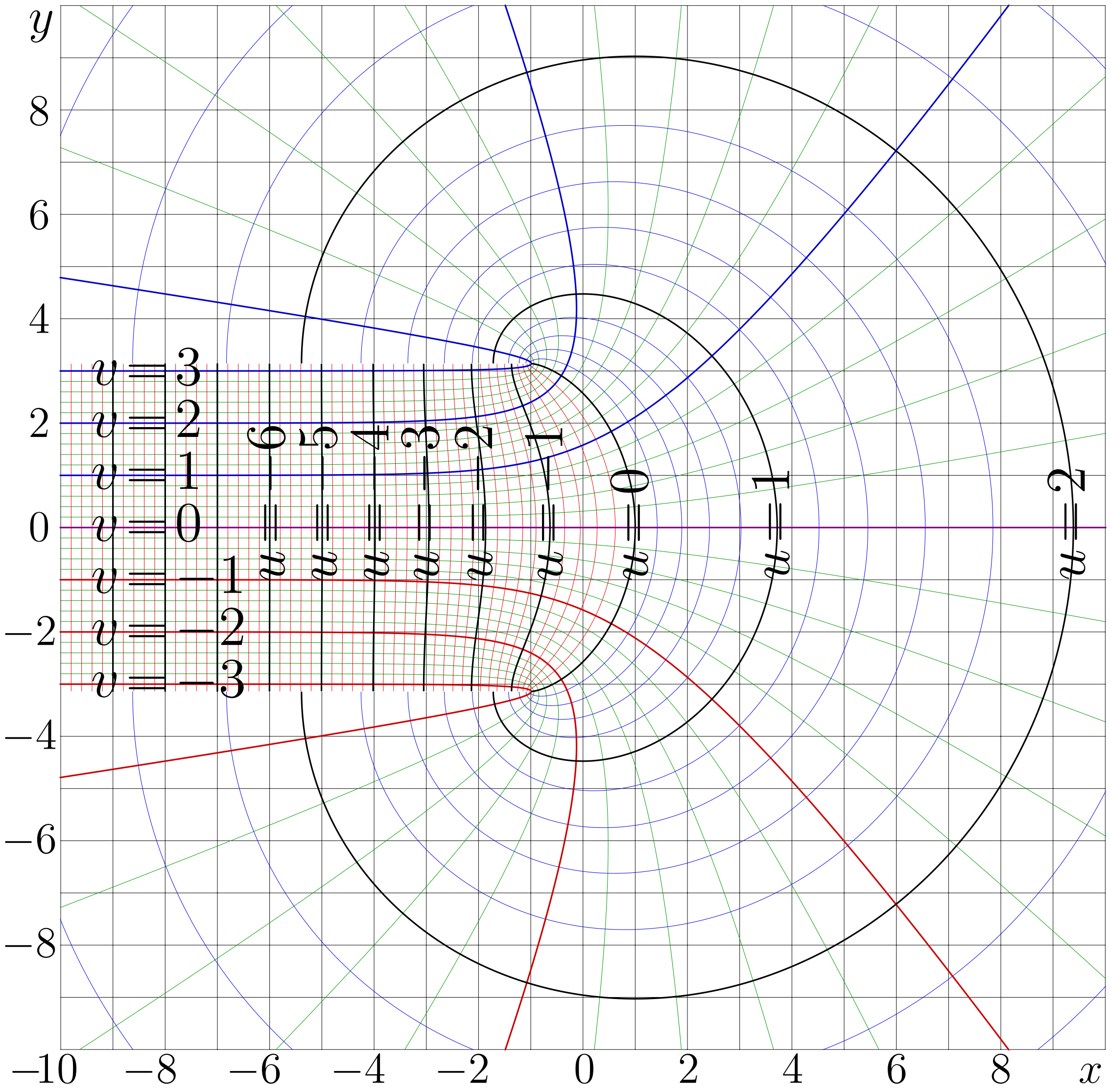
http://mizugadro.mydns.jp/t/index.php/File:TraMapT.jpg

Figure 20.2: $u+\mathrm{i}v = \mathrm{tra}(x+\mathrm{i}y)$ по формуле (20.1) [tramap]

Word "trap" can be interpreted also as gate, a way to the future successes in solving of various transfer equations and iterates of tricky functions. We see, function tra deserves the special name.

Complex map of function tra by (20.1) is shown in figure 20.2. In the right hand side of the figure, the map looks similar to that go exponent: at the background of the exponential growth, the linear addition in the right hand side of equation (20.1) does not look impressive. In the left hand side of the map, contrary, the exponent becomes negligible, and the lines of constant real part and those of constant imaginary part of the function form almost uniform grid of lines, parallel to the coordinate axes.

Similar property, at least in some part of the complex plane, is shown by the inverse function, id est, $\mathrm{ArcTra} = \mathrm{tra}^{-1}$. Complex map of ArcTra is shown in figure 20.3. I consider its evaluation in the next section.



http://mizugadro.mydns.jp/t/index.php/File:ArcTraMapT.jpg

Figure 20.3: $u+\mathrm{i}v=\mathrm{ArcTra}(x+\mathrm{i}y)$ [arctranmap]

2 ArcTra, inverse of trappmann

In order to iterate a function, I mean non-integer iterates, first, we need to learn to evaluate the integer iterates. For positive integer number of iterate, we may apply the function to the argument so many times as necessary. As for the negative iterates, we need the efficient algorithm for the inverse function. Let this inverse function be called ArcTra, in analogy tithe ArcSin, ArcTan and ArcBessel; in some wide range to values of the argument (that includes the real axis), the ArcTra should satisfy equation

$$\mathrm{tra}\Big(\mathrm{ArcTra}(z)\Big) = z \quad [\mathrm{traArcTra}] \tag{20.2}$$

Complex map of function ArcTra is shown in figure 20.3.

Perhaps, I should explain, why I describe in this Book so elementary thing, as building-up of inverse function. In century 20, I used to deal with students interested in nonlinear optics and quantum optics. Some of them were smart [28], but some students had problem even with linear optics. It costed to me certain efforts, to explain them that there is a lot of pretty "linear" science behind every so-called "nonlinear" phenomenon, as it is shown in figure 20.4. In the similar way, it is vain to discuss non-integer iterate, while even the negative integer iterates are not yet implemented. I describe the implementation of function ArcTra in this section.

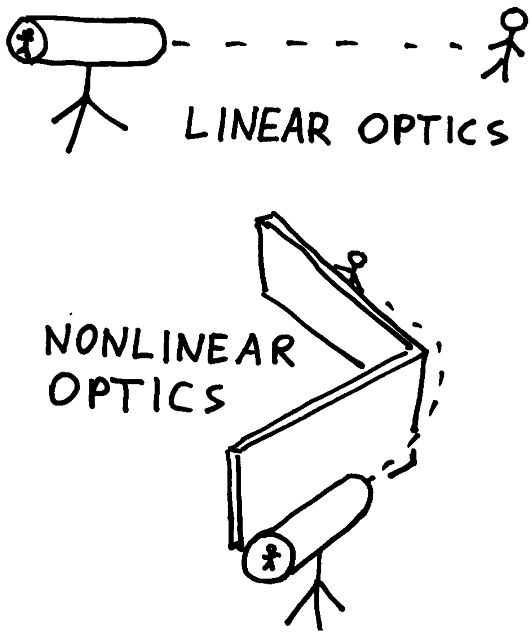


Figure 20.4: Linear versus nonlinear

During the USSR, there was some science there. The famous institutes of the so-called "Soviet School" were Fizfak (Физфак, Physics department of the Moscow State University) and Fiztech (Физтех, Moscow Phisics-Technical Institute). As one can guess, Fizfak used to deal with fundamental science, and Fiztech did with the applied one. In order to compare a specialist graduated from Fizfak to that graduated from Fiztech, in the USSR, the following example is popular. One, graduated from Fiztech, can calculate or ensemble everything. But he/she understands close to nothing. As for one graduated from Fizfak - Oooh.. he or she understands everything, although can calculate close to nothing. I remind that story for the analogy with figures of this section. Readers, who are interested in the beautiful pictures, may look at the coplex maps presented in this section. Then, these Readers can be qualified with specification "understand everything". As those graduated from Fizfak.

For the implementation of the Abel function for the transfer function tra, the inverse function $\mathrm{ArcTra} = \mathrm{tra}^{-1}$ is used. Unfortunately, as in the case of ArcSin and ArcFactorial, I could not find any complex double implementation of ArcTra, and I had to make it by myself. This is general rule: for the efficient implementation of the non-integer iterates of some transfer function [1] , the inverse function also should be implemented. In order to show the underwater stones, that may appear at calculation of superfunctions, in this section, the efficient implementation of function ArcTra is described.

It is not difficult to calculate the derivative of function tra:

$$\mathrm{tra}(z) = z + \exp(z) \quad , \quad \mathrm{tra}'(z) = 1 + \exp(z) \qquad \text{[traagain]} \tag{20.3}$$

The Newton method gives a good precision evaluating solution f of equation

$$\mathrm{tra}(f) = z \quad \text{[traf]} \tag{20.4}$$

$$f_{n+1} = f_n + \frac{z - \mathrm{tra}(f_n)}{\mathrm{tra}'(f_n)} \qquad \text{[arctranewton]} \tag{20.5}$$

$$f = \lim_{n\to\infty} f_n \quad \text{[arctralim]} \tag{20.6}$$

under a simple condition: the initial approximation f_0 indicates the correct branch of the resulting inverse function. It is general rule, that any non-trivial holomorphic function (and, especially, entire function) has some points, where its derivative is zero; and these points provide branches of the inverse function. For this reason, for the robust implementation, the correct choice of the initial approximation is essential: over-vice, the resulting function may return values from different branches in some almost random, almost unpredictable way. In order to indicate the correct branch, the expansions below are used.

Expansions of tra in vicinity of the saddle points $\pm\pi\mathrm{i}$ can be inverted, giving the expansions of ArcTra in vicinity of $-1\pm\mathrm{i}\pi$. These expansions determine the positions of the branch points, and the direction of the cut lines, seen in figure 20.3.

Consideration of the exponential in (20.3) as a small parameter and as a big parameter gives two more asymptotic expansions. Together with the Taylor expansion at unity and the expansions at the branch

[1] In principle, any holomorphic function can be declared as "transfer function". I still specify that tra is "transfer function", in order to indicate my intention to iterate it and to distinguish it from its superfunction and its Abel function.

points, these expansions cover all the complex plane with the appropriate primary approximations, providing several correct decimal digits of the primary estimate f_0 of value of function ArcTra. Then, few iterations by (20.5) already reach the maximal precision (15 digits), available for the complex double variables. This algorithm is used in the numerical implementations of ArcTra and AuTra. [2]

The easiest expansion is at the negative part of the real axis:

$$\mathrm{ArcTra}(z) \approx \mathrm{app}_4(z) = z - \mathrm{e}^z + \mathrm{e}^{2z} - \frac{1}{2}\mathrm{e}^{3z} \qquad (20.7)$$

The Readers are invited to plot the map of the agreement function

$$\mathcal{A}_4(z) = -\lg\left(\frac{|\mathrm{tra}(\mathrm{app}_4(z)) - z|}{|\mathrm{tra}(\mathrm{app}_4(z))| + |z|}\right) \qquad (20.8)$$

The logarithmic growth of function ArcTra can be caught with the asymptotic expansion This expansion provides the approximation

$$\mathrm{ArcTra}(z) \approx \mathrm{app}_3(z) = \ln(z)\left(1 + \frac{1}{z}\sum_{m=0}^{M} \frac{P_m(\ln(z))}{z^m}\right) \qquad (20.9)$$

where P_m is polynomial of mth order. In particular,

$$\begin{aligned}
P_0(\ell) &= -1 && (20.10)\\
P_1(\ell) &= \ \ 1 - \ell/2 && (20.11)\\
P_2(\ell) &= -1 + 3\ell/2 - \ell^2/3 && (20.12)\\
P_3(\ell) &= \ \ 1 - 3L + 11\ell^2/6 - \ell^3/4 && (20.13)\\
P_4(\ell) &= -1 + 5\ell - 35\ell^2/6 + 25\ell^3/12 - \ell^4/5 && (20.14)
\end{aligned}$$

The Taylor expansion of function tra at unity gives the approximation

$$\mathrm{ArcTra}(z) \approx \mathrm{app}_1(z) = \sum_{n=1}^{M} c_n (z-1)^n \qquad [\mathrm{arctrap1}]\ (20.15)$$

Some tens of coefficients c of this expansion can be calculated, inverting the Taulor expansion of function tea at zero; the beginning of the expansion can be written as follows:

[2] Details of the algorithm and the maps of the primary approximations for ArcTra are loaded to the special page `http://mizugadro.mydns.jp/t/index.php/ArcTra` .

