

## Doya Function and Tania function

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In the simple model of amplifier with gain saturation, the output signal is treated as function of the input signal. This function is interpreted as transfer function of the system. The idealized case of uniform (uniformly pumped) amplifier is considered. The specific realistic example of the transfer function is called “Doya function”. The distribution of signal inside the amplifier along the amplification is called “superfunction” with respect to the transfer function. For the specific case of Doya function, the superfunction is called “Tania function”; it is expressed as solution of differential equation  $Tania'(z) = Tania(z)/(1+Tania(z))$  with boundary condition  $Tania(0) = 1$ . The efficient algorithms for evaluation of the two functions are suggested. The complex maps and explicit plots of these functions are supplied. Possible applications of these two functions in the Laser Science are discussed.

**Keywords:** Holomorphic function, laser materials, transfer function, superfunction

### 1. Introduction

#### 1.1 Doya and Tania

In this article, two functions are described. They are called Doya and Tania.<sup>1)</sup> These functions are specific for laser science, and deserve to have the specific names.

Define function Doya with

$$\text{Doya}^z(a) = F(z) \quad (1)$$

where function  $F$  is solution of equation

$$F'(z) = \frac{F(z)}{1 + F(z)} \quad (2)$$

with boundary condition

$$F(0) = a \quad (3)$$

If the superscript at the name of function Doya is omitted, it is assumed to be unity;

$$\text{Doya}(z) = \text{Doya}^1(z) \quad (4)$$

Equation (2) corresponds to the simple model of the gain medium with saturation;<sup>2,3)</sup>  $z$  has sense of the normalized coordinate, and  $F(z)$  has sense of intensity, measured in units of saturation intensity.

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Define function Tania with

$$\text{Tania}(z) = \text{Doya}^z(1) \quad (5)$$

In such a way, Tania is solution  $F$  of equation (2) with boundary condition  $F(0) = 1$ . The solution of equation (2) deserves to have the specific name.

### 1.2 motivation

This article is motivated by the notes that the mathematical functions, used in the previous publication,<sup>1)</sup> are not supplied with the detailed descriptions. Actually, these functions were described at site TORI (Tools for Outstanding Research and Investigation); but 2013.02.27, namely during consideration of manuscript of,<sup>1)</sup> the site <http://tori.ils.uec.ac.jp/TORI><sup>4)</sup> had been severely vandalized<sup>7)</sup> (the wikimedia engine had been damaged), and the recovery happened to be very difficult, if at all. In order to mitigate the sequences of that attack and possible future attacks, the description of two functions, typical for the Laser Science, is presented below. These functions are Tania by (5) and Doya by (4).

### 1.3 About notations

Name of function Tania is chosen after the first name of the first author of the series of publications about propagation of monochromatic modulated waves in the gain medium with simple model of saturation.<sup>5,6)</sup> After to analyze properties of function Tania, it had been revealed, that in vicinity of the real axis (and, in particular, for real values of argument), Tania can be expressed through the known function WrightOmega,<sup>10-12)</sup> and also through the function LambertW.<sup>13,14)</sup> These relations are the following:

$$\text{Tania}(x) = \text{WrightOmega}(x+1) \quad (6)$$

and

$$\text{Tania}(x) = \text{LambertW}(\exp(x)) \quad (7)$$

However, the definition of Tania through the equations (2) and  $\text{Tania}(0) = 1$  has certain advantages. First, the integer value at zero simplifies comparison of Tania with other functions, that describes evolution of signal in the amplifiers; in particular, with function Shoka, that describes amplification of short pulses.<sup>15,16)</sup> Second, the behavior of function WrightOmega, as it is implemented in Mathematica, in the complex plane has many cutlines. These cutlines makes difficult the use of function WrightOmega in other applications, in particular, in the analysis of the fixed points of logarithm. While, this analysis is presented at TORI <http://tori.ils.uec.ac.jp/TORI/index.php/Filog>. As TORI happened to be vulnerable with respect to attacks by vandals, that description also will be presented as a separate publication.

For functions WrightOmega and LambertW, no efficient C++ algorithms were available;

the complex(double) implementation of Tania in C++ allows to use relations (6) and (7) for implementation of functions WrightOmega and LambertW. The C++ implementations of these functions are important, because the plotting of complicated pictures with Mathematica is slow, and often, the Mathematica's plots have poor resolution in compare to the direct graphics from C++ code, at the same size of the resulting picture. In addition, the C++ implementation allows the figures to be reproduced by colleagues who do not have Mathematica installed, while the C++ compilers are distributed for free. In such a way, the C++ implementation of functions simplifies verification and refutation of scientific concepts and correspond to the general methodology of Science, based on the TORI axioms, see.<sup>4,9)</sup> The draft of the TORI axioms had been described in the first issue of the Far East Journal for Mechanical Engineering.<sup>8)</sup>

Name of function Doya is chosen after the last name of Valerie Doya from the Universite de Nice. Discussions with Valerie Doya had stimulated the analysis of superfunctions, specific for the nonlinear optical materials. The optical amplifier with saturation is a special case of such a material; the Doya function appears as the transfer function of the idealized amplifier.

#### 1.4 Goal of this article

The general goal of this article is to pop-up the status of functions Tania and Doya closer to those of elementary functions. Below, the properties of functions Tania and Doya by equations (5) and (4) are described. The inverse function  $\text{ArcTania} = \text{Tania}^{-1}$  is considered. It happens to be expressed through elementary function. These properties are used to construct the efficient (id est, fast, stable and precise) algorithm for evaluation of Tania and Doya. The algorithms for the complex(double) precision are presented at TORI; the improvement of the precision is straightforward, these functions can be evaluated with so many decimal digits as necessary. The representation of physical quantities through these functions is exact, in the same sense, as exact is representation of the coordinate of the idealized harmonic oscillator through function sin.

Function Doya has sense of the transfer function of the optical amplifier, in the simple model, that still takes into account the saturation of gain. Then, function Tania can be interpreted as the superfunction<sup>17)</sup>. The use of superfunctions is expected to improve the precision of characterization of the optical materials. For demonstration of the efficiency of this algorithm, it is important to consider examples, where the exact, precise expression for the superfunction is already known. Functions Doya and Tania provide one of such examples.

In many cases, behavior of a superfunction in the complex plane is important to establish its uniqueness. For this reason, it is important to analyze properties of superfunctions in the complex plane (although, for the intermediate applications in the Laser Science, perhaps, only the real values of the argument are used). In this article, properties of functions Tania and Doya are analyzed for complex values of the argument.

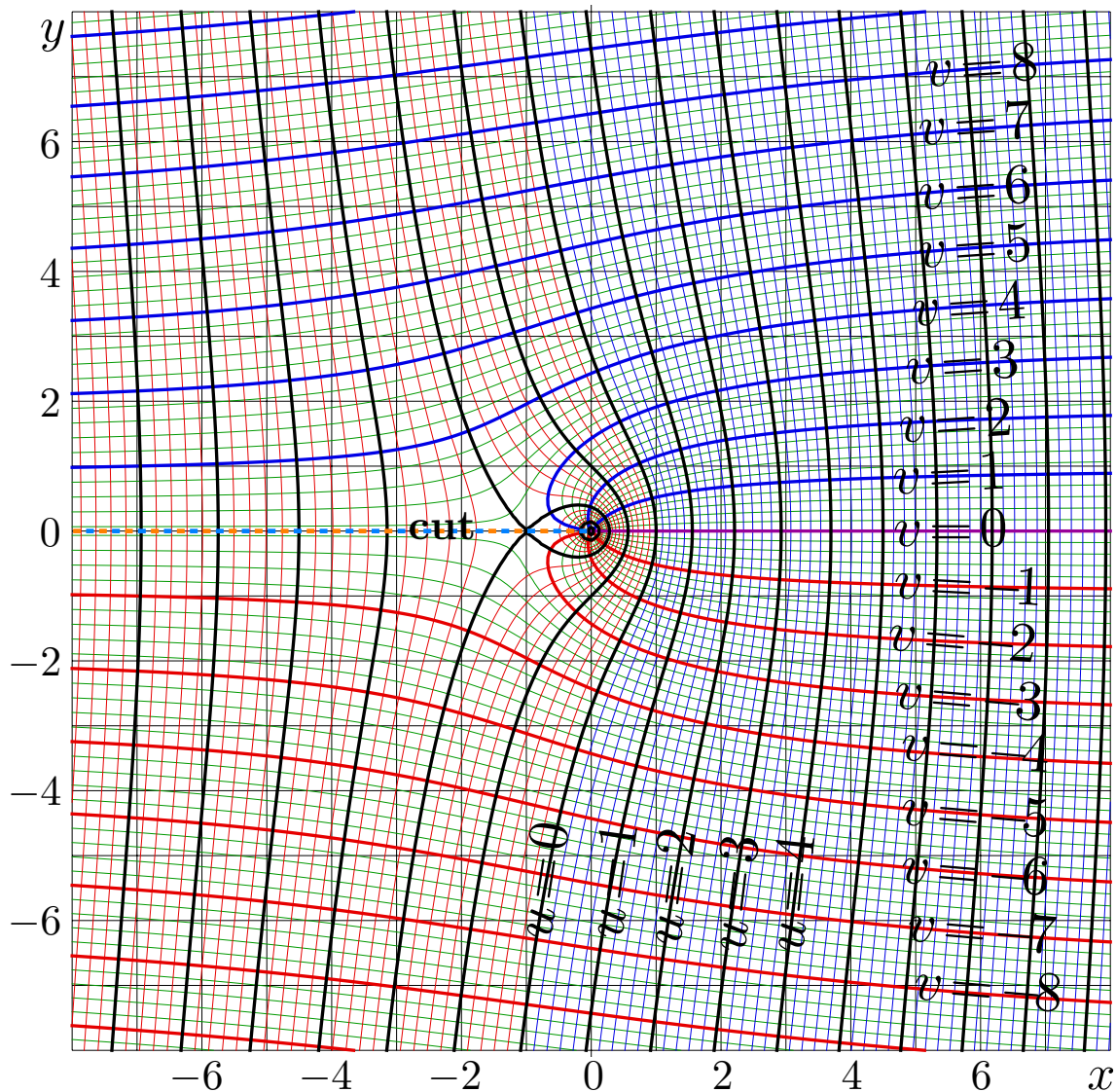


Fig. 1. Complex map of  $f = \text{ArcTania}(x+iy)$  ;  $u = \Re(f)$  ,  $v = \Im(f)$

**2. Arctania**

Let ArcTania be the inverse function of Tania; id est,  $\text{ArcTania} = \text{Tania}^{-1}$ . Complex map of this function is shown in Figure 1; the real-real plot is shown in Figure 2. ArcTania can be expressed in terms of elementary functions:

$$\text{ArcTania}(z) = z + \ln(z) - 1 \tag{8}$$

Expression (8) is valid for  $z \in \mathbb{C} \setminus \{z \in \mathbb{R}, z \leq 0\}$  and represents the specific case of the general rule, that the inverse function for the solution  $F$  of equation

$$F'(z) = H(F(z)) \tag{9}$$

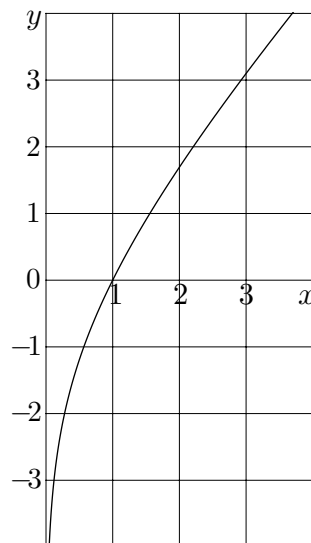


Fig. 2.  $y = \text{ArcTanya}(x)$

can be expressed with the integral (id est, quadrature) of the simple expression of function  $H$ . In order to see that, consider two variables,  $x$  and  $y$ , related with equation  $y = F(x)$ . Then, the differentials  $dx$  and  $dy$  for these variables are related with

$$dy = H(y) dx \quad (10)$$

which is in certain sense simpler than (9); this leads to

$$dx = \frac{dy}{H(y)} \quad (11)$$

The integration from  $x_0$  to  $x_1$  gives

$$x_1 - x_0 = \int_{y_0}^{y_1} \frac{dy}{H(y)} \quad (12)$$

where  $y_0 = F(x_0)$  and  $y_1 = F(x_1)$ . In particular, for

$$H(z) = \frac{z}{1+z} \quad (13)$$

and  $x_0 = 0$ ,  $y_0 = 1$ , the integral in (12) can be simplified, and the straightforward algebra leads to relation  $x = \text{ArcTania}(y)$  with function  $\text{ArcTania}$  defined with (8). In wide range of values of  $z$ , the relation  $\text{Tania}(\text{ArcTania}(z)) = z$  holds.

The complex map of function  $\text{ArcTania}$  is shown in Figure 1 with lines  $u = \Re(\text{ArcTania}(x+iy)) = \text{const}$  and lines  $v = \Im(\text{ArcTania}(x+iy)) = \text{const}$ .  $\text{ArcTania}$  has the cut among the negative part of the real axis. Before the attack 2013.02.27, this image and its C++ generator (id est, the C++ code that draws the curves) were available also at <http://tori.ils.uec.ac.jp/TORI/index.php/File:ArcTaniaMap.png>.

The explicit plot  $y = \text{ArcTania}(x)$  is shown in Figure 2. It shows the vertical asymptotic at  $x \rightarrow 0$  and almost linear growth at large positive values of abscissa.

Various expansions for function  $\text{Tania}$  and the resulting algorithm for the evaluation uses the simple representation (8) for the inverse function. Similar approach can be used for the phenomenological description of various hompogeneous laser media, where the gain is expressed as function of intensity.