$\mathrm{ArcTra}(1{+}z) = \frac{z}{2} - \frac{z^2}{16} + \frac{z^3}{192} + \frac{z^4}{3072} - \frac{13z^5}{61440} + \frac{47z^6}{1474560} + \frac{73z^7}{41287680} - \frac{2447z^8}{1321205760} + ..$
At $M=21$, for $|z|<1$, this approximation provides at least 12 significant figures.

Expansion of ArcTra at point $-1-\mathrm{i}\pi$ can be written as follows:

$$
\begin{aligned}
\mathrm{ArcTra}(z) \approx \mathrm{app}_2(z) &= \sum_{m=1}^{M} c^*_m \,(z{+}1{+}\mathrm{i}\pi)^{m/2} \\
&= -\mathrm{i}\pi + \mathrm{i}\sqrt{2}\sqrt{z{+}\mathrm{i}\pi{+}1} + \frac{1}{3}(z{+}\mathrm{i}\pi{+}1) \\
&\quad -\frac{\mathrm{i}(z{+}\mathrm{i}\pi{+}1)^{3/2}}{9\sqrt{2}} - \frac{2}{135}(z{+}\mathrm{i}\pi{+}1)^2 + .. \qquad (20.16)
\end{aligned}
$$

In more compact (and more efficient for the numerical implementation) form, this expansion can be re-written as follows:

$$
\begin{aligned}
\mathrm{ArcTra}\big(-1{-}\mathrm{i}\pi{+}2t^2\big) &= -\mathrm{i}\pi + 2\mathrm{i}t + \frac{2t^2}{3} - \frac{2\mathrm{i}t^3}{9} - \frac{8t^4}{135} \\
&\quad + \frac{\mathrm{i}t^5}{135} - \frac{32t^6}{8505} + \frac{139\mathrm{i}t^7}{42525} + .. \qquad (20.17)
\end{aligned}
$$

with obvious modification for the "conjugated" region

$$
\begin{aligned}
\mathrm{ArcTra}\Big(-1+\mathrm{i}\pi+2t^2\Big) &= \mathrm{i}\pi - 2\mathrm{i}t + \frac{2t^2}{3} + \frac{2\mathrm{i}t^3}{9} - \frac{8t^4}{135} \\
&\quad - \frac{\mathrm{i}t^5}{135} - \frac{32t^6}{8505} - \frac{139\mathrm{i}t^7}{42525} + .. \qquad (20.18)
\end{aligned}
$$

where t has sense of algebraic function of argument z of the arctrappmann,

$$
t = \sqrt{\frac{z+1-\mathrm{i}\pi}{2}} \quad \text{[arctrait]} \qquad (20.19)
$$

The cut of the square root function in (20.19) automatically determines the cut lines of function ArcTra, seen in figure 20.3. I invite the Reader to plot the maps of the agreement functions

$$
\mathcal{A}_m(z) = -\lg\left(\frac{|\mathrm{tra}(\mathrm{app}_m(z)) - z|}{|\mathrm{tra}(\mathrm{app}_m(z))| + |z|}\right) \qquad (20.20)
$$

in the complex plane $z=x+\mathrm{i}y$ for $m=1,2,3,4$, id est, for the four primary approximations suggested in this section.

When the approximations above are implementd and called with names arctra1, arctra2, arctra3, arctra4, the numeral implementation of function ArcTra can follow the algorithm below:

```
z_type arctran(z_type z) { DB x=Re(z), y=Im(z);
if( x>2.) return arctra3(z);
DB Y=fabs(y);
if(Y<M_PI){ if(x<-1.5) return arctra4(z);
            if(Y<2.) return arctra1(z); }
if( Y>5. || x<-4. ) return arctra3(z);
if( y>0. ) return arctra2(z);
return conj( arctra2(conj(z)) );
}
```

Alternatively, function ArcTra can be expressed through the Tania function (5.3):

$$\mathrm{ArcTra}(z) = z - \mathrm{Tania}(z-1) \quad [\mathrm{ArcTraTania}] \tag{20.21}$$

In such a way, in this Book, function Tania is used already 3 times in three pretty different ways: First, in the simple model of the laser amplifier, then, in the representation of function flog, and, en fin, now, as an alternative implementation of ArcTra.

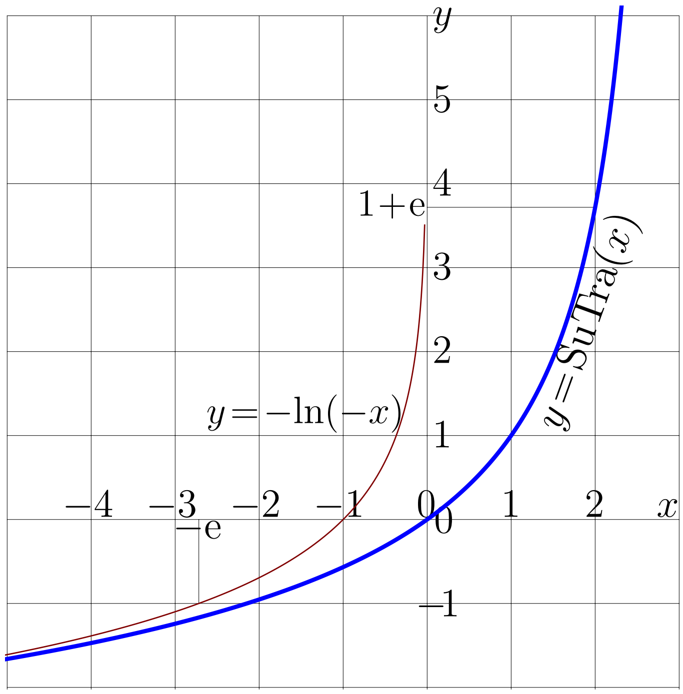
At least on vicinity of the real axis, function Tania can be expressed also through other special functions namely, LambertW and WrightOmega, see (5.2). In principle, one could use those representations instead of the implementation described above. However, at large z, Tania shows the growth similar to the proportional; so, the numerical implementation of ArcTra through Tania may cause loss of precision due to the rounding errors. However, the special implementation of ArcTra specifid above, had been used for the testing of the numerical implementation of function Tania.

While both tra and ArcTra are described and implemented, function tra can be iterated, using the representation through the superfunction and the abelfunction. I call these functions SuTra and AuTra; then, a usually

$$\mathrm{tra}^n(z) = \mathrm{SuTra}\Big(n + \mathrm{AuTra}(z)\Big) \quad [\mathrm{tranz}] \tag{20.22}$$

In such a way I announce the future consideration: Function SuTra is described in the next section, and function AuTra is treated soon after that section.

3 Supertrappmann



http://mizugadro.mydns.jp/t/index.php/File:SuTraPlo3T.jpg

Figure 20.5: $y = \mathrm{SuTra}(x)$ and $y = -\ln(-x)$

As it is mentioned above, the Trappmann function tra has no fixed points. For this reason, the methods described in the previous chapters of this Book, cannot be applied "as is". Henryk Trappmann had expected, that it will be difficult, to construct superfunction for the function tra, if at all.

However, the simple and efficient way to construct and evaluate SuTra, which is superfunction for the Trappmann function, exist. Plot of function SuTra is shown in figure 20.5 and compared to the graphic of its asymptotic. Below I describe the construction of this function.

Many colleagues, instead of to trace the deduction, ask first "How did you guess?" Following such an interest, first, I describe the hint, that leads to the efficient representation of function SuTra. For the transfer function tra, and its superfunction F, the transfer equation can be written as follows:

$$F(z+1) = \mathrm{tra}(F(z)) = F(z) + \exp(F(z)) \qquad \text{[trantrap]} \quad (20.23)$$

and re-written in the following form:

$$F(z+1) - F(z) = \exp(F(z)) \qquad \text{[trantrap1]} \qquad (20.24)$$

In the left hand side of equation (20.24), I see something, that looks similar to derivative of function F. This similarity can be expressed with approximate equation

$$\frac{\mathrm{d}F}{\mathrm{d}z} \approx \frac{1}{\exp(-F)} \qquad (20.25)$$

that gives

$$\int \exp(-F)\mathrm{d}F \approx \int \mathrm{d}z \qquad (20.26)$$

$$\exp(-F) \approx -z \tag{20.27}$$

This relation indicates, that expansion of superfunction F may begin with the logarithmic term, namely the term, shown in figure 20.5. However, the "heuristic solution" above is not a solution at all; in the best case, it is only the asymptotic approximation. Substitution of the approximation of function F into the transfer equation (20.23) gives, of course, some residual. This residual indicates the form of the next term of the expansion; and so on. In such a way, I guess the form of the asymptotic solution F:

$$F(z) = F_M(z) + O\left(\frac{\ln(-z)}{z}\right)^{M+1} \quad \text{[sutraFMO]} \tag{20.28}$$

where M is natural number, and

$$F_M(z) = -\ln(-z) + \sum_{m=1}^{M} P_m\Big(\ln(-z)\Big)\, z^{-m} \quad \text{[sutraFM]} \tag{20.29}$$

where

$$P_m(z) = \sum_{n=1}^{m} a_{m,n} z^n \quad \text{[sutraP]} \tag{20.30}$$

Substituting this expansion into the transfer equation (20.23), I collect terms with equal powers of z and equal powers of $\ln(-z)$. This gives both, verification of the form of the asymptotic expansion (20.29),(20.30) and values of coefficients a. Several coefficients a, calculated in this way, are shown in Table 20.1.

Table 20.1: Coefficients a in the expansion (20.28),(20.29)

0	$-\frac{1}{2}$	$a_{1,2}$	$a_{1,3}$	$a_{1,4}$	$a_{1,5}$	$a_{1,6}$	$a_{1,7}$	$a_{1,8}$
$\frac{1}{6}$	$-\frac{1}{4}$	$\frac{1}{8}$	$a_{2,3}$	$a_{2,4}$	$a_{2,5}$	$a_{2,6}$	$a_{2,7}$	$a_{2,8}$
$\frac{7}{48}$	$-\frac{7}{24}$	$\frac{3}{16}$	$-\frac{1}{24}$	$a_{3,4}$	$a_{3,5}$	$a_{3,6}$	$a_{3,7}$	$a_{3,8}$
$\frac{647}{4320}$	$-\frac{35}{96}$	$\frac{5}{16}$	$-\frac{11}{96}$	$\frac{1}{64}$	$a_{4,5}$	$a_{4,6}$	$a_{4,7}$	$a_{4,8}$
$\frac{1427}{8640}$	$-\frac{4163}{8640}$	$\frac{25}{48}$	$-\frac{17}{64}$	$\frac{25}{384}$	$-\frac{1}{160}$	$a_{5,6}$	$a_{5,7}$	$a_{5,8}$
$\frac{1380863}{7257600}$	$-\frac{1883}{2880}$	$\frac{5963}{6912}$	$-\frac{653}{1152}$	$\frac{305}{1536}$	$-\frac{137}{3840}$	$\frac{1}{384}$	$a_{6,7}$	$a_{6,8}$
$\frac{3278773}{14515200}$	$-\frac{2171723}{2419200}$	$\frac{97603}{69120}$	$-\frac{3961}{3456}$	$\frac{537}{1024}$	$-\frac{263}{1920}$	$\frac{49}{2560}$	$-\frac{1}{896}$	$a_{7,8}$
$\frac{251790467}{914457600}$	$-\frac{35981749}{29030400}$	$\frac{1049251}{460800}$	$-\frac{920881}{414720}$	$\frac{69953}{55296}$	$-\frac{13381}{30720}$	$\frac{4123}{46080}$	$-\frac{363}{35840}$	$\frac{1}{2048}$

Using Mathematica, coefficients a can be found with the following code:

```
T[z_] = z + Exp[z];
Clear [n, m, M];
P[m_, L_] := Sum[a[m, n] L^n, {n, 0, m}]; P[m, L];
F[z_]=-Log[-z]+a[1,1] Log[-z]/z+Sum[P[m,Log[-z]]/z^m,{m,2,M}]
M = 12;
F1x = F[-1/x + 1];
Ftx = T[F[-1/x]];
s[1] = Series[F1x - Ftx, {x, 0, 2}];
t[1] = Extract[Solve [Coefficient[s[1], x^2] == 0, {a[1, 1]}], 1]
A[1, 1] = ReplaceAll[a[1, 1], t[1]];
su[1] = t[1]

m = 2; s[m] = ReplaceAll[Series[F1x - Ftx, {x, 0, m + 1}], su[m]];
t[m] = Coefficient[ReplaceAll[s[m], Log[x] -> L], x^(m + 1)];
u[m] = Collect[t[m], L];
v[m] = Table[Coefficient[u[m] L, L^(n + 1)] == 0, {n, 0, m}];
w[m] = Table[a[m, n], {n, 0, m}];
ad[m] = Extract[Solve[v[m], w[m]], 1];
su[m + 1] = Join[su[m], ad[m]];
ReplaceAll[ReplaceAll[F[x], su[m + 1]], Log[-x] -> L]

m = 3; s[m] = ReplaceAll[Series[F1x - Ftx, {x, 0, m + 1}], su[m]];
t[m] = Coefficient[ReplaceAll[s[m], Log[x] -> L], x^(m + 1)];
u[m] = Collect[t[m], L];
v[m] = Table[Coefficient[u[m] L, L^(n + 1)] == 0, {n, 0, m}];
w[m] = Table[a[m, n], {n, 0, m}];
ad[m] = Extract[Solve[v[m], w[m]], 1];
su[m + 1] = Join[su[m], ad[m]];
ReplaceAll[ReplaceAll[F[x], su[m + 1]], Log[-x] -> L]
```

and so on $m\!=\!4$, $m\!=\!5$, etc. I do not arrange here the loop "For" with respect to m, in order to keep the code explicit and simplify the tracing step by step.