### 3. Tania function

In principle, the  $\text{Tania}$  function can be evaluated, inverting  $\text{ArcTania}$  numerically. At a poor initial approach, generally, such evaluation is slow and does not correspond to the goal to pop up the  $\text{Tania}$  function to the status of special function. The suitable initial approximation is important for the precise numerical adjustment of solution  $f$  of equation  $\text{ArcTania}(f) = z$  for given  $z$ . In this section, the approximations of the  $\text{Tania}$  function with elementary functions are suggested and supplied with their complex maps.

The complex map of  $\text{Tania}$  function is shown in Figure 3, in the similar way as  $\text{ArcTania}$  is shown in Figure 1. In Figure 3,  $f = \text{Tania}(x+iy)$  is shown in the  $x, y$  plane with lines

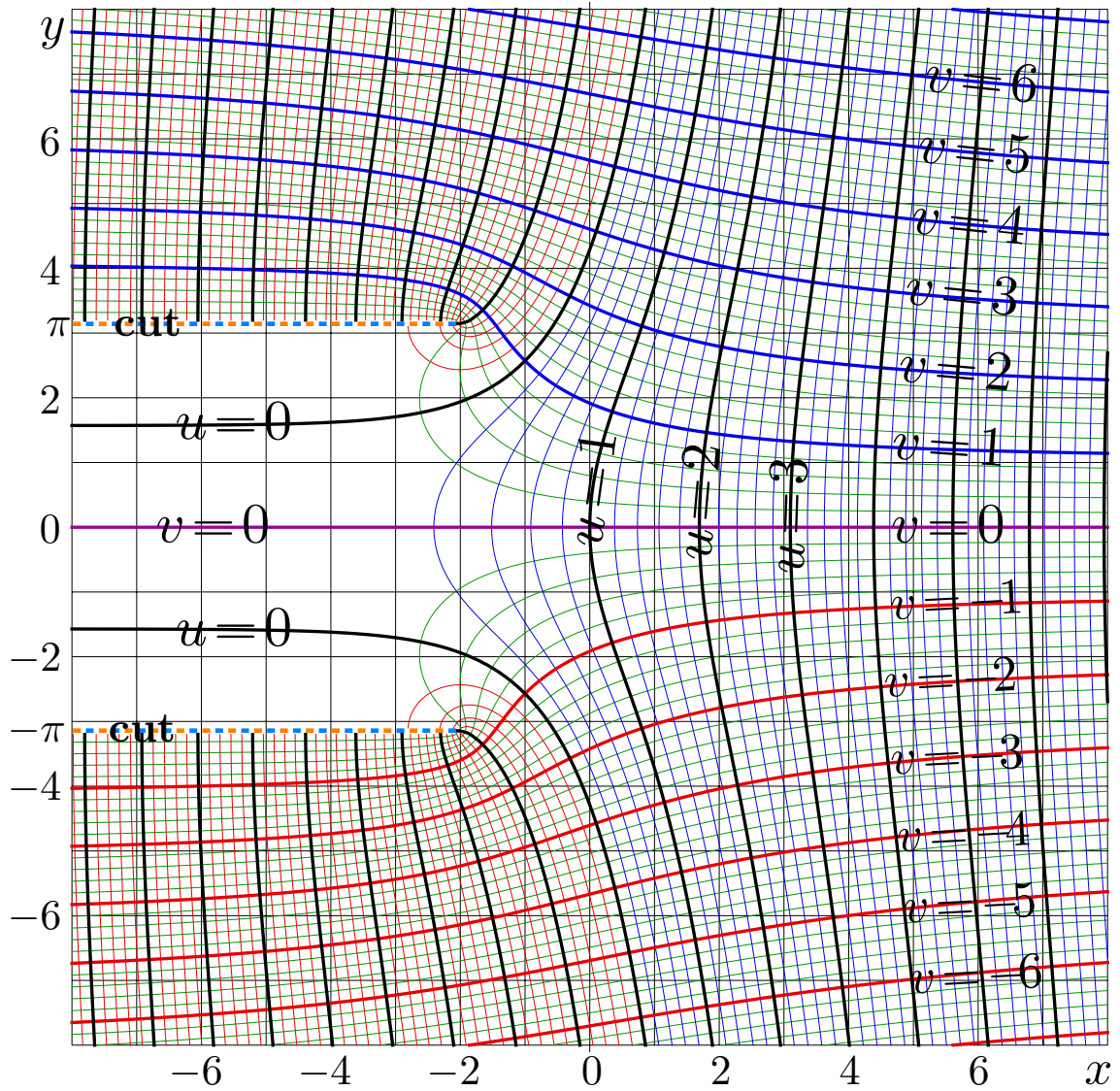


Fig. 3. Complex map of  $f = \text{Tania}(x+iy)$ ;  $u = \Re(f)$ ,  $v = \Im(f)$

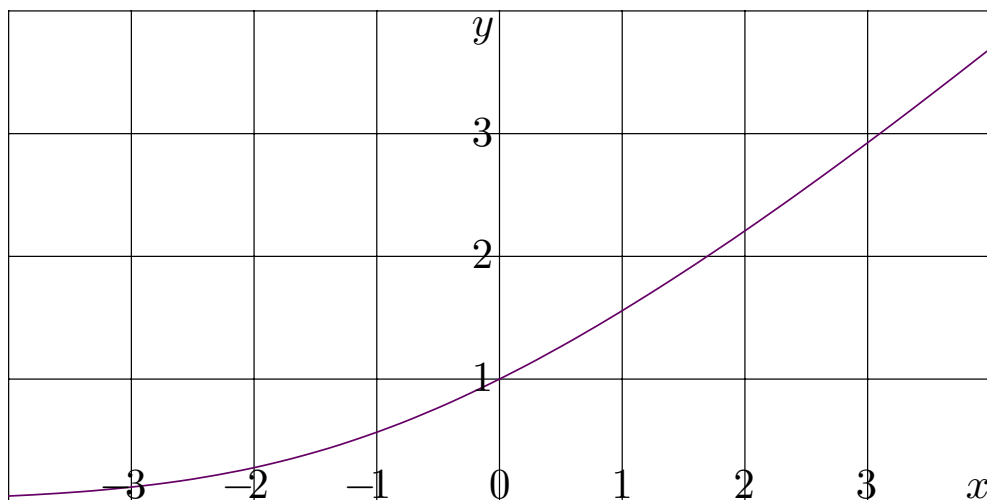


Fig. 4.  $y = \text{Tania}(x)$

$u = \Re(f) = \text{const}$  and lines  $v = \Im(f) = \text{const}$ .

Function Tania has two branch points, at  $-2 \pm \pi i$ . Following the TORI tradition, the cut lines are oriented along the real axis toward negative values of real part of the argument; in the figure, to the left.

The real-real plot of Tania function is shown in Figure 4. At negative values of the argument, Tania function shows almost exponential growth. The graphic passes through point (0,1). At large values of the argument, Tania function shows almost linear growth.

In order to plot Figures 3 and 4, the C++ implementation of the Tania function with 14 decimal digits is used. Before the attack 2013.02.27, that implementation was available for free at <http://tori.ils.uec.ac.jp/TORI>. The implementation uses the expansions below.