Expression (20.29) can be considered as primary approximation of superfunction of the Trappmann function (20.1). Then, the exact solution F of the transfer equation appears as limit

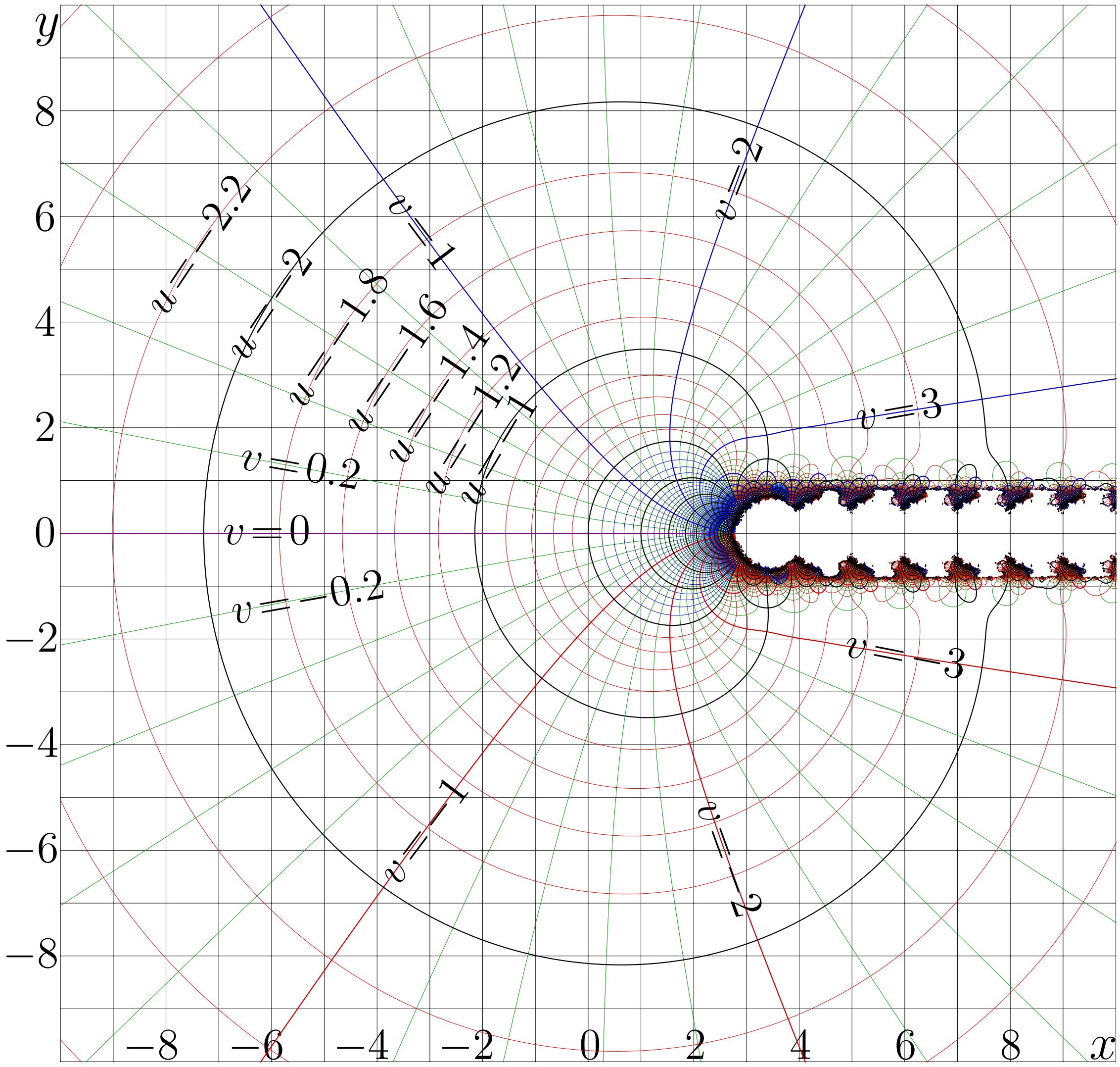
$$F(z) = \lim_{k\rightarrow\infty} \mathrm{tra}^k\big(F_M(z\!-\!k)\big) \quad \text{[exa]} \tag{20.31}$$

In order to get superfunction SuTra, that satisfies also the additional condition

$$\mathrm{SuTra}(0) = 0 \quad \text{[SuTra0]} \tag{20.32}$$

I set

$$\mathrm{SuTra}(z) = F(z + x_0) \quad \text{[SuTraDef]} \tag{20.33}$$



http://mizugadro.mydns.jp/t/index.php/File:SuTraMapT.jpg

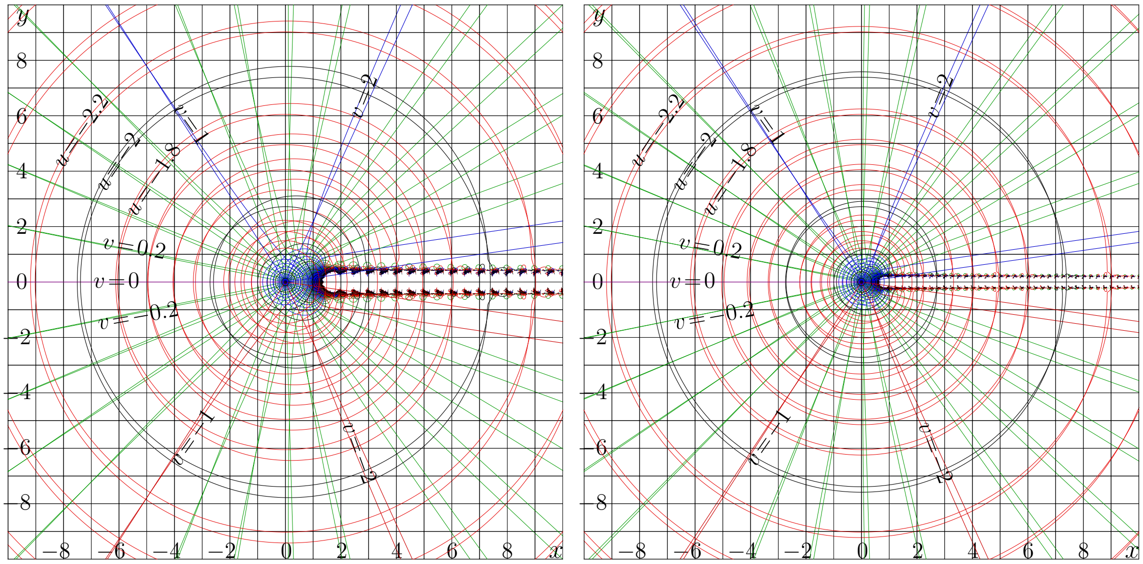
Figure 20.6: $u+\mathrm{i}v = \mathrm{SuTra}(x+\mathrm{i}y)$ по формуле (20.33)   [sutramap]

where $x_0 \approx -1.1259817765745026$ is real solution of equation

$$F(x_0) = 0 \quad [\mathrm{x00}] \tag{20.34}$$

Figure 20.5 shows $y = \mathrm{SuTra}(x)$ versus x. For comparison, the thin curve shows the leafing term of the asymptotic representation of SuTra, id est, $y = -\ln(-x)$. In the left hand side of the figure, graphic of SuTra approaches its asymptotic.

Complex map of function SuTra is shown in figure 20.6. Being far from the positive part of the real axis, function SuTra looks similar to function $z \to -\ln(-z)$, as it is suggested by the leading term of its asymptotic representation. Lines of the constant real part look similar to circles, while lines of constant imaginary part look similar to the straight lines; this make the map to look similar to the map of logarithm.



http://mizugadro.mydns.jp/t/index.php/File:Sutralomap.jpg

Figure 20.7: Complex maps of functions f_2 by (20.35) and f_4 (20.36), overlapped with map of function f_∞ by (20.37)

I would like to show, how the logarithmic function $z \to -\ln(-z)$ can be approximated with the entire function. Figure 20.7 shows complex maps of the entire functions

$$f_2(z) = \mathrm{SuTra}(2z) + \ln(2) \tag{20.35}$$

$$f_4(z) = \mathrm{SuTra}(4z) + \ln(4) \tag{20.36}$$

for comparison, each of these maps is overlapped with the map of

$$f_\infty(z) = -\ln(-z) \tag{20.37}$$

In the right hand side picture of figure (20.7), the levels of different functions are so close, that it is difficult to see, which of functions does each level correspond to. For all z, except zero and positive part of the real axis, sequence

$$\Phi_n(z) = -\Big(\mathrm{SuTra}(-nz) + \ln(n)\Big) \quad \text{[phin]} \tag{20.38}$$

at big n approximates $\ln(z)$. Up to my knowledge, expression (20.38) provides the range of approximation of logarithm with entire functions, that is wider than that of all approximations ever suggested before publication [88].

I have no idea, what for the approximation of logarithm with entire function can be used. This approximation appears here as a by-product at construction of function SuTra. But is someone needs such an approximation, it is done and it is here, formula (20.38).

4 Implementation of SuTra and the testing

The key parameters of any approximation are the range of applicability, the precision and the number of elementary operations that should be performed for each evaluation. If the range of applicability is wide (for example, cover the range of holomorphism of the approximated function), the precision is close to the that allowed at the computer representation of numbers, and the complex maps can be plotted in real time, then I qualify the solution as “exact”. This implies, that, if, for some reason, the precision needs to be improved, the implementation of the high precision code is straightforward, id est, can be realised with the same formulas.

If some function is supplied with definition, the properties are described, and the efficient precise algorithm for the evaluation is presented, I treat such a function as a “special function”. Then, if the solution of some problem is explicitly expressed in terms of the special functions, I call it “exact solution”. I place this explanation for my old coauthor, who until now believes, that π is approximate number. However, the fundamental mathematical constants are, contrary, exact numbers, in the sense mentioned above; neither π, nor values of superfunctions constrcted in this Book are exceptions: they are exact and can be evaluated with any required precision.

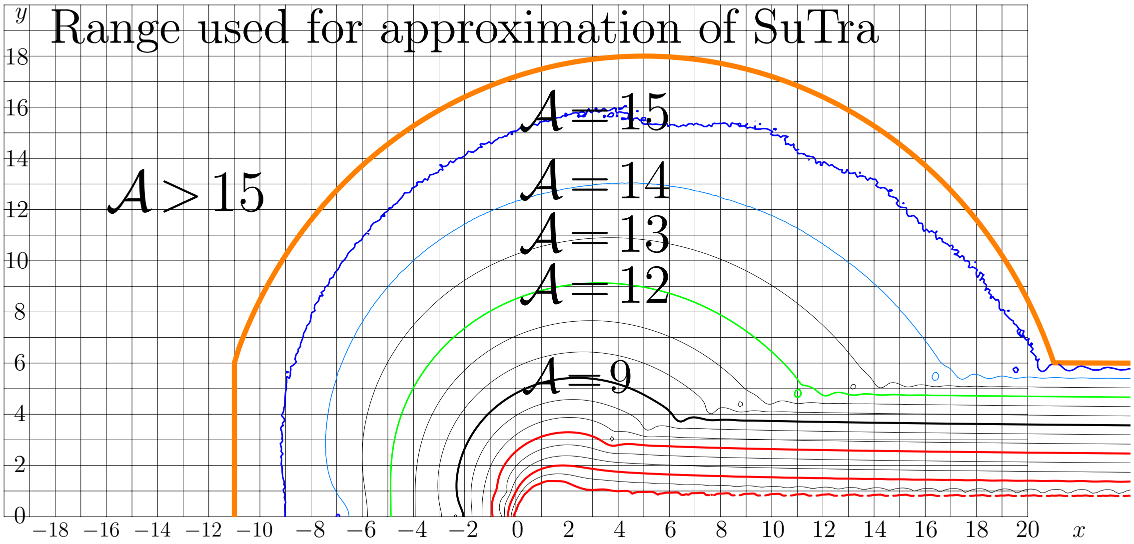
Following the ideology of the preamble of this section, I consider here the range of values of the argument z, at which the approximation $F_M(z{+}x_0)$ by (20.29) can be considered as good approximation of $\mathrm{SuTra}(z)$. Then I describe the algorithm, based on this analysis.