### 3.1 Tania at large values of the argument

The complex maps of Tania and ArcTania above show that asymptotically each of these functions becomes similar to its argument. Being far from the branch points (and outside the cut lines), the grid of the lines of constant real and constant imaginary part looks similar to that of the identity function. This indicates that the leading term of the expansion at infinity of Tania( $z$ ) should be just  $z$ . Such an expansion is suggested in this section.

At large values of the argument, Tania can be expanded as follows:

$$\begin{aligned} \text{Tania}(z) = & z + 1 - \ln(z) + \frac{\ln(z) - 1}{z} + \frac{\ln(z)^2 - 4\ln(z) + 3}{2z^2} \\ & + \frac{2\ln(z)^3 - 15\ln(z)^2 + 30\ln(z) - 17}{6z^3} + O\left(\frac{\ln(z)}{z}\right)^4 \end{aligned} \quad (14)$$

At  $|z| \gg 1$ , the truncated series (14) can be used for the evaluation of Tania( $z$ ), while the argument is not between the cut lines in figure 3. Similar expansion can be written also as follows:

$$\begin{aligned} \text{Tania}(z) = & (z+1) - \ln(z+1) + \frac{\ln(z+1)}{z+1} + \left(\frac{\ln(z+1)}{z+1}\right)^2 \left(\frac{1}{2} - \ln(z+1)^{-1}\right) \\ & + \left(\frac{\ln(z+1)}{z+1}\right)^3 \left(\frac{1}{3} - \frac{3}{4}\ln(z+1)^{-1} + \ln(z+1)^{-2}\right) \\ & + \left(\frac{\ln(z+1)}{z+1}\right)^4 \left(\frac{1}{4} - \frac{11}{5}\ln(z+1)^{-1} + 3\ln(z+1)^{-2} - \ln(z+1)^{-3}\right) \\ & + O\left(\frac{\ln(z+1)}{z+1}\right)^5 \end{aligned} \quad (15)$$

Such an expression shows that the effective small parameter of the expansion is  $\frac{\ln(z+1)}{z+1}$ . The complex map of the truncation of expression in the right hand side of (15) is plotted in the figure at right in the same notations as in the previous maps.

In figure 5, the shaded region indicates the range, where the precision of the approximation (15) is less than 3. In particular, the approximation fails between the cut lines of function

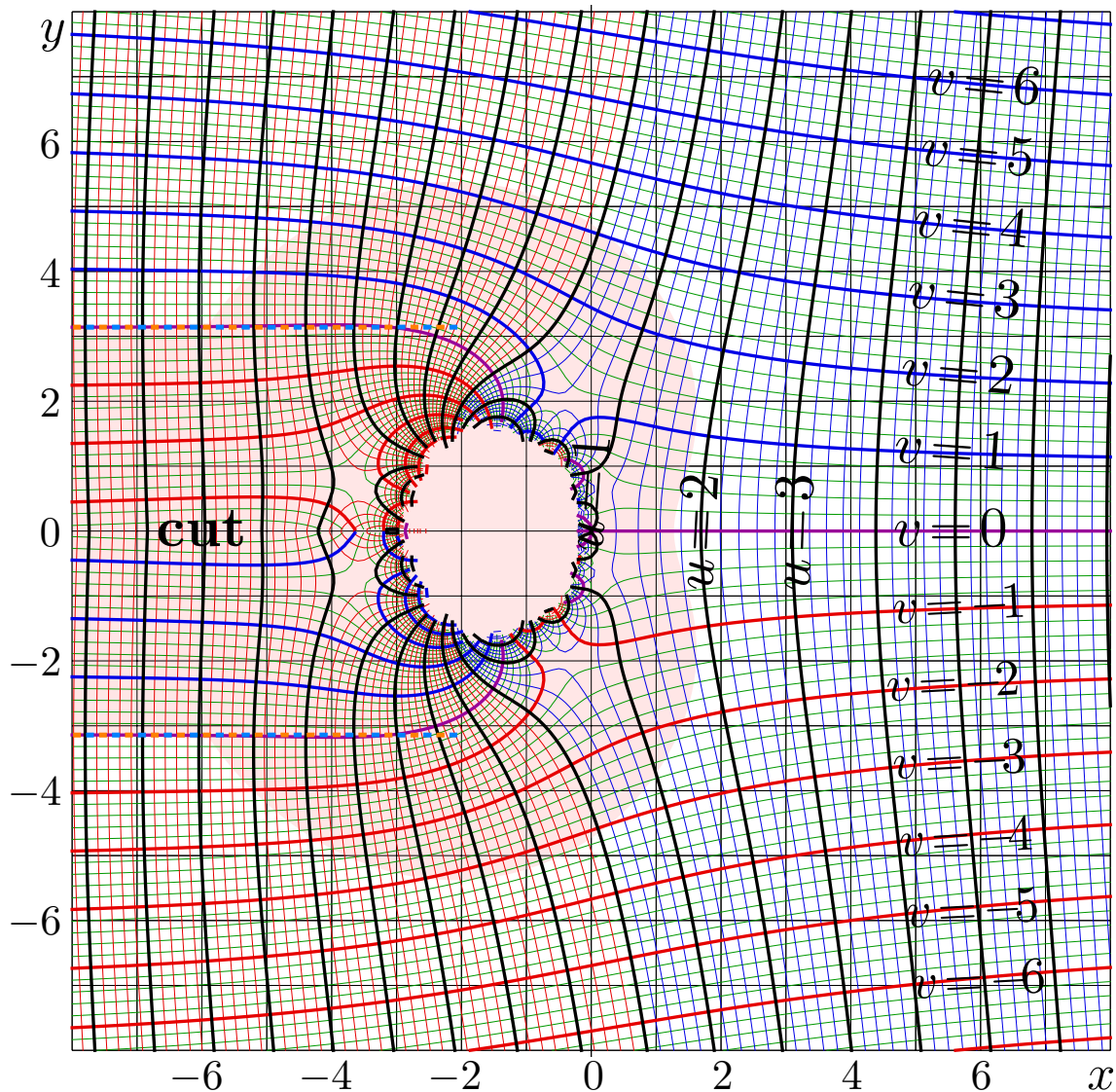


Fig. 5. Map of approximation (15) of Tania function at large values of the argument in the same notation as in previous maps

Tania, and, in addition, has an additional cut line along the negative part of the real axis (That the Tania function does not have).