For the practical reasons (in order to get the complex double implementation, that can be easily reproduced and verified), I took 12 terms in the expansion, id est, $M\!=\!12$. For the dozen terms, the ageement

$$\mathcal{A}(z) = -\lg\!\left(\frac{|F_M(z{+}x_0) - \mathrm{SuTra}(z)|}{|F_M(z{+}x_0)| + |\mathrm{SuTra}(z)|}\right) \qquad \text{[sutraA](20.39)}$$

had been analysed. Levels of this function are shown in figure 20.8.

For the precision complex double, values outside the “thick” contour in figure 20.8 can be used “as is”. For other values, formula



http://mizugadro.mydns.jp/t/index.php/File:Sutraamap.jpg

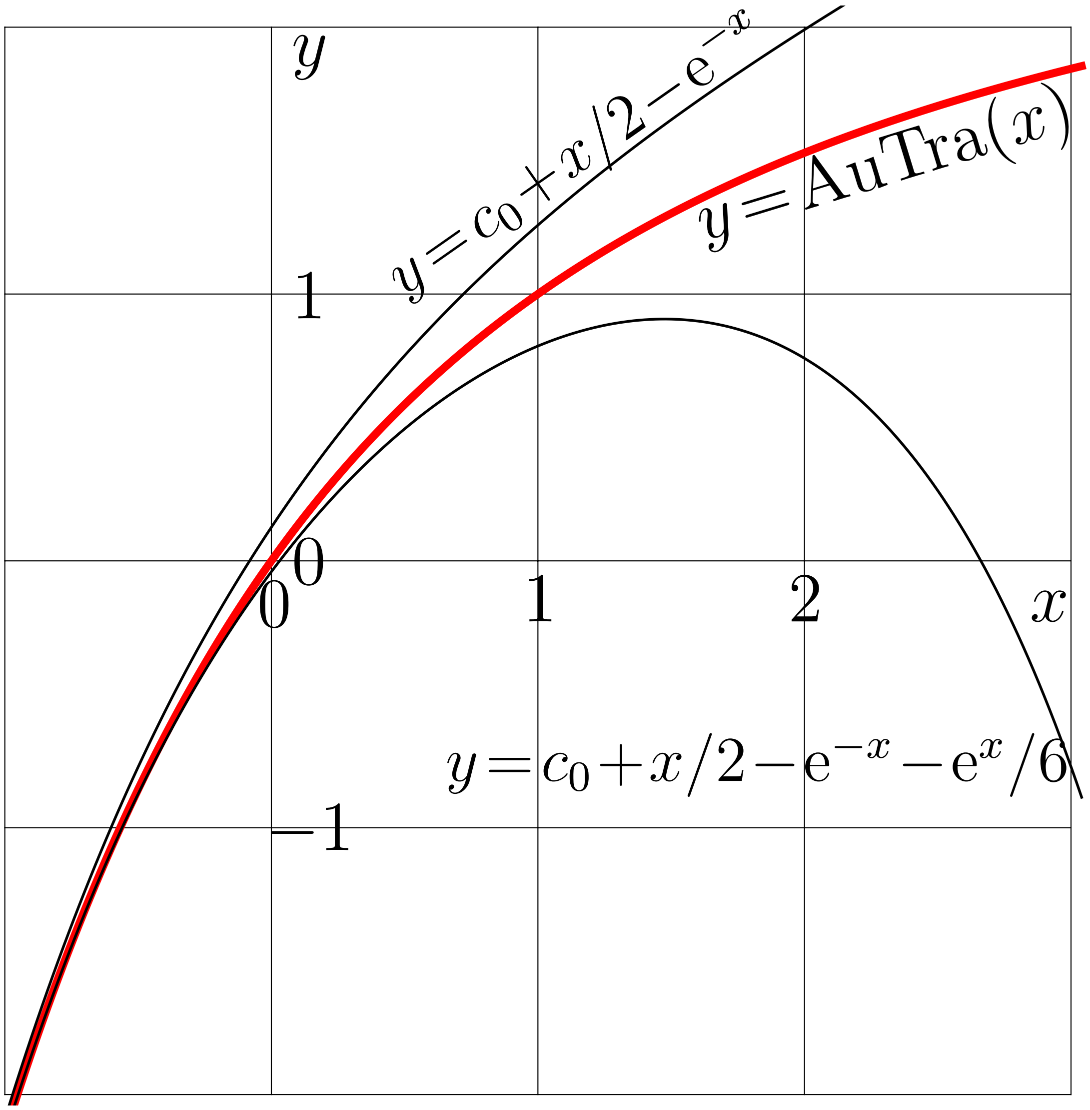
Figure 20.8: Map of $\mathcal{A}=\mathcal{A}(x+\mathrm{i}y)$ by (20.39)

$$\mathrm{SuTra}(z)=\mathrm{tra}^k(\mathrm{SuTra}(z-k)) \qquad [\mathrm{SuTrak}] \tag{20.40}$$

is used with appropriate k, in such a way, that the inner argument is outside the "thick" contour in figure 20.8. This thick contour is not part of the map; it is formed with the segment along the line $x=-11$, arc with centre at point $(5,0)$ of radius 18, and the half-line along $y=6$. Outside the thick contour in figure 20.8, the precision of evaluation of function SuTra is limited mainly by the rounding errors. Values outside the "thick contour" are used for the numerical implementation directly. For other values, formula 20.40 is used with appropriate value of k. This algorithm is used for the numerical implementation of function SuTra, `http://mizugadro.mydns.jp/t/index.php/Sutran.cin`

At the evaluation of function SuTra, function tra should be evaluated of order of ten times. (However, this depend on the initial value). Function tra is fast, because $\mathrm{tra}(z)=z+\exp(z)$. In such a way, the evaluation of SuTra is only for an order of magnitude slower, than evaluation of other special functions like exp, erfc or BesselJ. This is one of reasons why I qualify SuTra as special function.

For iterates of the Trappmann function, I need also the inverse function, AuTra = SuTra − 1. This function is shown in figure 20.9 and described in the next section.



http://mizugadro.mydns.jp/t/index.php/File:Autraplot.jpg

Figure 20.9: $y\!=\!\mathrm{AuTra}(x)$ by (20.47) and two its asymptotics by (20.44)

5 AuTra, abelfunction of trappmann

This section describes evaluation of the inverse function of SuTra, id est, the Abel function for the Trappmann function tra. I call this abelfunction AuTra. Its explicit plot is shown in figure 20.9; for comparison, in the same picture, the two asymptotics of function AuTra are shown.

Complex map of function AuTra is shown in figure 20.11.

For the transfer function tra, the abeldunction G satisfies the Abel equation

$$G(\mathrm{tra}(z)) = G(z) + 1 \quad \text{[abeltraeq]} \tag{20.41}$$

In order to see the asymptotic expansion of the solution G, I rewrite this equation as

$$G(z+\mathrm{e}^z) - G(z) = 1 \quad \text{[abeltraeq1]} \tag{20.42}$$

and expand the left hand side, using e^z as small parameter:

$$G'(z)\mathrm{e}^z + G''(z)e^{2z}/2 + .. \approx 1 \quad \text{[abeltraeq3]} \tag{20.43}$$

This expansion allows to guess the asymptotic solution:

$$G(z) \approx \frac{z}{2} - \mathrm{e}^{-z} - \frac{\mathrm{e}^z}{6} + \frac{\mathrm{e}^{2z}}{16} - \frac{19\mathrm{e}^{3z}}{540} + \frac{\mathrm{e}^{4z}}{48} - \frac{41\mathrm{e}^{5z}}{4200} + \mathcal{O}(\mathrm{e}^{6z}) \tag{20.44}$$

The coefficients of the expansion above are calculated and evaluated with the Mathematica code below:

```
tra[z_]=z+Exp[z];
g0[z_] = z/2 - Exp[-z] + Sum[c[n] Exp[n z], {n, 1, 20}]
n = 1; s[n] = Series[g0[Log[t]] + 1 - g0[tra[Log[t]]], {t,0,n+1}]
u[n] = Extract[Solve[Coefficient[s[n], t^(n+1)] == 0, c[n]], 1]
g[n, z_] = ReplaceAll[g0[z], u[n]]
For[n = 1, n < 20, n++;
   s[n] = Series[ g[n-1,Log[t]]+1-g[n-1,tra[Log[t]]],{t,0,n+1}];
   u[n] = Extract[Solve[Coefficient[s[n], t^(n+1)] == 0, c[n]],1];
   g[n,z_] = ReplaceAll[g[n-1, z], u[n]];        ]
g[n, z]
Table[Coefficient[g[n, z], Exp[n z]], {n, 1, 20}]
N[Table[Coefficient[g[n, z], Exp[n z]], {n, 1, 20}], 18]
```

The same coefficients can be obtained also by the inversion of the asymptotic expansion of function SuTra. Note, that the asymptotic expansion of AuTra is simpler, than that of SuTra.

For some fixed integer M, define the primary approximation as truncation of the series above:

$$G_M(z) = \frac{z}{2} - \mathrm{e}^{-z} + \sum_{m=1}^{M} c_m \mathrm{e}^{mz} \quad \text{[autraGM]} \tag{20.45}$$

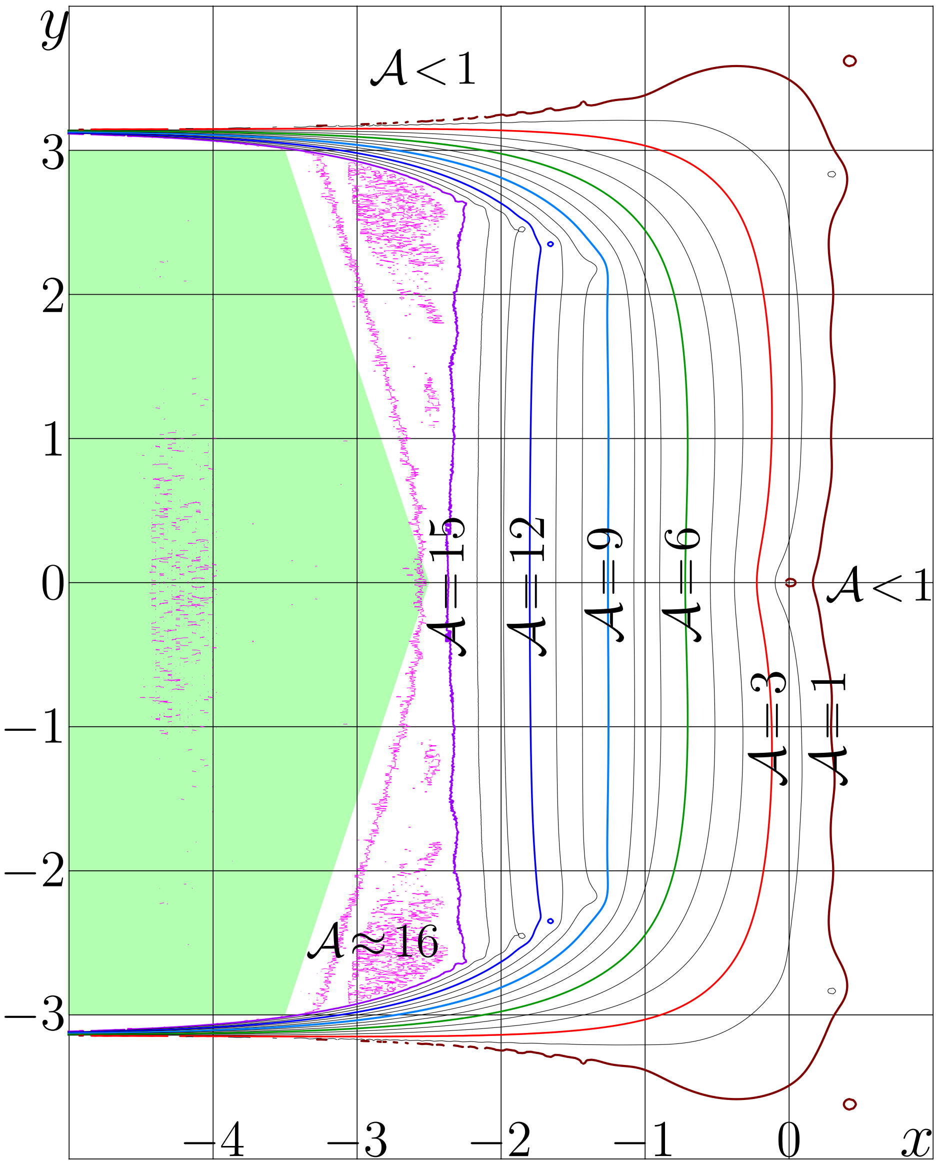
Define function G as limit

$$G(z) = \lim_{n\to\infty} \left(G_M(\mathrm{ArcTra}^n(z)) + n\right) \quad \text{[autraG]} \tag{20.46}$$

AuTra can be expressed through G with

$$\mathrm{AuTra}(z) = G(z) - G(0) \approx G(z) + 1.1259817765745026 \tag{20.47}$$

The constant $G(0)$ can be interpreted also as negative of coefficient x_0 in equation (20.34), id est, $x_0 = -G(0)$.



http://mizugadro.mydns.jp/t/index.php/File:Autran0m9tes64t.jpg

Figure 20.10: Map of agreement $\mathcal{A} = A(x+\mathrm{i}y)$ by (20.50); range (20.49) is shaded $\mathcal{A} = A(x+\mathrm{i}y)$ по формуле (20.50) [autraAgreMap]

For the numerical implementation, we need to choose the appropriate number M in and to determine the number n of iterations approximating limit in equation (20.46). The reasonable choice is $M=9$. Then, the primary approximation

$$\mathrm{SuTra}(x+\mathrm{i}y) \approx g_9(x+\mathrm{i}y) \tag{20.48}$$

is used for the region defined with condition

$$|y|<3 \ \text{ and } \ |y|/3+x<3.5 \tag{20.49}$$

This region is shaded in the figure 20.10 Also, at the same figure, the map of agreement A is shown,

$$\mathcal{A}(z)=-\lg\!\left(\frac{|\mathrm{SuTra}(g_M(z))-z|}{|\mathrm{SuTra}(g_M(z))|+|z|}\right) \tag{20.50}$$

Map in figure (20.10) can be considered also as the numerical verification of relation

$$\mathrm{SuTra}(\mathrm{AuTra}(z)) = z \quad [\mathrm{SuTraAu}] \tag{20.51}$$

In the shaded range, for the numerical implementation, the relation (20.51) holds with at least 15 decimal digits; as it is supposed to be while AuTra is abelfunction, corresponding to superfunction SuTra for the transfer function tra.

Region, where the primary approximation is used, can be optimised, approaching to the level $\mathcal{A}=15$ and improving the algorithm. I suggest, the Reader can do the as an exercise; I hope, the reader will not forget to test the improved algorithm.

The primary approximation (20.48) can be used only in the narrow range of values of the argument shaded in figure autraAgreMap. If the initial argument $z = x + \mathrm{i}y$ happens to be outside the shaded region, then, the function $\mathrm{ArcTra} = \mathrm{Tra}^{-1}$ is applied n times with such n, that the argument comes to the shaded range. Then, approximation

$$\mathrm{AuTra}(z) \approx g_9(\mathrm{ArcTra}^n(z)) + n \qquad [\mathrm{AuTrag9}] \tag{20.52}$$

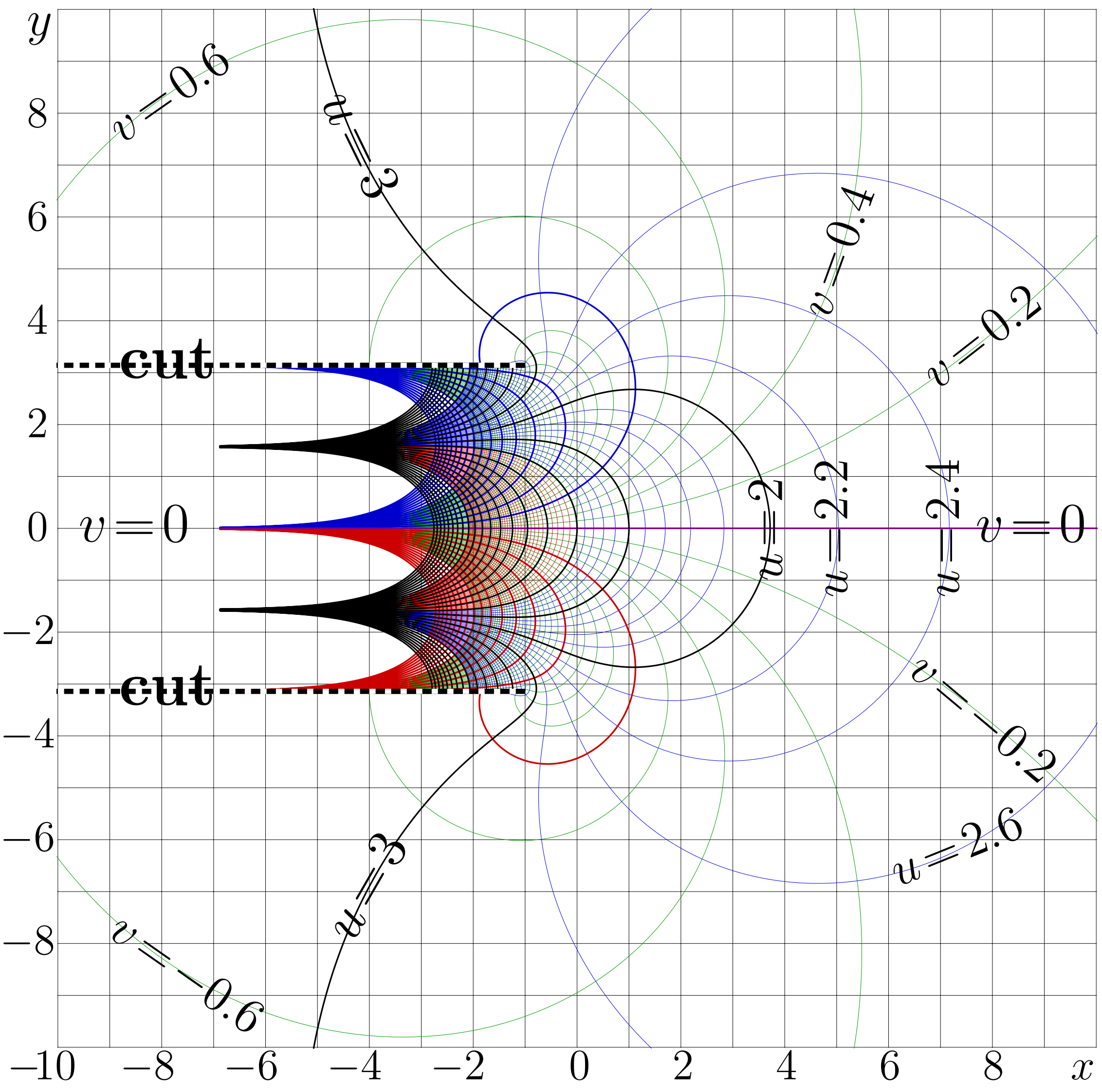
is used to evaluate the function.

The cuts of the range of holomorphism of function ArcTra determine also the cuts the range of holomorphism of function AuTra. These cuts are seen in figures 20.3 and 20.11. At large $|z|$, function $\mathrm{AuTra}(z)$ shows the slow growth, except the half-strip $\Re(z) < 0$, $|\Im(z)| < \pi$. In this half-strip, at large negative values of $\Re(z)$, function $|\mathrm{AuTra}(z)|$ increases exponentially. In particular, this refers to the real values of the argument. This behaviour agrees with that of SuTra shown in figure 20.5.

Readers are invited to download the generators of the figures of this chapter, together with implementations of SuTra and AuTra, and investigate numerically the ranges of applicability of identities

$$\mathrm{AuTra}(\mathrm{SuTra(z)}) = \mathrm{z}\ ,\ \ \mathrm{SuTra}(\mathrm{AuTra(z)}) = \mathrm{z} \tag{20.53}$$

With functions SuTra and AuTra, described in this chapter, one can evaluate the non-integer iterates of function tra. These iterates are described in the next section.



http://mizugadro.mydns.jp/t/index.php/File:AuTraMapT.jpg

Figure 20.11: $u+\mathrm{i}v = \mathrm{AuTra}(x+\mathrm{i}y)$ [autraMap]

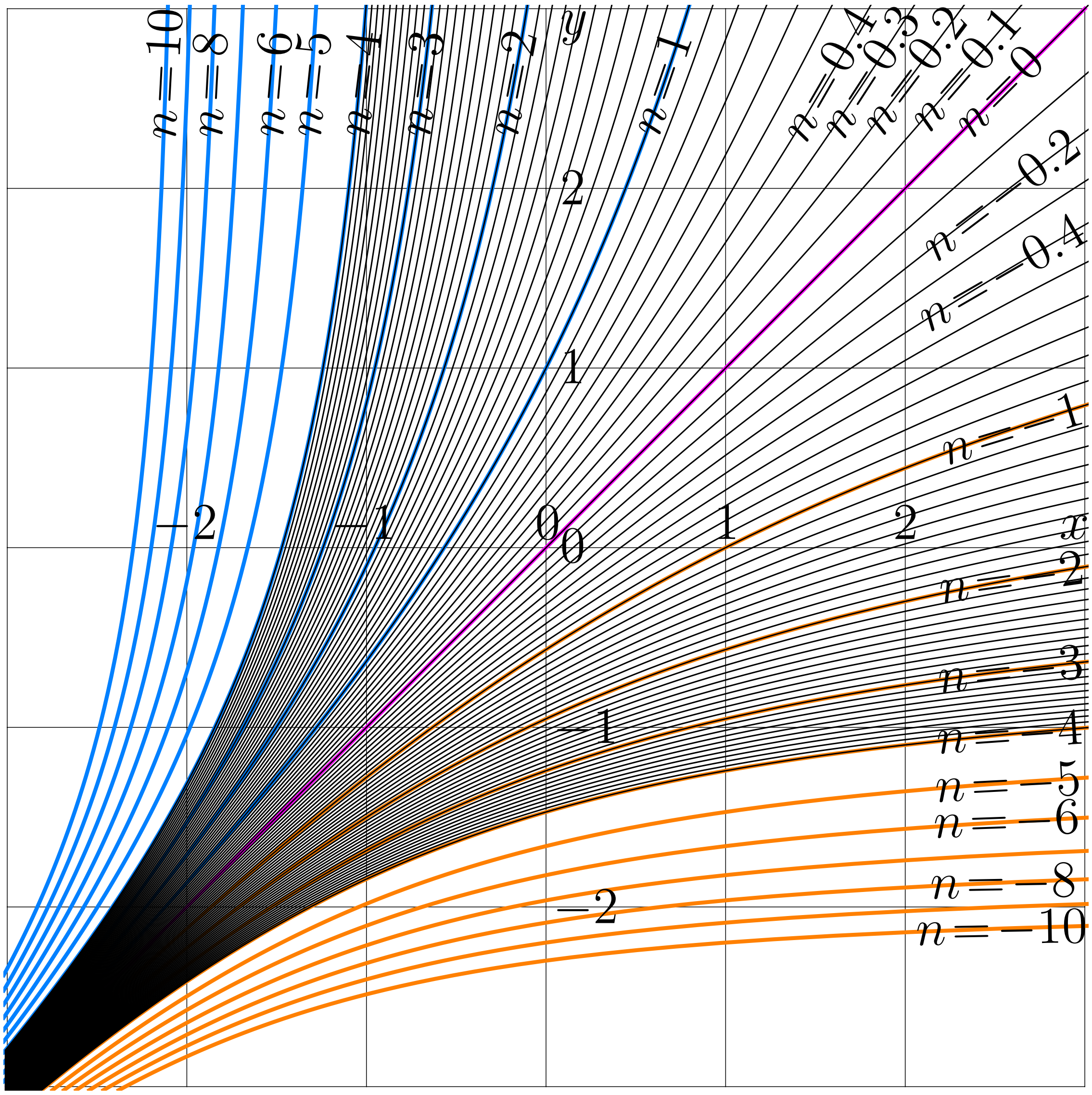
6 Iterates of trappmann

With functions SuTra and AuTra, the iterate of the Trappmann function can be expressed as follows:

$$\mathrm{tra}^n(z) = \mathrm{SuTra}\Big(n + \mathrm{AuTra}(z)\Big) \quad \text{[traite]} \qquad (20.54)$$

As usually, the number n of iterate has no need to be integer. For real value of the argument and some real values of the number n, the iterates of the Trappmann function are shown in figure 20.12.