Outside the shaded region, the truncated series with the last term of order of

$$\left(\frac{\ln(z+1)}{z+1}\right)^4 \quad (16)$$

return more than 3 significant decimal digits. The precision of any approximation of Tania function is evaluated as follows:

$$\text{Precision}(z) = -\lg\left(\frac{|\text{Tania}(z) - \text{approximation}(z)|}{|\text{Tania}(z)| + |\text{approximation}(z)|}\right) \quad (17)$$

Outside the shaded range in Figure 5, the truncated series (15) provides at least 3 significant decimal digits. Then, the evaluation can be adjusted by the Newton method; the 3-d iteration



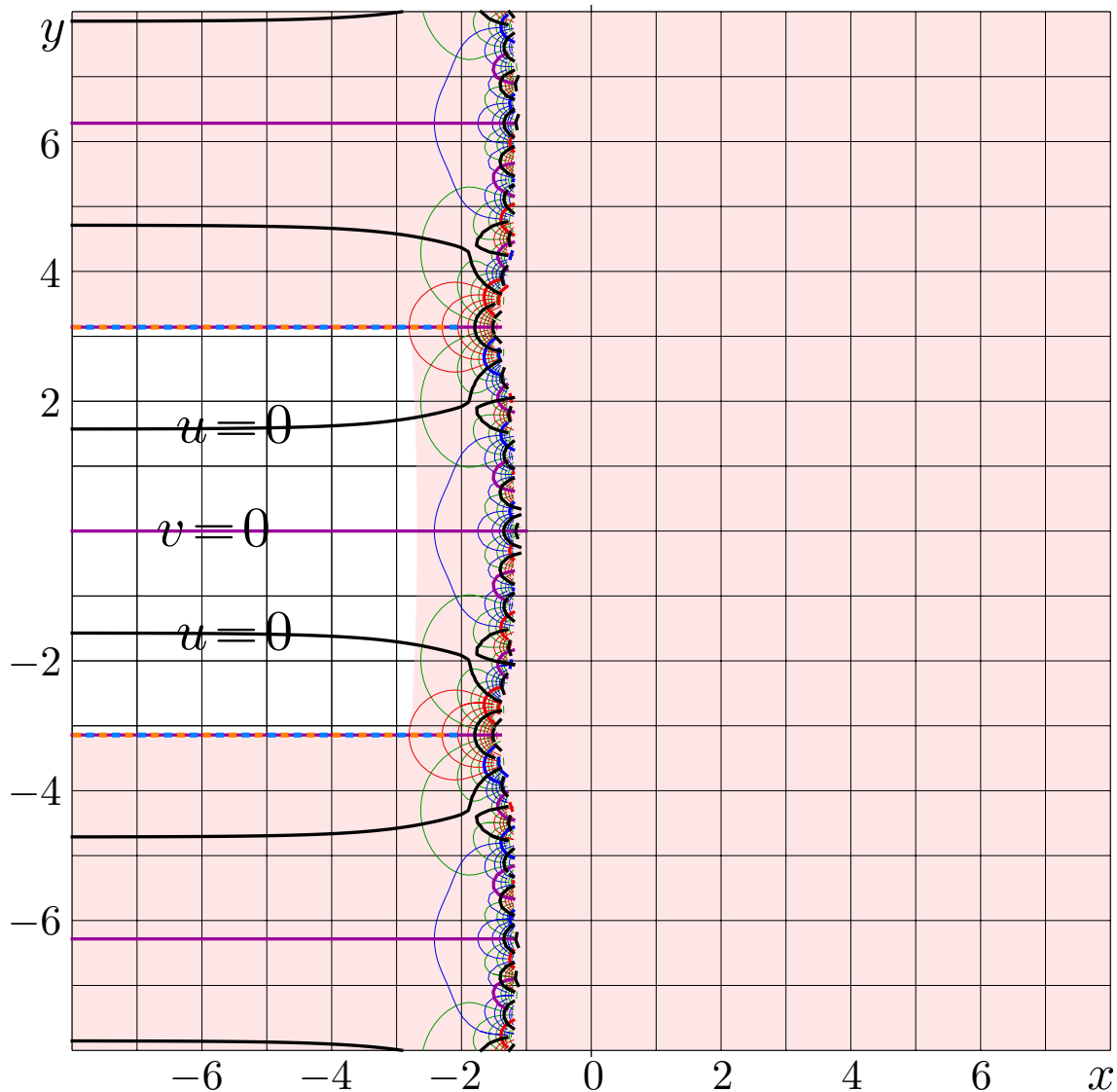


Fig. 6. Complex map of the approximation by (18); in the shaded region the approximation returns less than 3 correct digits.

provides of order of 14 significant figures.

Approximation (15) covers the most of the complex plane. However, it fails at small values of the argument. Some additional representations are necessary for the evaluation of Tania at the branch points and in the strip along negative part of the real axis. These representations are considered in the following subsections.

### 3.2 Asymptotic expansion of Tania between the cut lines

The expansions (14) and (15) are not valid between the cut lines. For this range, for large values of the argument, the special expansion, shown in Figure 6, is suggested in this subsection.

For  $-\Re(z) \gg 1$  at  $|\Im(z)| < \pi$ , the solution  $f$  of equation  $\text{ArcTania}(f) = z$  can be expanded

as follows:

$$F(z) \sim \varepsilon - \varepsilon^2 + \frac{3}{2}\varepsilon^3 - \frac{8}{3}\varepsilon^4 + \frac{125}{24}\varepsilon^5 + O(\varepsilon^6) \quad (18)$$

where  $\varepsilon = \exp(1+z)$ .

Such expansion can be obtained iterating assignment

$$F = \varepsilon \exp(-f) \quad (19)$$

with the initial approach  $f = \varepsilon + O(\varepsilon^2)$ .

The example of the truncated series is shown in Figure 6. The same notations for the complex map are used. In the shaded region, the precision is less than 3. In particular, the expansion (18) does not approximate  $Tania(z)$  at  $\Re(z) > -2$ . However, the truncated series can be used for the precise evaluation of  $Tania(z)$  at  $\Re(z) < -3$ ,  $\Im(z) < \pi$ .

Together with the expansion 15, the expansion (18) covers the most of the complex plane, leaving only a finite-size domain that includes the branch points and the origin of the coordinates. The expansion at these points are considered below.

### 3.3 *Tania in vicinity of its branch point*

The expansions above do not approximate function  $Tania$  at vicinity of its branch points. The expansion at the upper branch point gives the approximation shown in Figure 7 is suggested below.

$Tania(z)$  has two cut lines at  $\Re(z) \leq -2$ ,  $\Im(z) = \pm\pi$ . For the expansion of  $Tania(z)$  at the upper branch point  $z = -2 + \pi i$ , the convenient small parameter is

$$t = i\sqrt{\frac{2}{9}(z + 2 - \pi i)} \quad (20)$$

then, the expansion of  $Tania$  can be written as follows:

$$Tania(z) = -1 + 3t - 3t^2 + \frac{3}{4}t^3 + \frac{3}{10}t^4 + \frac{9}{160}t^5 - \frac{3}{70}t^6 - \frac{1251}{22400}t^7 - \frac{9}{280}t^8 + \dots \quad (21)$$

The Mathematica routine **Series** easily calculates some tens of the first coefficients of the expansion. The series seems to converge while  $|z + 2 - \pi i| < 2\pi$ ; in vicinity of another branch point, id est, at  $z \approx -2 - i$ , such an approximation is not valid.

The figure at right shows complex map of the truncated series by (21) in the same notations as in the previous maps. The shaded region indicates the domain where the precision of the approximation is less than 3; in the white spot the approximation by the polynomial of 5th order with respect to  $t$  gives at least three significant figures.

In vicinity of the origin of coordinates, for example, at  $-4 < z < 6$ , the truncated series (with term of order of  $t^9$  and higher dropped) provides of order of 2 decimal digits, and the deviation of line  $v=0$  from the abscissa axis is clearly seen in the figure.

The expansion at the branch point is asymptotic; the more terms are taken into account, the narrower is the range of approximation. From the point of view of implementation, it

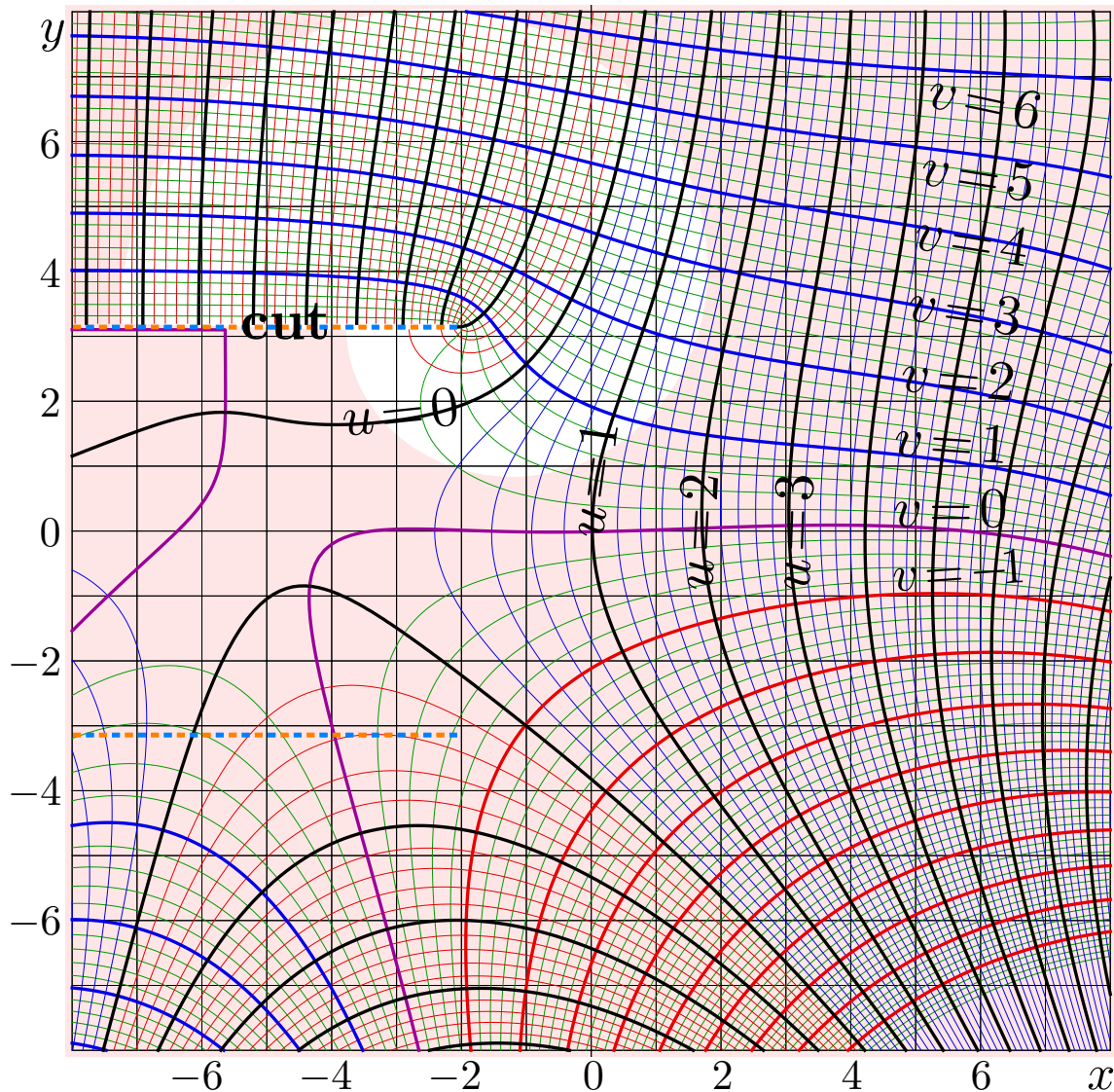


Fig. 7. Approximation of Tania based on expansion (20), (21)

has sense to take into account not so many terms, and to adjust the solution by the Newton method; the ArcTania function and its derivative are fast to evaluate.

Similar expansion of Tania at point  $-2 - \pi i$  can be obtained by complex conjugation of expressions (20) and (21).

### 3.4 Tania at small values of the argument

The expansions above, in principle, allow to cover all the complex plane with the approximations of the Tania function. But the expansion at the branch points may be not convenient while dealing with Tania of a real argument. For this case, the Taylor expansions of Tania at the real axis are suggested in this section. Complex map of the corresponding approximation of Tania function is shown in Figure (??).