Iterates of function $\mathrm{tra}(z) = z+\mathrm{e}^z$ look similar to those of other growng transfer functions. These iterates provide the smooth transfer from the function tra to its inverse function $\mathrm{ArcTra} = \mathrm{tra}^{-1}$, and zeroth iterate is the identity function.



http://mizugadro.mydns.jp/t/index.php/File:TraitT.jpg

Figure 20.12: $y = \mathrm{tra}^n(x)$ versus x for various n by (20.54)

7 Relation to other functions

Function SuTra can be expressed through function SuZex

$$\mathrm{SuTra}(z) = \mathrm{SuZex}(\ln(z)) \quad [\mathrm{SuTraSuZex}] \tag{20.55}$$

and function AuTra can be expressed through function AuZex

$$\mathrm{AuTra}(z) = \exp(\mathrm{AuZex}) \quad [\mathrm{AuTraAuZex}] \tag{20.56}$$

One can look at expansions of SuZex and AuZex from chapter 12 and see, that expansions for functions SuTra and AuTra can be obtained with relations (20.55) and (20.56). However, these representations may loss some precision, especially in the regions where the logarithm has low derivative, and the exponential has high derivative. So, for the numerical tests, I use the special representations from the previous sections.

In order to make the deep test of relations, suggested in this Book, the robust representations are required for all the functions involved. For these reasons, the special representations and implementations for functions ArcTra, SuTra and AuTra are described above. The readers are invited to make the asymptotical analysis for the relations above. Also, the numerical verification can be used. For moderate values of the argument, the representation of SuTra through SuZex loss only few decimal digits in the precision. I load the descriptions of functions mentioned in this section,
`http://mizugadro.mydns.jp/t/index.php/ArcZex`
`http://mizugadro.mydns.jp/t/index.php/AuZex`
`http://mizugadro.mydns.jp/t/index.php/SuZex`
`http://mizugadro.mydns.jp/t/index.php/ArcTra`
`http://mizugadro.mydns.jp/t/index.php/SuTra`
`http://mizugadro.mydns.jp/t/index.php/AuTra`,
together with their complex double numerical implementations.

On this I finish the consideration of the Trappmann function and its iterates. And at this point I finish consideration of examples of transfer functions, superfunctions, abelfunctions and the non-integer iterates. Perhas, I should explain, why I had spent so many efforts on this. I think, the best explanation is to remind the old folkloric story below.

One-legged friend of one Taylor asked him to sew the special pants with one leg. He payed well for the custom pants, but he needed also the pants for his dog, who, as himself, had lost one leg long time ago.Taylor sewed the pants for that 3-leg dogs. The pants were beautiful, and friend of friend asked him the same for his normal, 4-leg dog.. The story is long, the starfishes and octopuses are mentioned there. En fin, the Taylor had elaborated tools to sew pantaloons for creatures with arbitrary number n of legs. And if tomorrow some extraterrestrials with n legs come, the Taylor already has pantaloons for them.

I typed the story above in order to explain better, what is scientific research and what do the researchers. You may consider this as a kind of joke, but if someone needs to approximate, for example, the logarithmic function with entire functions, - then, like the Taylor mentioned, I already have such an approximation; it can be expressed through SuTra by formula (20.38). This may be considered as a small specific addition to various motivations suggested in the Preface.

On this point I stop the main body of the Book and go to overview the results presented above. This makes the content of the next chapter.

Chapter 21

Conclusion

This chapter discusses the same, as the Introduction. The main difference of this chapter from the introduction is, that I assume, that the reader already has browsed at least some of the previous chapters, as it is shown in figure 21.1, and understands sense of some notations used. In order to show the need of conclusion, I remind the old leyend.

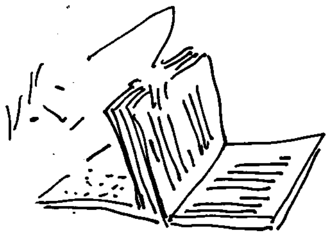


Figure 21.1:

One emperor wanted to study history. He ordered the Ministry of Science to develop a full course of the world-wide history. The greatest scientists were working on this tutorial during many years. Finally, the heavy truck arrived with thousands volumes of "Complete Course of the World History". The Emperor realised that all his life will not be sufficient to read this course. The Emperor asker the President of the Ministry of Science to shorten the course. The historians worked on the second edition during few years, and then, in a big pack, "Trilogy of the World History" had been deliverer to the Emperor. But the Emperor already had weak eyes, and he could not read that Trilogy. Again, the historians had to shorten the course. A year later, the Top Historician came to the Emperor and gave him the pamphlet "A Brief History of the Imperial Family." The Emperor was old and ill, and could not even read that brochure. He asked Historian whether the brochure can be reduced. The Historian answered: "*No new edition is necessary. I'll tell you right now: People were born, suffered and died.*"

Several colleagues had told me, that this Book is too thick, and asked, if it can be shorter. As the historian in the story above, I follow the requests and describe the topic of superfunctions briefly. Below, the main results of the Book are collected in a single section.

1 Basic results

Holomorphic functions can be iterated. The iterates can be expressed through the superfunction and the abelfunction; then, the number of iterate has no need to be integer.

In order to iterate some holomorphic function T, first, I declare it as "transfer function". Then, I construct for the superfunction F, that is solution of the transfer equation

$$F(z+1) = T\Big(F(z)\Big)$$

I need also the inverse function, $G = F^{-1}$. I call it "Abel function" or "abelfunction". It satisfies the Abel equation

$$G\Big(T(z)\Big) = G(z) + 1$$

When superfunction F and the Abel function G are established, the iterates of function T can be expressed as

$$T^n(z) = F\Big(n + G(z)\Big)$$

where the number n of iterate has no need to be integer; the transfer function can be iterated some non-integer, rational or even complex number of times. However, for the integer n, the conventional expression of iterates holds:

$$T^n(z) \ = \ \underbrace{T\Big(T(..T(z)..)\Big)}_{n \text{ evaluations of function } T}$$

Solution of the transfer equation is not unique. If some solution F is found, then, one additional solution f can be constructed with modification of the argument,

$$f(z) = F\Big(z + \theta(z)\Big)$$

where θ is periodic holomorphic function with period unity. Accordingly, the new abelfunction g can be established. Then, the new superfunction and the corresponding abelfunction will provide new, different iterates of the transfer function.

Variety of superfunctions can be narrowed, if we establish, postulate the asymptotic behaviour of the superfunction in the complex plane. The superfunction with simple asymptotic behaviour I treated as principal. Other superfunctions can be expressed with modification of argument of the principal superfunction. Holomorphic periodic function grows at least exponentialy (as sin and cosine to in the direction of imaginary axis); even a small periodic modification is easy to reveal in the complex plane. For this reason, for uniqueness, it is important to build-up the superfunctions for the complex argument, even if they are supposed to be used for the real argument. The criterion of holomorphism indicates, which of superfunctions is expected to have the physical sense and should be considered as "true" one.

Some superfunctions and abelfunctions have special names. These functions are collected in Table 3.1. Some of them are widely known; one may use them without to know, that they are superfunctions.

In principle, for any holomorphic function F, one can built-up the inverse function $G = F^{-1}$ and define $T(z) = F\big(1 + G(z)\big)$. Then, such function T can be treated as transfer function with known superfunction F, abelfunction G and non-integer iterates $T^n(z) = F\big(n + G(z)\big)$.

The inverse problem, id est, construction of superfunction F for some given transfer function T, is considered in this Book. For this construction, the key question is about the fixed points of the transfer function, id est, about solutions L of the equation

$$L = T(L)$$

As physicist, I am interested mainly in the real-holomorphic functions, for which $T(z^*) = T(z)^*$. In order to reduce variety of solutions of the transfer equation, I postulate, that the superfunction approaches the fixed point L at the infinity.

Methods of asymptotic expansions of the superfunctions are suggested in the Book. At infinity, the superfunction is postulated to approach the fixed point of the transfer function. In many cases, this approach is exponential; and, in many cases, the exact superfunction appears as the limit at the multiple application of the transfer function to the asymptotic solution with displaced argument.

It may happen, that all the fixed points L of the transfer function T are complex, not real. In particular, this is case of the natural exponent. Then, the superfunction can be expressd through the Cauchy integral and solution of the corresponding integral equation. Historically, the first complex map of superfunction of exponent, had been constructed with this representation. Complex map of tetration is shown in figure 14.12 and at the First page of the cover of this Book. Tetration tet appears as solution of the transfer equaiotn

$$\mathrm{tet}_b(z+1) = \mathrm{e}^{\mathrm{tet}_b(z)} \ , \ \mathrm{tet}_b(0) = 1$$

$\mathrm{tet}_b(z)$ is supposed to be bounded in the strip $|\Re(z)| \le 1$. For real $b>1$ and real x, dependence $y = \mathrm{tet}_b(x)$ is shown in figure 17.1. Tetration can be constructed also for the complex base; the example with $b = 1.52598338517+0.0178411853321\,\mathrm{i}$ is shown in figure 18.3.

It may happen, that the transfer function T has no fixed point at all. One example of such a function is the Trappmann function

$$T(z) = \mathrm{tra}(z) = z + \exp(z)$$

However, even for this function, the superfunction can be constructed. It is called SuTra; its map is shoe in figure 20.6. This is entire function with logarithmic asymptotic; up to my knowledge, before publication [88], no one such function had been suggested.

Since 2010, I claim, that I can construct the superfunction F, abel-function G and non-integer iterates for any growing real-holomorphic transfer function T. This Book describes the sequence of attempts to negate, refute this claim. All these attempts failed: I could not find the transfer function, for which I cannot construct the superfunction. In support of my claim, the Book presents examples of transfer function with real fixed point(s), examples with complex fixed points, and the example of the transfer function without any fixed point. While all the tests are successful, the ability of construction of superfunctions can be interpreted as a scientific fact.

I just have mentioned, what is done. But, as usually, a lot of can be done about superfunctions. The next section is dedicated to this.

2 Future work

I tried to collect that I know about superfunctions and iterates, in this Book. However, always some phantasies remain, what else would be interesting to do. Some of hese phantasies are collected below.

The more efficient, more simple and more rigorous proofs of the existence and uniqueness of superfunction may require the future work, and also the additional conditions, that should be applied in order to provide the uniqueness of superfunction.

Application of superfunctions into the laser science may be subject for the future analysis. Especially, this refers to the laser science, where the physical sense of superfunctions and the non-integer iterates is especially explicit.

The future work may be related to the specific case of superfunctions, namely, ackermanns. Figure 19.7 shows the graphics for the first five ackermanns: addition of constant, multiplication to constant, exponent, tetration and pentation. I think, similar graphics can be constructed also for higher ackermanns.

The important suggestion for the future work is related with the automatisation of construction of superfunctions. I mean the automatic algorithm, that begins with the transfer function, searches for the appropriate fixed points of this function (if the transfer function has fixed points), chooses the appropriate asymptotic for the superfunction, use it to build-up the superfunction, build-up the corresponding abelfunction and calculates the non-initeger iterates. The software Mathematica already has name for such a procedure; it is called Nest. Up to date of preparation of this Book, the routine Nest can deal only with very special case, namely, when the number of iterate is expressed with positive integer constant; in other cases, the call is interpreted as error. The upgrade of that routine for the real and complex number of iterate would be intersting.

Phantasies and curiosity should be motivations for any serious scientific research. I collect the tools that can be used in this work. Many of them are described in this Book.

3 Notations

I try to use the same notations through the whole Book. In order to approach this, the notations are different from those used in the original publications. Some notations are collected in tables 21.1, 21.2. I collect the most important notations, and those, that often cause confusions.

Table 21.1: Notations, Latin alphabet

$\mathcal{A}(m,z)=\mathcal{A}\Big(m-1,\mathcal{A}(m,z-1)\Big)$	canonical ackermann (19.14)
$A_{b,n}$	ackermann to base b of order n, (19.7)
a_n	Coefficinets in various asymptotic expansions
$\mathrm{AuExp}_b=\mathrm{SuExp}_b^{-1}$	growing abelexponent to base b
AuTra	Abel function for the Trappmann function
AuSin	abelsine
Arc	prefix indicating the inverse function
b	base of ackermanns, (19.1)
c_n	coefficients in expansions, for ex., (10.47) or (20.15)
C_n	coefficients in expansions, for ex., (7.12)
$\mathrm{e}=\exp(1)\approx 2.7$	base of natural logarithms
exp	exponent to base e
$\exp_b$	exponent to base b
F	Name, used to denote superfunctions
$\mathrm{Factorial}(z)=z!$	holomorphic solution of equations $z!=z\,(z-1)!$
f	name used for various functions
$G=F^{-1}$	name used to denote abelfundtions
$h=\mathrm{Elu}_s^{0.5}$	half iterate of logistic operator
$\mathrm{i}=\sqrt{-1}$	imaginary unit $\mathrm{i}^2=-1$
i	variable, usually it takes integer values
K	plug for the contour integral rfK, or e^k
k	increment in the asymptotic representation of superfunction (6.3)
L	fixed point of transfer function T, id est, solution of $T(L)=L$
ln	logatirhm to base e
$\log_b$	Logarithm to base b
M	number of terms in truncated series
O	some function that grows-up not faster than its argument
P	period or asymptotic period; also is used as inverse of function Q
Q	is used as inverse function of P in table 3.1
Super	prefix to name of function, indicating its sperfunction
s	parameter of the logistic operator (7.1)
T	transfer function
tet	natural tetration (14.28), [54]
tet_b	solution f of equaitons $f(z+1)=b^{f(z)}$, $f(0)=1$, see fig. 17.1
u	real part of function in the complex map
v	imaginary part of function in the complex map
x	used as dummy parameter, that takes real values
y	used as dummy parameter, that takes real values
z	used as dummy parameter, that takes complex values; often $z=x+\mathrm{i}y$

Table 21.2: Notations, Greek alphabet

α , β , γ	used as parameters in the expansions
$\eta = \exp(1/\mathrm{e}) \approx 1.444667861$	Henryk base
$\varepsilon = \exp(kz)$	small parameter in the expansion of superfunctions
θ	holomorphic periodic function with period unity
ζ	$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$, Riemann function (8.9), [13, 18]
Ω	contour of integration in the Cauchy formula (14.21)

4 Afterwards

The first (Russian) version of this Book happened thicker, than I had expected. and could be much thicker; because all the time, there is the evil illusion, that some additonal small formula should greatly simplify the understanding. In addition, with each formula from the book, several additional picture can be plotted. However, I think, for the Reader, it will be much better, to plot some picture, than to see the gallery of similar figures in the Book.

I tried to make this Book shorter, than its first Russian version. Actually, the Book happened to be longer, thicker, because I include here the chapter about the Nemtsov function $\mathrm{Nem}_q(x) = x + x^3 + qx^4$; this is important example of the exotic iteration of transfer function with the specific expansion at the fixed point.

Some things are still dropped out from the Book. I did not include the holomorphic extension of tetration beyond the cut lines. I did not include figures to iterates of the exponent to complex base. And I did not include many other figures, assuming, that the Readers can download the generators of the figures and plot all modifications they need.

The Readers are invited to download algorithms, figures and their generators from `http://mizugadro.mydns.jp/t/index.php/Category:Book`; with these tools, the colleague may continue from that points of research, where I am now. I beleve, this is correct style of making science, where all the results are available for researchers. This is supposed to simplify the verification and refutation (if anything is seriously wrong), as well as revealing and correction of mistakes in formulas and bugs in the algorithms. I invite the colleagues to expose their results in a similar way.

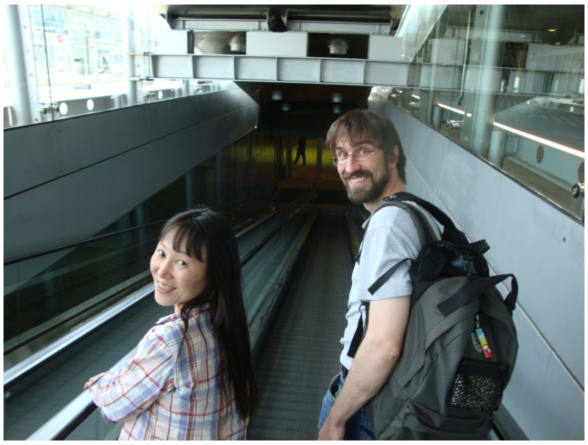


Figure 21.2: Near relatives, whose existence is essential for writing of this Book

5 Acknowledgement

I am grateful to relatives (Figure 21.2) for the tolerance with respect to this Book, it took much more time and efforts, than expected.

I am grateful to colleagues, who helped me to collect the literature on superfunctions (Figure 21.3) and arrange the server, database and mediawiki: without these instruments, I would get lost among a thousand files used for generation of this Book.

Figure 21.3: Bookorm [117]

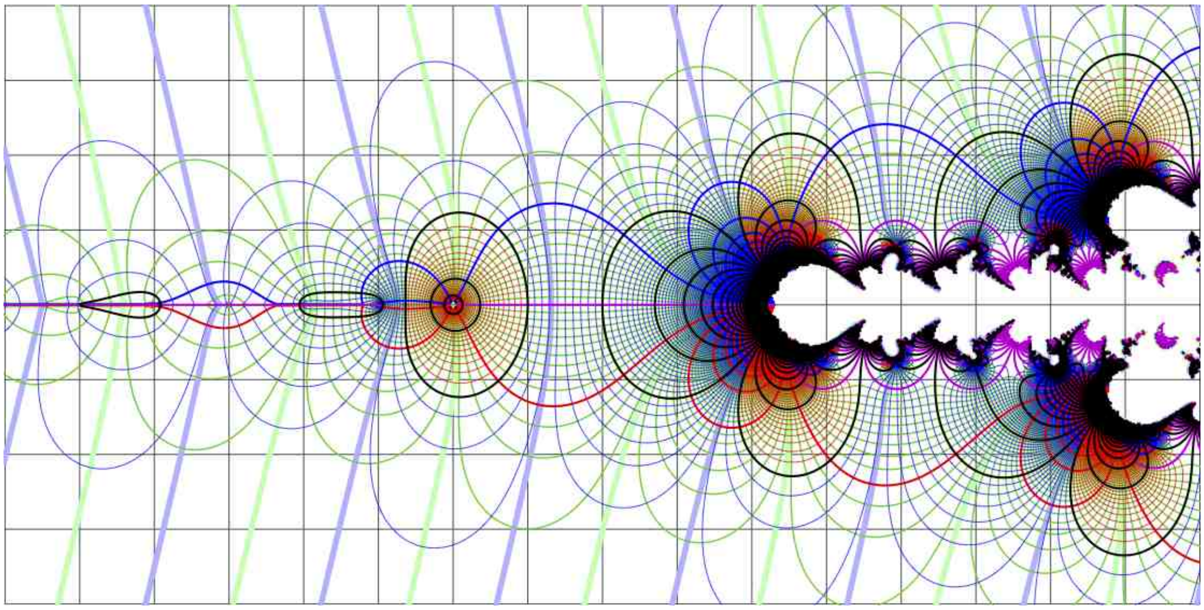
The goal was to get possibility to answer the questions on superfunctions with either *Nobody knows this!* or *Das ist in meinem Buch!* (see figure 21.4). Perhaps, this goal is not reachable; the new algorithms are expected to be reported [93]. Tanks to colleagues who keep doing superfunctions.

Figure 21.4: Das ist in meinem Buch! [ainu]

Chapter 22

Supplement

1 About the cover



At the front cover of this book, the map is shown, from `http://mizugadro.mydns.jp/t/index.php/File:Tetma.jpg`

This is one of versions of figure 14.12; Some details and labels are removed in order to simplify the aesthetic view.

The cover of the Russian version is loaded as `http://mizugadro.mydns.jp/t/index.php/File:Covervi.jpg`

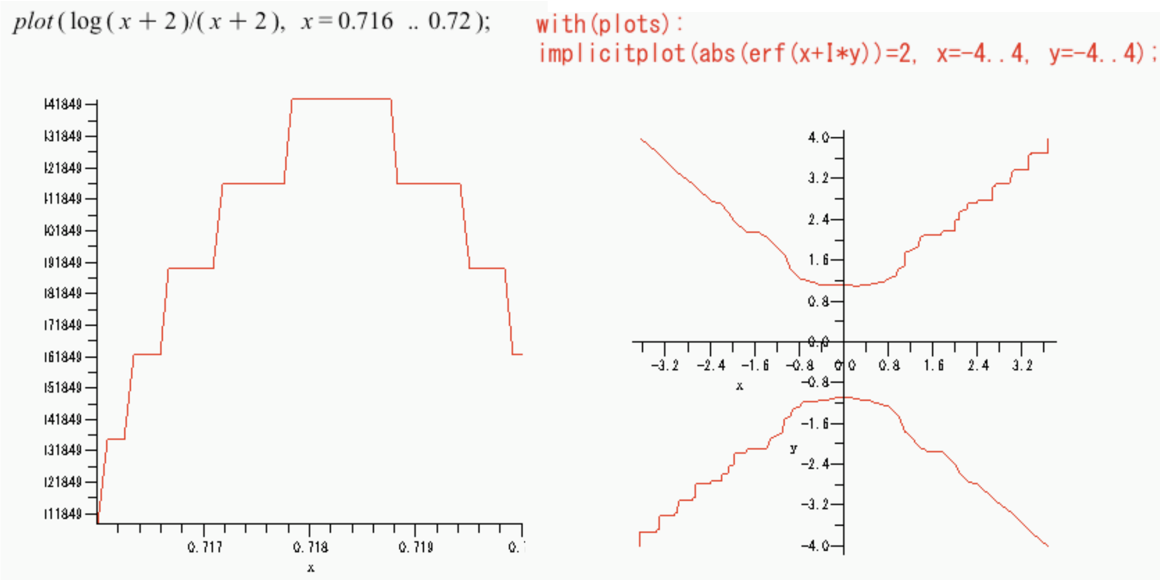


Figure 22.1: The reader should have Maple-10 installed, in order to reproduce the plots above with the codes supplied at the top of each picture

2 Maple and tea

Here is the company C of computers; they master:
Per each two years, their PCs run twice faster.

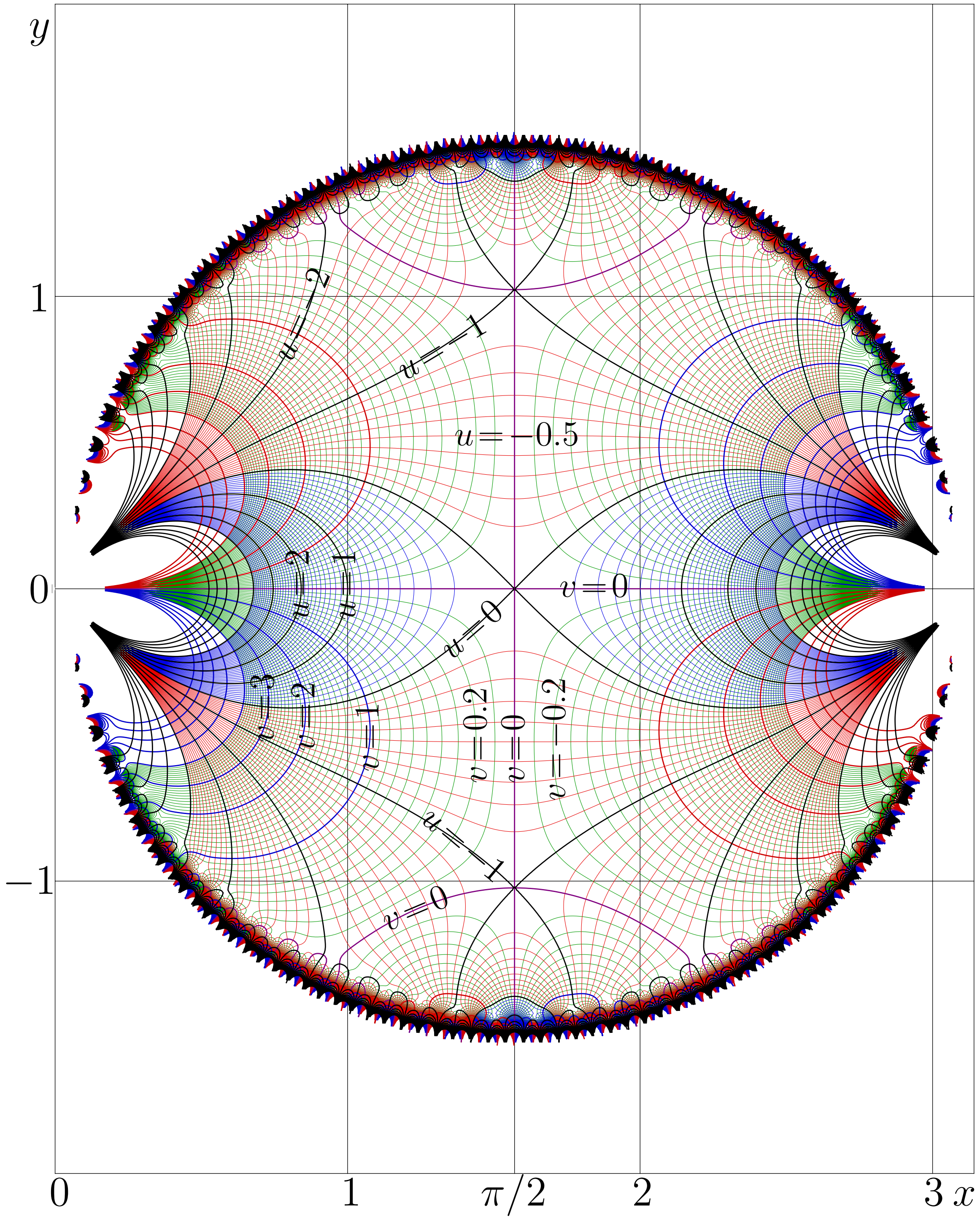
Here is company S of the soft; they work hard as plowers,
So, each new release runs 0.7 times slower;
The soft is to run at the user's PC,
Which is made, of course, by the company C.