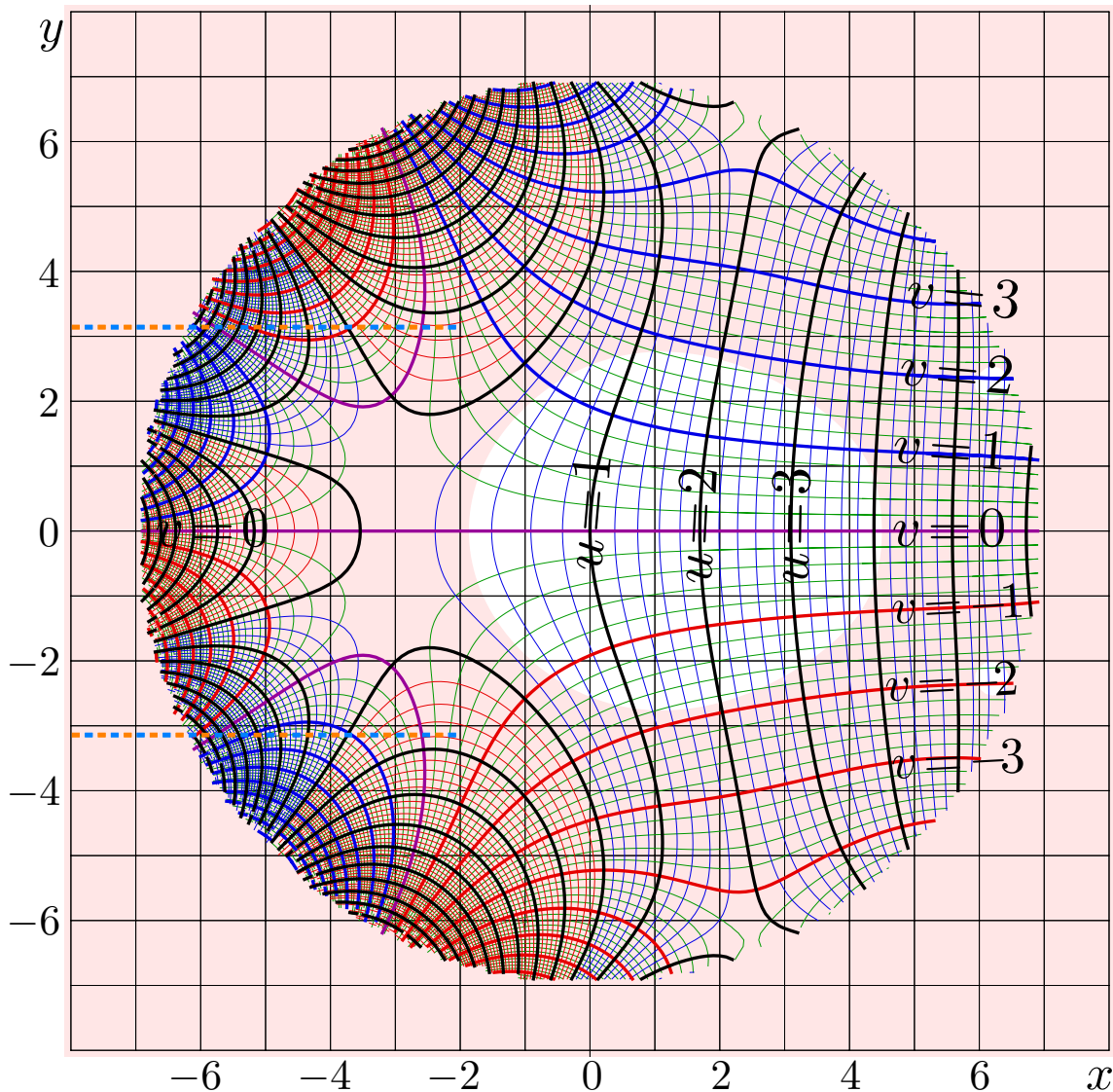


Fig. 8. Approximation of Tania function with the Taylor expansion expansion (22)

The Taylor expansion of Tania at zero

$$\text{Tania}(z) = 1 + \frac{z}{2} + \frac{z^2}{16} - \frac{z^3}{192} - \frac{z^4}{3072} + \frac{13z^5}{61440} - \frac{47z^6}{1474560} + \frac{73z^7}{41287680} + \dots \quad (22)$$

converges at  $|z| < \sqrt{4 + \pi^2} \approx 3.724$ ; this series can be obtained with the Mathematica's operator InverseSeries, converting the expansion

$$\text{ArcTania}(z) = 2(z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \frac{(z-1)^4}{4} + \frac{(z-1)^5}{5} + \frac{(z-1)^6}{6} + O(z-1)^7 \quad (23)$$

However, more terms can be added to the series in (22) and (23). At moderate values of  $|z| \lesssim 1$ , the expansion (22) can be used for the efficient (quick and precise) evaluation of  $\text{Tania}(z)$ ; in order to get 14 significant decimal digits, it is sufficient to take 20 terms.

The complex map of the truncated expansion (22) as polynomial of 7th power of  $z$  is shown in Figure 8 for  $|z| < 7$ . The range where such an approximation of  $\text{Tania}(z)$  gives less

than 3 significant figures is shaded. The same notations as in the previous figures are used.

In the similar way, the expansion of  $Tania(z)$  at  $z = -2 + 1/e \approx -1.632$  can be written as follows:

$$\begin{aligned} Tania(z) = & \frac{1}{e} + \frac{1/e}{1+1/e} \left( z + 2 - \frac{1}{e} \right) + \frac{1/e}{2(1+1/e)^3} \left( z + 2 - \frac{1}{e} \right)^2 \\ & + \frac{(1-2/e)/e}{6(1+1/e)^5} \left( z + 2 - \frac{1}{e} \right)^3 + \dots \end{aligned} \quad (24)$$

The series converges at  $|z + 2 - \frac{1}{e}| < \sqrt{\pi^2 + 1/e^2} \approx 3.1445$ .

### 3.5 *Tania implemented*

Together, the truncated expansions above cover all the complex plane with at least 3-digit approximations  $f_0(z)$ . The third iteration with Newton method

$$f_{n+1} = \frac{z - \text{ArcTania}(f_n)}{\text{ArcTania}'(f_n)} \quad (25)$$

divides the evaluation  $f_3$  that approximates  $Tania(z)$  with at least 14 significant figures in the whole complex plane. Only few terms are used in the truncated expansions for the numerical implementation; this makes the algorithm of evaluation fast and robust. In particular, all figures in this article can be plotted in real time; without to press a key, to have a tea.

The position of cut lines indicate the part of integration of equation (2). The contour of integration may go from 0 to  $\Im(z)$  along the imaginary axis, and then from  $\Im(z)$  to  $z$  along the straight line, parallel to the real axis. However, any continuous topological modifications of the contour of integration may be applied, while the modification does not touch the cut lines in Figure 4.

The following improvement of the precision of the implementation of  $Tania$  is straightforward. In this sense, function  $Tania$  is exact, in the same way as exact is, for example, function  $\sin$ . On one hand, for plotting graphics, and in particular, the complex map of function  $Tania$  in Figure 3, the 14 digit precision are not necessary; four digits would be sufficient to get the camera-ready copy of all the pictures. In the similar way, for comparison with experiments, the  $Tania$  of real argument is expected to be used. On the other hand, after this article, function  $Tania$  should be considered as a robust tool, that can be used without to analyze, how does it work. It is especially important for comparison with other superfunctions, while the asymptotic behavior in the complex plane seems to be important criterion do choose namely physical solution among various superfunctions.

## 4. Doya

Function  $Tania$  may have sense of distribution of intensity along the propagation of a wide uniform beam (neglecting the diffraction) through the uniformly pumped gain medium with simple kinetic model. However, for applications in the experimental laser science, another

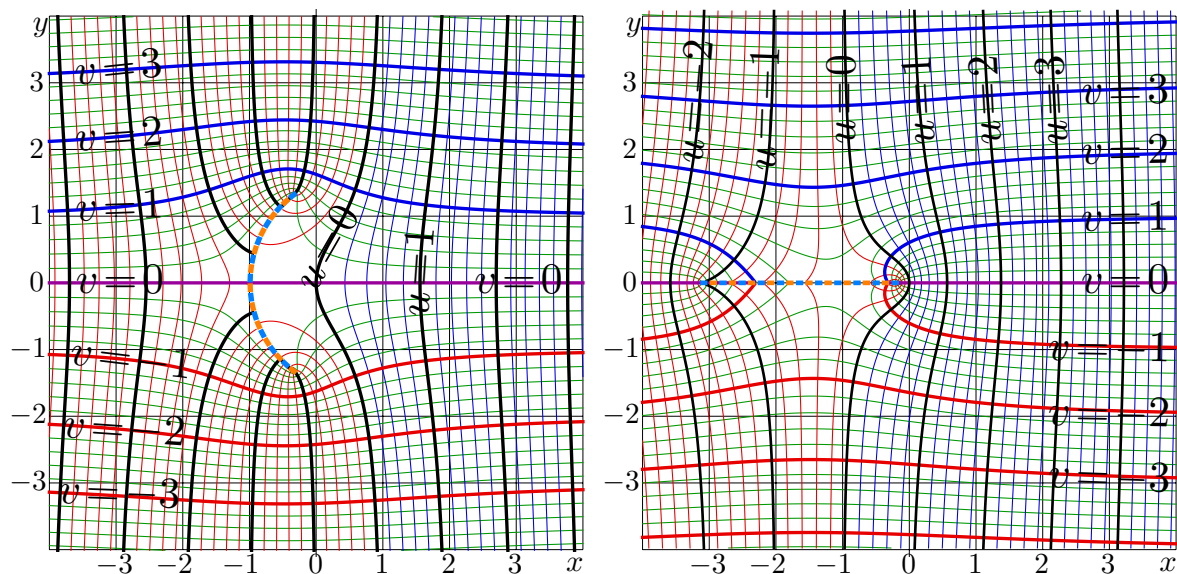


Fig. 9. Complex maps of functions  $f = \text{ArcDoya}(x+iy)$ , left, and  $f = \text{Doya}(x+iy)$ , right; lines  $u = \Re(f) = \text{const}$  and lines  $v = \Im(f) = \text{const}$  are shown.

function is also important, namely, the Doya function by (1)-(4). This function has sense of the transfer function of the idealized amplifier. Properties of this function are described in this section. Complex maps of functions  $\text{Doya} = \text{Doya}^1$  and  $\text{ArcDoya} = \text{Doya}^{-1}$  are shown in Figure 9. The real-real plot of function  $\text{Doya}^t$  is shown in Figure 10 for various real values of parameter  $t$ .

#### 4.1 Implementation through Tania

As function Tania is already implemented, the  $t$ th iteration of the Doya function can be expressed as follows:

$$\text{Doya}^t(z) = \text{Tania}\left(t + \text{ArcTania}(z)\right) \quad (26)$$

In this equation,  $z$  has sense of the input signal, and  $t$  has sense of the length of the amplifier. Then,  $\text{Doya}^t(z)$  has sense of the output signal. Representation (26) is used to plot the complex maps in Figure 9 and the real-real plot of  $y = \text{Doya}^t(x)$  in Figure 10.

In the representation (26), parameter  $t$  has sense of number of iteration of function Doya. However, this parameter has no need to be integer; the length of the amplified can be arbitrary. At the complex implementation of functions Tania and ArcTania, parameter  $t$  may have even complex values; the function can be iterated even complex number of times.

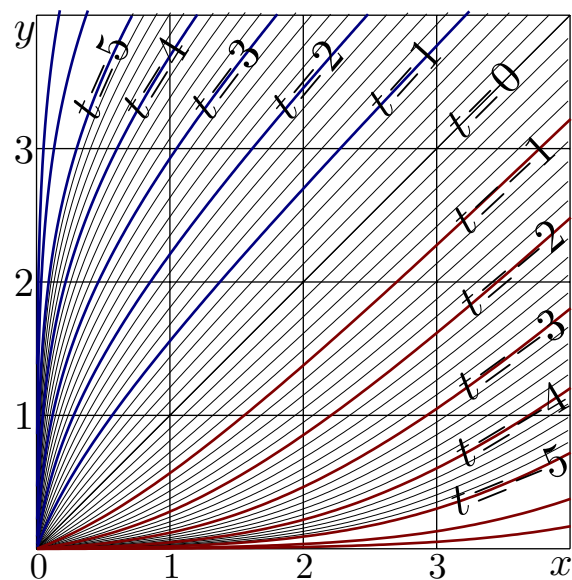


Fig. 10.  $y = \text{Doya}^t(x)$  versus  $x$  for various  $t$

#### 4.2 Tania as superfunction of function Doya

From representation (26), it follows, that function Tania can be interpreted as superfunction<sup>17)</sup> for the transfer function Doya; the transfer equation

$$\text{Doya}(\text{Tania}(z)) = \text{Tania}(z+1) \quad (27)$$

holds for wide range of values of  $z$ . This justifies the use of the upper superscript in notation  $\text{Doya}^t$ . In this case, parameter  $t$  can be interpreted as number of iteration of function Doya. In particular,  $\text{Doya}^0$  is the identify function, and  $\text{Doya}^{-1} = \text{ArcDoya}$  is just inverse function of Doya.

In wide range of values of variables, the group relation takes place:

$$\text{Doya}^a(\text{Doya}^b(z)) = \text{Doya}^{a+b}(z) \quad (28)$$

Such a relation has analogy with notations used in Quantum Mechanics. Powers of linear operators are combined in the same way, as equation (28); then  $z$  should be replaced to the wave function. Generalization of formalism of superfunctions from the case of single complex argument  $z$  the multidimensional case may be subject of the future research.

#### 4.3 Application of Doya function

While, for the specific application in the Laser Science, the only real values of  $t$  have simple physical meaning. But the history of science indicates, that, as some function is implemented for integer and then real values of some parameter or argument, soon or late, the complex values also find their applications. The goal of this article is to provide the Tool. I expect, the complex values of number of iterations  $t$  or those of argument of function Doya will find their applications, as complex values of the argument of function sin are sometimes used. For this reason, the functions Tania and Doya are constructed for complex values of the argument. It is easier (and methodologically correct) to make the universal tool at once, than to redo it, when the new area of the applications appear.

The Doya function is important example for the general method of construction of superfunctions, because, for the Doya function, the physically-meaningful superfunction is already known, it is Tania function. In such a way, we can solve the transfer equation (27) with given transfer function Doya numerically, and compare the solution with the exact solution Tania.<sup>1)</sup> However, in order to make such a demonstration clear, the efficient numerical implementation of functions Tania and Doya should be available, and their properties should be described.

Function Doya may be used for fitting of the experimental data; it is easier to use the special function ("almost elementary function") than the numerical solution of the differential equation (2). The similar approach is expected to be useful also for other transfer functions and their superfunctions.

## 5. Conclusions

Two functions, Tania and Doya, are defined with (1)-(5). Tania can be interpreted as superfunction of function Doya, satisfying the transfer equation (27). Function Tania has sense of distribution of intensity of light in optical amplifier with simple model of saturation. Function Doya has sense of the transfer function of such an amplifier. The real-real plots and the complex maps of these functions are supplied. The efficient numerical algorithms in C++ for the evaluation of these functions are developed through the asymptotic expansions. The algorithms suggested can be used as prototypes for the implementation as incorporated functions in the programming languages.

Functions Doya and Tania are important example to show the robustness of the general algorithm of building of superfunctions, in particular, for reconstruction of distribution of signal along the amplifier from its transfer function. While Doya is the transfer function, then, the physically-meaningful superfunction should be Tania. In recent paper,<sup>1)</sup> it is shown, that for the two specific examples, the general algorithm does well. The description of properties of functions Tania and Doya, presented in this paper, is important for the easy reproduction of these examples and testing of general algorithm of reconstruction of superfunction. Such a reconstruction is expected to improve the precision of characterization of the laser materials from the measurement of the transfer function of a bulk sample.

The efficient implementation of functions Tania and Doya should make them useful for fitting of experimental data with physically-meaningful special functions.

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