Here is user U, buys from C and from S,
the newest model and the latest release.
U presses some key, say, key number K;
The soft S responds, with some delay,
during some time, say, during time t,
to let user U to have some tea.

How many releases per year does sell S to U
to let him have tea, while he has nothing to do,
just waiting response by the soft to key K,
keeping the same time t of delay,
neutralizing the efforts of company C
to run faster U's task at his modern PC?

This problem above is not correct, because
The tea-concern, together with C, of course,
support the efforts of the company S
to make even bigger the newest release,
To force U buy more tea and a newest PC,
to boost business of C and the concern of tea.

(2007)



http://mizugadro.mydns.jp/t/index.php/File:Ausintay40t50.jpg

Figure 22.2: Map of the Taylor expansion f of AuSin with 40 terms in the standard notations. $u+\mathrm{i}v = f(x+\mathrm{i}y)$.

3 Taylor expansion of AuSin

In chapter 13, the Taylor expansion of function AuSin at $\pi/2$ is mentioned, (12.24). In figure 22.2, the complex map of the truncated series with 40 terms is shown. This map should be compared to Figure 12.4. Such an expansion can be used to boost the precise evaluation of AuSin for values of the argument in vicinity of $\pi/2$. Also, coefficients of this expansion can be used for evatuation of coefficients of expansion of SuSun at zero; for example, wight routine InverseSeries.

4 Sites

Goal to his Book is not to attract attention of Reader to the cites below, but to provide the description of superfunctions, that releases the reader from need to browse the sites below.

http://allmybase.com/dropbox/tetration.pdf
http://cdn.bitbucket.org/bo198214/bunch/downloads/main.pdf H.Trappmann, D.Kouznetsov. 5+ methods for real analytic tetration. June 28, 2010.
http://en.citizendium.org/wiki/Cauchy-Riemann_equations
http://en.citizendium.org/wiki/Superfunction
http://en.citizendium.org/wiki/Tetrational
http://en.citizendium.org/wiki/Holomorphic_function
http://en.wikipedia.org/wiki/Abel_equation
http://en.wikipedia.org/wiki/Abel_function
http://en.wikipedia.org/wiki/Cauchy's_integral_formula
http://en.wikipedia.org/wiki/Hellmuth_Kneser
http://en.wikipedia.org/wiki/Niels_Henrik_Abel
http://en.wikipedia.org/wiki/Superfunction
http://en.wikipedia.org/wiki/Tetration
http://math.eretrandre.org/tetrationforum/index.php
http://math.stackexchange.com/tags/tetration
http://mathworld.wolfram.com/Tetration.html
http://oeis.org/wiki/Tetration
http://mizugadro.mydns.jp/t/index.php/Abel_function
http://mizugadro.mydns.jp/t/index.php/ArcShoka
http://mizugadro.mydns.jp/t/index.php/AuSin
http://mizugadro.mydns.jp/t/index.php/AuTra
http://mizugadro.mydns.jp/t/index.php/AuZex
http://mizugadro.mydns.jp/t/index.php/Complex_map
http://mizugadro.mydns.jp/t/index.php/Doya_function
http://mizugadro.mydns.jp/t/index.php/Factorial
http://mizugadro.mydns.jp/t/index.php/Holopmorphic_extension_of_Collatz_Subsequence
http://mizugadro.mydns.jp/t/index.php/Keller_function
http://mizugadro.mydns.jp/t/index.php/LambertW
http://mizugadro.mydns.jp/t/index.php/Logistic_sequence
http://mizugadro.mydns.jp/t/index.php/Regular_iteration
http://mizugadro.mydns.jp/t/index.php/Shoka_function
http://mizugadro.mydns.jp/t/index.php/Superfunction
http://mizugadro.mydns.jp/t/index.php/Superfactorial
http://mizugadro.mydns.jp/t/index.php/SuSin
http://mizugadro.mydns.jp/t/index.php/SuTra
http://mizugadro.mydns.jp/t/index.php/SuTra
http://mizugadro.mydns.jp/t/index.php/Table_of_superfunctions
http://mizugadro.mydns.jp/t/index.php/Tania_function
http://mizugadro.mydns.jp/t/index.php/Tetration
http://www.proofwiki.org/wiki/Definition:Superfunction
http://www.proofwiki.org/wiki/Definition:Tetration
http://www.tetration.org/Tetration/index.html D.Geisler. What lies beyond exponentiation?
http://www.youtube.com/watch?v=z-mfxP1Tmfw Kasane Teto. Tetration ↑↑. 2012.

5 New notations

In future, the new notations will be requested. Below, I suggest tentative names for them:

Transferation. For given function f, construction of the transfer function $T(z) = f\big(1+f^{-1}(z)\big)$. For this transfer function T, function f is superfunction and f^{-1} is abelfunction.

Transation. For given function f, construction of the transfer function $T(z) = f^{-1}\big(1+f(z)\big)$. For this transfer function T, function f^{-1} is superfunction and f is abelfunction.

Superation. For given function f, construction of superfunction F as solution of equation $F(z+1) = f\big(F(z)\big)$. In this case, f appears as transfer function.

Supation. For given function f, construction of abelfunction G as solution of the Abel equation $G\big(f(z)\big) = G(z)+1$. In this case also f appears as a transfer function.

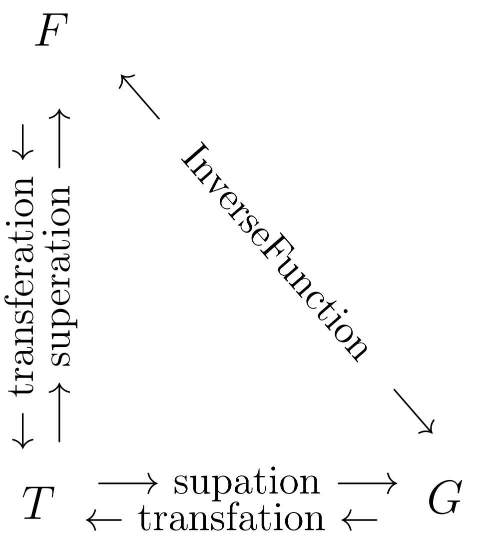


Figure 22.3: T, F and $G=F^{-1}$

Transferation is inverse operation with respect to suppuration. Transfation is inverse operation with respect to supation. This is shown in figure 22.3 for the transfer function T, superfunction F and abelfunction G.

This Book is dedicated to superation and supation, and also to the additional requirements, that provide the uniqueness of superfunction F and abelfunction G.

In this Book, I do not use the new words shown in figure 22.3; while, there is no need to use them. In the similar way, there was no need in the special terms for differentiation and integration, until these operations became routines. However, the terms shown in figure 22.3 will be requested as soon as the automatic construction of superfunctions and abelfunctions will be realised. In language Mathematica, for the superation, there exist name Nest (until year 2017, this routine is supported only for natural values of the number of iterate).

I am not sure, that namely the names from figure 22.3 will be usual. For this reason, I do not use these names in the Book. However, the operations, mentioned in figure 22.3 exist; they will require some names.

6 Extended abstract

The Book is dedicated to construction of superfunction F and abelfunction G for given transfer function T. The formalism of superfunctions is considered with examples, that are presented in ready-to-use form. The reader is supposed to know something about the complex numbers and elementary functions.

The superfunction is solution of the transfer equation
$F(z+1) = T\big(F(z)\big)$
The abelfunction $G = F^{-1}$ satisfies the Abel equation
$G\big(T(z)\big) = G(z) + 1$
In order to provide the uniqueness of solution, the additional requirements on F are applied, referring to its behaviour in the complex plane.

The nth iterate of function T is denoted with superscript:

$$T^n(z) \;=\; \underbrace{T\Big(T\big(..T(z)..\big)\Big)}_{n \text{ evaluations of function } T}$$

With superfunction F and abelfunction G, the iterate is expressed as $T^n(z) = F\big(n+G(z)\big)$. In this representation, the number n of iterate has no need to be integer.

Examples of superfunctions are considered and collected as Table 3.1. Superfunctions are constructed for sin, factorial, exponential, tetration and other functions.
Many explicit plots and complex maps for these functions, superfunctions, and iterates are included. The figures are loaded also to TORI together with their generators at
`http://mizugadro.mydns.jp/t/index.php/Category:Book`

The formalism of superfunctions greatly extends the set of functions available for applications in the scientific research.

7 Keywords

In this section, I suggest the essence of the notations

T Transfer function

$T\big(F(z)\big) = F(z{+}1)$ Transfer equation, superfunction

$G\big(T(z)\big) = G(z) + 1$ Abel equation, abelfunction

$F\big(G(z)\big) = z$ Identity function

$T^n(z) = F\big(n + G(z)\big)$ nth iterate

$F(z) = \dfrac{1}{2\pi \mathrm{i}} \displaystyle\oint \frac{F(t)\,\mathrm{d}t}{t - z}$ Cauchy integral

$\mathrm{tet}_b(z{+}1) = b^{\mathrm{tet}_b(z)}$ tetration to base b

$\mathrm{tet}_b(0) = 1$, $\mathrm{tet}_b\big(\mathrm{ate}_b(z)\big) = z$

$\mathrm{ate}_b(b^z) = \mathrm{ate}_b(z) + 1$ arctetration to base b

${\exp_b}^n(z) = \mathrm{tet}_b\big(n{+}\mathrm{ate}_b(z)\big)$ nth iterate of function $z \mapsto b^z$

$\mathrm{Tania}'(z) = \dfrac{\mathrm{Tania}(z)}{\mathrm{Tania}(z){+}1}$ Tania function, $\mathrm{Tania}(0){=}1$

$\mathrm{Doya}(z) = \mathrm{Tania}\big(1{+}\mathrm{ArcTania}(z)\big)$ Doya function

$\mathrm{Shoka}(z) = z + \ln(\mathrm{e}^{-z}{+}\mathrm{e}{-}1)$ Shoka function

$\mathrm{Keller}(z) = \mathrm{Shoka}\big(1{+}\mathrm{ArcShoka}(z)\big)$ Keller function

$\mathrm{tra}(z) = z + \exp(z)$ Trappmann function

$\mathrm{zex}(z) = z\ \exp(z)$ Zex function

Bibliography

[1] https://en.wikisource.org/wiki/Gargantua/The_Author%27s_Prologue_to_the_First_Book Gargantua by François Rabelais, translated by Thomas Urquhart and Peter Antony Motteux. *..Most noble and illustrious drinkers, and you thrice precious pockified blades (for to you, and none else, do I dedicate my writings),*

[2] http://moliere-in-english.com/bourgeois.html J.B. Moliere. The Bourgeois Gentleman. (1670) English adaptation by Timothy Mooney. Act Two, Scene Four. *.. These forty years now, I've been speaking in prose without knowing it! ..*

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English translation:
According to the results of the student vote, winners were the icon with the image of the lever, raising the Earth, and the current from the well-known emblem of the root of factorial, inscribed in the letter Φ. This icon is created by the student Department of Biophysics, A.Sarvazyanom; it is attracted by its simplicity and expressiveness. Then the icon emblem has undergone harsh criticism from the Faculty of management, since it has no physical meaning, mathematically absurd and ideologically vacuous. Yielding to the requests of the Komsomol, the party committee (Professor A.I.Kuzovnikov) agreed that the icon to release a series of small, only one year, so that in the future to create a real symbol for the Faculty students. Now, paradoxically, the emblem of the first icon was included in the official form and faculty printing. ..

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Tools for evaluation of superfunctions, abelfunctions and non-integer iterates of holomorphic functions are collected. For a giver transfer function T, the superfunction is solution F of the transfer equation F(z+1)=T(F(z)) . The abelfuction is inverse of F. In particular, superfunctions of factorial, exponent, sin are suggested. Also, the holomorphic extensions of the logistic sequence and those of the Ackermann functions are considered. Among ackermanns, the tetration (mainly to the base b>1) and natural pentation (to base b=e) are presented. The efficient algorithm for the evaluation of superfunctions and abelfunctions are described. The graphics and complex maps are plotted. The possible applications are discussed. Superfunctions significantly extend the set of functions that can be used in scientific research and technical design. Generators of figures are loaded to the site TORI, http://mizugadro.mydns.jp for the free downloading. With these generators, the Readers can reproduce (and modify) the figures from the Book. The Book is intended to be applied and popular. I try to avoid the complicated formulas, but some basic knowledge of the complex arithmetics, Cauchy integral and the principles of the asymptotical analysis should help at the reading.

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